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EISENSTEIN-SERIES ON REAL, COMPLEX, AND QUATERNIONIC HALF-SPACES

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The real, complex, and quaternionic half-spaces are introduced in certain analogy with the Siegel half-space. The modified symplectic group acts on the attached half-space in the usual way. At first properties of these half-spaces considered as symmetric spaces are derived. Then a fundamental domain with respect to the modified modular group, which consists of integral modified symplectic matrices, is constructed. The behavior of convergence of the corresponding Eisenstein-series is determined carefully. The Fourier-coefficients of the Eisenstein-series are calculated explicitly, whenever the degree is sufficiently small.

Introduction. The present paper deals with half-spaces, which are built in analogy with the Siegel half-space, and the corresponding non-analytic Eisenstein-series. The roots can be traced back to C. L. Siegel's paper "Die Modulgruppe in einer einfachen involutorischen Algebra" [**30**]. A special case of these investigations is considered and continued by the examination of the Riemannian geometry as well as the attached Eisenstein-series.

To be more precise, throughout this paper let $\mathbf{F}$ stand for $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, where $\mathbf{H}$ is the skew-field of real Hamiltonian quaternions. Just as in [**16**] let $r = r(\mathbf{F}) = \dim_{\mathbf{R}} \mathbf{F}$ and denote the standard basis of $\mathbf{F}$ over $\mathbf{R}$ by $1 = e_1, \ldots, e_r$. Given $a = \sum_{j=1}^{r} a_j e_j \in \mathbf{F}$, $a_j \in \mathbf{R}$, put $\operatorname{Re}(a) := a_1$ and let $a \mapsto \bar{a} = 2\operatorname{Re}(a) - a$ denote the canonical conjugation in $\mathbf{F}$. Then $A^{(n)}$, resp. $A \in \operatorname{Mat}(n; \mathbf{F})$, means that A is an $n \times n$ matrix with entries in $\mathbf{F}$ and A' denotes the transpose of A. The letter I is reserved for the identity matrix and 0 for the zero matrix of appropriate size. $\operatorname{GL}(n; \mathbf{F})$ stands for the group of units in the ring $\operatorname{Mat}(n; \mathbf{F})$.

The half-space $\mathscr{H}(n; \mathbf{F})$ consists of all $Z \in \operatorname{Mat}(n; \mathbf{F})$ such that $Z + \overline{Z}'$ becomes a positive definite Hermitian matrix. Thus $i\mathscr{H}(n; \mathbf{C})$ equals the Hermitian half-space, which was investigated by H. Braun [**3**]. But the remaining cases are related, because $\mathscr{H}(n; \mathbf{H})$ can always be embedded into the Hermitian half-space of degree $2n$.

The attached modified symplectic group $\operatorname{MSp}(n; \mathbf{F})$ consists of the automorphs of the symmetric matrix $Q = \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$, $I = I^{(n)}$, having the signature (n, n) and acts on $\mathscr{H}(n; \mathbf{F})$ in the usual way. The real modified symplectic group was already investigated by C. L. Siegel **[28]**, M. Koecher **[14]**, III, §1, and H. Maaß **[23]** in different contexts. Considering the symplectic group

$$(0.1) \qquad \operatorname{Sp}(n; \mathbf{F}) = \{M \in \operatorname{Mat}(2n; \mathbf{F}); \overline{M}'JM = J\},$$
$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad I = I^{(n)},$$

as in **[16]**, one has

$$(0.2) \qquad \begin{pmatrix} e_2 I & 0 \\ 0 & I \end{pmatrix} \operatorname{MSp}(n; \mathbf{C}) \begin{pmatrix} e_2 I & 0 \\ 0 & I \end{pmatrix}^{-1} = \operatorname{Sp}(n; \mathbf{C}).$$

$\operatorname{MSp}(n; \mathbf{F})$ is obviously conjugate to the indefinite unitary group $\mathrm{U}^n(2n, \mathbf{F})$ in **[34]**, p. 377, and to $\mathrm{O}(n, n)$, $\mathrm{U}(n, n)$, resp. $\operatorname{Sp}(n, n)$, if $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, resp. $\mathbf{H}$, in Helgason's notation (cf. **[8]**, p. 340).

Nevertheless the notion of modified symplectic group may be justified by the connection with C. L. Siegel's paper **[30]**. Consider $\mathbf{F} = \mathbf{R}$, $\mathbf{H}$ and an arbitrary $\mathbf{R}$-involution ι of $\operatorname{Mat}(n; \mathbf{F})$. According to **[1]**, X, Theorem 11, there exists $F \in \operatorname{GL}(n; \mathbf{F})$ such that $\overline{F}' = \pm F$ and

$$\iota(X) = F\overline{X}'F^{-1} \quad \text{for } X \in \operatorname{Mat}(n; \mathbf{F}).$$

In this general situation C. L. Siegel **[30]** defined the symplectic group Σ. In our notation we gain

$$(0.3) \qquad \Sigma = \begin{cases} \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix} \operatorname{Sp}(n; \mathbf{F}) \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}^{-1} & \text{if } \overline{F}' = F, \\ \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix} \operatorname{MSp}(n; \mathbf{F}) \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix}^{-1} & \text{if } \overline{F}' = -F. \end{cases}$$

The special case $\mathbf{F} = \mathbf{H}$, $n = 1$, $F = (e_3)$ was recently treated by E. Kähler **[10]**.

The Riemannian geometry and the description of the geodesics can be pointed out along the lines of Siegel's classical work **[29]**, where the case $\mathbf{F} = \mathbf{C}$ is due to H. Klingen **[12]**. If dZ denotes the matrix of differentials, then

$$ds^2 = \tfrac{1}{2} \operatorname{trace}(Y^{-1}dZY^{-1}\overline{dZ}' + dZY^{-1}\overline{dZ'}Y^{-1}), \qquad Y := \tfrac{1}{2}(Z + \overline{Z}'),$$

proves to be a positive definite quadratic differential form. The modified symplectic transformations become isometries. Thus $\mathscr{H}(n; \mathbf{F})$ endowed with ds^2 turns out to be a Riemannian globally symmetric space of the noncompact type, which is irreducible except for

$\mathbf{F} = \mathbf{R}$, $n = 1, 2$ and which fails to be Hermitian, whenever $\mathbf{F} = \mathbf{R}$, $n \neq 2$, resp. $\mathbf{F} = \mathbf{H}$, $n \geq 1$.

$\mathscr{H}(1; \mathbf{C})$ equals the right half-plane in $\mathbf{C}$. Moreover $\mathscr{H}(1; \mathbf{H})$ becomes a model of the four-dimensional hyperbolic space, which was recently treated by E. Kähler [**10**]. Kähler's paper was the starting point of these investigations. The present paper arose from the attempt of combining Kähler's approach with the investigations of Eisenstein-series on the three-dimensional hyperbolic space by J. Elstrodt, F. Grunewald and J. Mennicke [**6**] as well as with Siegel's methods. Therefore this paper can also be understood as an extension of [**6**].

Choosing a special order for $\mathbf{F} = \mathbf{R}, \mathbf{C}, \mathbf{H}$, namely $\mathbf{Z}$, the Gaussian integers and the quaternions of Hurwitz, the modified modular group is defined to consist of all integral modified symplectic matrices. By means of the Euclidean algorithm a simple set of generators of the modified modular group can be determined. Following the classical procedure as in the case of the Siegel half-space, a fundamental domain is obtained, which has a cusp only at infinity.

The last two paragraphs deal with the corresponding non-analytic Eisenstein-series. Let Γ_n denote the modified modular group and Γ_n^∞ the subgroup of all matrices, whose C-block equals 0. Given $Z \in \mathscr{H}(n; \mathbf{F})$ and $M \in \Gamma_n$ set $Y_M = \frac{1}{2}(M\langle Z\rangle + \overline{M\langle Z\rangle}')$. Then the Eisenstein-series is given by

$$E_n^{\mathbf{F}}(Z, s) = \sum_{M:\,\Gamma_n^\infty\backslash\Gamma_n} (\det Y_M)^s, \qquad Z \in \mathscr{H}(n; \mathbf{F}),$$

and converges locally uniformly in Z and s. The abscissa of absolute convergence equals $\mathrm{Re}(s) = \frac{1}{n} \cdot d$, where d denotes the dimension of the real vector space of all skew-Hermitian matrices. One can define a modified Siegel ϕ-operator and obtains the same result, namely

$$E_n^{\mathbf{F}}(\cdot, s) \,|_s \phi = E_{n-1}^{\mathbf{F}}(\cdot, s),$$

as known from the classical case.

The investigations of $E_n^{\mathbf{R}}(\cdot, s)$ by H. Maaß [**23**] are extended and partially strengthened. The Eisenstein-series $E_n^{\mathbf{C}}(\cdot, s)$ were also examined by G. Shimura [**27**]. But one has to distinguish carefully between $E_n^{\mathbf{H}}(\cdot, s)$ and the analytic Eisenstein-series on the half-space of quaternions in [**16**], since the domains of definition are completely different.

Moreover coincidences between different classes of symmetric spaces for "small" values of n (cf. [**8**], p. 351–353) correspond to identities between the associated Eisenstein-series. Therefore Eisenstein-series on the upper half-plane in $\mathbf{C}$ as well as Eisenstein-series for $\mathrm{GL}(4;\mathbf{Z})$ (cf. [**31**]) come to light.

Finally the Fourier-expansions of Eisenstein-series are investigated. Just as in the case of the Siegel half-space, one cannot expect explicit formulas for arbitrary degree. But if the degree is sufficiently "small", the explicit description of the Fourier-coefficients succeeds. As one can expect from the upper half-plane (cf. [**19**], [**20**]), resp. the three-dimensional hyperbolic space (cf. [**6**]), resp. from Eisenstein-series for $\mathrm{GL}(n;\mathbf{Z})$ (cf. [**31**]), the Fourier-coefficients involve the modified Bessel function and certain weighted divisor sums.

Although a great deal of work can be done along the lines of classical patterns, one has to be cautious with the analogy. On several occasions the cases $\mathbf{F} = \mathbf{R}$ or $\mathbf{F} = \mathbf{H}$ or even $n = 1$ have to be treated in a different way. Thus an explicit description might be useful.

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1. Real, complex, and quaternionic half-space. Considering the symmetric matrix

$$Q := \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad I = I^{(n)},$$

we define

$$\mathrm{MSp}(n;\mathbf{F}) := \{M \in \mathrm{Mat}(2n;\mathbf{F}); \overline{M}'QM = Q\}$$

and call $\mathrm{MSp}(n;\mathbf{F})$ the *modified symplectic group of degree n over* $\mathbf{F}$. Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{MSp}(n;\mathbf{F})$ we always assume $A, B, C, D \in \mathrm{Mat}(n;\mathbf{F})$. Clearly $M \in \mathrm{MSp}(n;\mathbf{F})$ is equivalent to $\overline{M}' \in \mathrm{MSp}(n;\mathbf{F})$ as well as to

$$(1.1) \qquad A\overline{B}' + B\overline{A}' = C\overline{D}' + D\overline{C}' = 0, \quad A\overline{D}' + B\overline{C}' = I.$$

In this case one has

$$(1.2) \qquad M^{-1} = Q\overline{M}'Q = \begin{pmatrix} \overline{D}' & \overline{B}' \\ \overline{C}' & \overline{A}' \end{pmatrix}.$$

The definition contains one trivial case, namely

$$\text{(1.3)} \qquad \mathrm{MSp}(1;\mathbf{R}) = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}; 0 \neq a \in \mathbf{R}\right\} \cup \left\{\begin{pmatrix} 0 & b \\ b^{-1} & 0\end{pmatrix}; 0 \neq b \in \mathbf{R}\right\}.$$

Again in the general situation we want to describe special elements. Therefore we need the real vector space

$$\mathrm{Alt}(n;\mathbf{F}) := \{X \in \mathrm{Mat}(n;\mathbf{F}); \overline{X}' = -X\}$$

of all skew-Hermitian matrices, which has the dimension $\frac{1}{2}rn(n+1) - n$. Then the matrices

$$\text{(1.4)} \qquad Q = \begin{pmatrix} 0 & I \\ I & 0\end{pmatrix}, \quad \begin{pmatrix} I & S \\ 0 & I\end{pmatrix}, \quad S \in \mathrm{Alt}(n;\mathbf{F}),$$
$$\begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1}\end{pmatrix}, \quad U \in \mathrm{GL}(n;\mathbf{F}),$$

belong to $\mathrm{MSp}(n;\mathbf{F})$ in view of (1.1).

Moreover consider the subgroup

$$\mathrm{MSp}(n;\mathbf{F})_\infty := \left\{M = \begin{pmatrix} A & B \\ C & D\end{pmatrix} \in \mathrm{MSp}(n;\mathbf{F}); C = 0\right\}.$$

Then (1.1) immediately yields

$$\text{(1.5)} \qquad \mathrm{MSp}(n;\mathbf{F})_\infty = \left\{\begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1}\end{pmatrix}\begin{pmatrix} I & S \\ O & I\end{pmatrix}; U \in \mathrm{GL}(n;\mathbf{F}), S \in \mathrm{Alt}(n;\mathbf{F})\right\}.$$

Given $0 < j < n$ we define the usual embedding

$$\mathrm{MSp}(j;\mathbf{F}) \times \mathrm{MSp}(n-j;\mathbf{F}) \to \mathrm{MSp}(n;\mathbf{F}), \quad (M_1, M_2) \mapsto M_1 \times M_2,$$

$$\text{(1.6)} \qquad \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1\end{pmatrix} \times \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2\end{pmatrix} := \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2\end{pmatrix}$$

(cf. **[16]**, p. 44). If $M = \left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in \mathrm{MSp}(n;\mathbf{F})$ with rank $C = j$, one can proceed as in the classical situation (cf. **[4]**, 3.12, **[16]**, II.1.4) in order to obtain $K, L \in \mathrm{MSp}(n;\mathbf{F})_\infty$ such that

$$\text{(1.7)} \qquad M = K(Q^{(2j)} \times I)L,$$

where $j = 0, n$ can be interpreted unmistakably.

LEMMA 1.1. (a) *The group* $\mathrm{MSp}(n;\mathbf{F})$ *is generated by the matrices*

$$Q^{(2)} \times I, \quad \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}, \quad S \in \mathrm{Alt}(n;\mathbf{F}),$$
$$\begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1} \end{pmatrix}, \quad U \in \mathrm{GL}(n;\mathbf{F}).$$

(b) *Let* $\mathbf{F} = \mathbf{R}$, n *odd, or* $\mathbf{F} = \mathbf{C}, \mathbf{H}$, $n \geq 1$. *Then* $\mathrm{MSp}(n;\mathbf{F})$ *is also generated by the matrices* (1.4).

Proof. (a) Apply (1.7).
(b) If $\mathbf{F} = \mathbf{C}, \mathbf{H}$, compute

$$Q^{(2)} \times I = \left(\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)^2 \begin{pmatrix} I & S \\ 0 & I \end{pmatrix}\begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1} \end{pmatrix},$$

where $S = \left(\begin{smallmatrix} e_2 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathrm{Alt}(n;\mathbf{F})$, $U = \left(\begin{smallmatrix} e_2 & 0 \\ 0 & I \end{smallmatrix}\right) \in \mathrm{GL}(n;\mathbf{F})$. If $\mathbf{F} = \mathbf{R}$, $n = 1$ use (1.3). In the case $\mathbf{F} = \mathbf{R}$, $n = 2m + 1$, $m \geq 1$, compute

$$Q^{(2)} \times I = \left(\begin{pmatrix} I & S \\ 0 & I \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\right)^3 \begin{pmatrix} U' & 0 \\ 0 & U^{-1} \end{pmatrix},$$

where $S = \left(\begin{smallmatrix} 0 & 0 \\ 0 & J \end{smallmatrix}\right) \in \mathrm{Alt}(n;\mathbf{R})$, $U = \left(\begin{smallmatrix} 1 & 0 \\ 0 & J \end{smallmatrix}\right) \in \mathrm{GL}(n;\mathbf{R})$, $J = J^{(2m)}$. □

The case $\mathbf{F} = \mathbf{R}$ has to be treated in a different way. Note that $\mathrm{Sp}(n;\mathbf{R}) \subset \mathrm{SL}(2n;\mathbf{R})$, whereas (1.5) and (1.7) yield the surprising formula

$$\det M = (-1)^j, \quad j = \operatorname{rank} C, \tag{1.8}$$

whenever $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{MSp}(n;\mathbf{R})$. Thus $\mathrm{MSp}(n;\mathbf{R}) \cap \mathrm{SL}(2n;\mathbf{R})$ becomes a normal subgroup of $\mathrm{MSp}(n;\mathbf{R})$ of index 2. If n is even, this subgroup is generated by the matrices (1.4).

Combining (0.2) and (0.3) with Siegel's procedure **[30]**, it becomes obvious how the attached half-space has to be defined. Consider the real vector space

$$\mathrm{Sym}(n;\mathbf{F}) := \{X \in \mathrm{Mat}(n;\mathbf{F}); \overline{X}' = X\}$$

of the dimension $n + \frac{1}{2}rn(n-1)$ as well as the open subset $\mathrm{Pos}(n;\mathbf{F})$ consisting of all positive definite matrices in $\mathrm{Sym}(n;\mathbf{F})$. Then set

$$\begin{aligned}\mathscr{H}(n;\mathbf{F}) &= \mathrm{Alt}(n;\mathbf{F}) + \mathrm{Pos}(n;\mathbf{F}) \\ &= \{Z \in \mathrm{Mat}(n;\mathbf{F}); Z + \overline{Z}' \in \mathrm{Pos}(n;\mathbf{F})\}.\end{aligned}$$

We always assume that each $Z \in \mathscr{H}(n;\mathbf{F})$ is given in the form

$$Z = X + Y, \quad X \in \mathrm{Alt}(n;\mathbf{F}), \quad Y \in \mathrm{Pos}(n;\mathbf{F}).$$

DEFINITION. $\mathscr{H}(n;\mathbf{F})$ is called the *real, complex*, resp. *quaternionic half-space of degree n*, whenever $\mathbf{F}=\mathbf{R}$, $\mathbf{C}$, resp. $\mathbf{H}$.

The definition especially yields

$$\mathscr{H}(1;\mathbf{R})=\mathbf{R}^{+}=\{y\in\mathbf{R};y>0\},$$
$$\mathscr{H}(1;\mathbf{H})=\left\{z=\sum_{j=1}^{4}z_je_j;z_j\in\mathbf{R},z_1>0\right\}.$$

Note that in the cases $\mathbf{F}=\mathbf{R}$, $\mathbf{H}$ there is a decisive difference between $\mathscr{H}(n;\mathbf{F})$ and the half-space $H(n;\mathbf{F})$ defined in **[16]**, p. 46. But there are also close relations, namely

$$(1.9)\qquad H(n;\mathbf{C})=i\cdot\mathscr{H}(n;\mathbf{C})=\mathrm{Sym}(n;\mathbf{C})+i\,\mathrm{Pos}(n;\mathbf{C}).$$

Given $a=\sum_{j=1}^{4}a_je_j\in\mathbf{H}$ define

$$\check{a}=\begin{pmatrix}a_1e_1+a_2e_2 & a_3e_1+a_4e_2\\ -a_3e_1+a_4e_2 & a_1e_1-a_2e_2\end{pmatrix}\in\mathrm{Mat}(2;\mathbf{C})$$

and $\check{A}=(\check{a}_{kl})\in\mathrm{Mat}(2n;\mathbf{C})$ for $A=(a_{kl})\in\mathrm{Mat}(n;\mathbf{H})$ (cf. **[16]**, p. 14,15, 46). Then (1.9) leads to

$$(1.10)\quad i\check{Z}=i\check{X}+i\check{Y}\in H(2n;\mathbf{C}),\ \text{whenever } Z=X+Y\in\mathscr{H}(n;\mathbf{H}).$$

Note that i and e_2 may be identified for $\mathbf{F}=\mathbf{C}$. Furthermore (0.2) implies

$$(1.11)\quad \begin{pmatrix}iI & 0\\ 0 & I\end{pmatrix}\left\{\check{M};M\in\mathrm{MSp}(n;\mathbf{H})\right\}\begin{pmatrix}iI & 0\\ 0 & I\end{pmatrix}^{-1}\subset\mathrm{Sp}(2n;\mathbf{C}),$$

where $I=I^{(2n)}$. Moreover we have the obvious relations

$$(1.12)\qquad \begin{gathered}\mathscr{H}(n;\mathbf{R})\subset\mathscr{H}(n;\mathbf{C})\subset\mathscr{H}(n;\mathbf{H}),\\ \mathrm{MSp}(n;\mathbf{R})\subset\mathrm{MSp}(n;\mathbf{C})\subset\mathrm{MSp}(n;\mathbf{H}).\end{gathered}$$

We need the abbreviation $A[B]:=\overline{B}'AB$, whenever A is an $n\times n$ and B an $n\times m$ matrix, as well as $|\det A|:=|\det\check{A}|^{1/2}$, whenever $A\in\mathrm{Mat}(n;\mathbf{H})$ (cf. **[16]**, p. 15, I.3.4, I.3.5).

PROPOSITION 1.2. *The half-space $\mathscr{H}(n;\mathbf{F})$ is an open convex subset of* $\mathrm{Mat}(n;\mathbf{F})$, *which is contained in* $\mathrm{GL}(n;\mathbf{F})$. *Given $Z=X+Y\in\mathscr{H}(n;\mathbf{F})$, one has*

$$|\det Z|^2=\det Y\cdot\det(Y+Y^{-1}[X]).$$

Proof.

$$\begin{aligned}|\det Z|^2 &= |\det Z|\,|\det \overline{Z}'| = \det Y\cdot|\det(X+Y)|\cdot|\det(-Y^{-1}X+I)|\\ &= \det Y\cdot\det(Y-XY^{-1}X).\end{aligned}$$

The remaining parts are obvious. □

Next we consider the action of the modified symplectic group on the attached half-space.

THEOREM 1.3. *Let* $L, M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{MSp}(n;\mathbf{F})$ *and* $Z = X + Y \in \mathscr{H}(n;\mathbf{F})$. *Then the following hold*:

(a) $M\{Z\} := CZ + D \in \mathrm{GL}(n;\mathbf{F})$.

(b) $M\langle Z\rangle := (AZ+B)(CZ+D)^{-1} = X_M + Y_M \in \mathscr{H}(n;\mathbf{F})$.

(c) $Y_M = Y[M\{Z\}^{-1}]$, $Y_M^{-1} = Y^{-1}[\overline{X}'\overline{C}' + \overline{D}'] + Y[\overline{C}']$.

(d) $(LM)\{Z\} = L\{M\langle Z\rangle\}\cdot M\{Z\}$.

The group $\mathrm{MSp}(n;\mathbf{F})$ *acts transitively on* $\mathscr{H}(n;\mathbf{F})$. *Two transformations* $Z \mapsto M\langle Z\rangle$ *and* $Z \mapsto L\langle Z\rangle$ *coincide if and only if*

$$L = \rho M, \quad \textit{where} \quad \rho \in \operatorname{center}\mathbf{F}, |\rho| = 1.$$

Proof. (a) Apply (1.5), (1.7) and Proposition 1.2.

(b), (c) According to (a) we obtain $X_M \in \mathrm{Alt}(n;\mathbf{F})$, $Y_M \in \mathrm{Sym}(n;\mathbf{F})$ satisfying $M\langle Z\rangle = X_M + Y_M \in \mathrm{Mat}(n;\mathbf{F})$. Thus we gain

$$2Y_M = M\langle Z\rangle + \overline{M\langle Z\rangle}' = 2Y[(M\{Z\})^{-1}]$$

in view of (1.1). Hence $Y_M \in \mathrm{Pos}(n;\mathbf{F})$ follows. The remaining parts can be derived by easy calculations. □

Clearly the definition yields

$$Z \in \mathscr{H}(n;\mathbf{F}) \Rightarrow \overline{Z}' \in \mathscr{H}(n;\mathbf{F}). \tag{1.13}$$

In the cases $\mathbf{F} = \mathbf{C}$, $n \geq 2$, and $\mathbf{F} = \mathbf{H}$, $n = 2$, additionally

$$Z \in \mathscr{H}(n;\mathbf{F}) \Rightarrow Z' \in \mathscr{H}(n;\mathbf{F})$$

holds. Now we are going to describe the combination of (1.13) with the action of $\mathrm{MSp}(n;\mathbf{F})$ on $\mathscr{H}(n;\mathbf{F})$. Given $M = \left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \mathrm{MSp}(n;\mathbf{F})$ one easily verifies

$$\tilde{M} := M\left[\begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}\right] = \begin{pmatrix} A & -B\\ -C & D\end{pmatrix} \in \mathrm{MSp}(n;\mathbf{F}).$$

Then a calculation using (1.1) and Theorem 1.3 implies

PROPOSITION 1.4. *Given* $Z, W \in \mathscr{H}(n;\mathbf{F})$ *and* $M \in \mathrm{MSp}(n;\mathbf{F})$, *one has*

$$\text{(a)}\quad \overline{M\langle \overline{Z}'\rangle}' = \tilde{M}\langle Z\rangle.$$

$$\text{(b)}\quad M\langle Z\rangle + \overline{M\langle W\rangle}' = \overline{M\{W\}}'^{-1}(Z+\overline{W}')(M\{Z\})^{-1}.$$

$$\text{(c)}\quad M\langle Z\rangle - M\langle W\rangle = \overline{\tilde{M}\{\overline{W}'\}}'^{-1}(Z-W)(M\{Z\})^{-1}$$
$$= \overline{\tilde{M}\{\overline{Z}'\}}'^{-1}(Z-W)(M\{W\})^{-1}.$$

Following C. L. Siegel **[30]** we obtain a bijection between the half-space and the set of positive definite modified symplectic matrices. Put

$$\mathscr{P}(n;\mathbf{F}) := \mathrm{MSp}(n;\mathbf{F}) \cap \mathrm{Pos}(2n;\mathbf{F}).$$

THEOREM 1.5. *The map*

$$\kappa\colon \mathscr{H}(n;\mathbf{F}) \to \mathscr{P}(n;\mathbf{F}),\quad Z = X + Y \mapsto \begin{pmatrix} Y^{-1} & 0 \\ 0 & Y \end{pmatrix}\left[\begin{pmatrix} I & -X \\ 0 & I \end{pmatrix}\right],$$

is bijective and satisfies

$$(*)\qquad \kappa(M\langle Z\rangle) = \kappa(Z)[M^{-1}]$$

for all $M \in \mathrm{MSp}(n;\mathbf{F})$ *and* $Z \in \mathscr{H}(n;\mathbf{F})$.

Proof. $\kappa(Z) \in \mathscr{P}(n;\mathbf{F})$ follows from (1.1). The surjectivity of κ is obtained by the method of completing squares (cf. **[16]**, I.3.2). Since κ is obviously injective, the first part is proved.

In order to demonstrate $(*)$ we may confine ourselves to $\mathbf{F} = \mathbf{H}$ and to the generators (1.4) of $\mathrm{MSp}(n;\mathbf{H})$. An explicit calculation using Theorem 1.3 completes the proof. □

There also exists a bounded domain, which is birationally equivalent to the half-space. Consider the generalized unit disc

$$\mathscr{D}(n;\mathbf{F}) := \{W \in \mathrm{Mat}(n;\mathbf{F}); I - \overline{W}'W \in \mathrm{Pos}(n;\mathbf{F})\}.$$

The generalized Cayley transformation yields that the maps

$$\mathscr{H}(n;\mathbf{F}) \to \mathscr{D}(n;\mathbf{F}),\quad Z \mapsto (Z-I)(Z+I)^{-1},$$
$$\mathscr{D}(n;\mathbf{F}) \to \mathscr{H}(n;\mathbf{F}),\quad W \mapsto (W+I)(-W+I)^{-1},$$

are bijective and inverse to each other.

As a consequence one obtains a good description of the stabilizer

$$\mathrm{Stab}(Z) := \{M \in \mathrm{MSp}(n;\mathbf{F}); M\langle Z\rangle = Z\}, \qquad Z \in \mathscr{H}(n;\mathbf{F}).$$

We need the unitary group

$$\mathscr{U}(n;\mathbf{F}) := \{U \in \mathrm{Mat}(n;\mathbf{F}); \overline{U}'U = U\overline{U}' = I\}.$$

Then an explicit calculation yields

PROPOSITION 1.6.

$$\begin{aligned}\mathrm{Stab}(I) &= \mathrm{MSp}(n;\mathbf{F}) \cap \mathscr{U}(2n;\mathbf{F}) \\ &= \left\{\begin{pmatrix} A & B \\ B & A\end{pmatrix}; A, B \in \mathrm{Mat}(n;\mathbf{F}), A\overline{B}' + B\overline{A}' = 0, A\overline{A}' + B\overline{B}' = I\right\} \\ &= \left\{\frac{1}{2}\begin{pmatrix} I & I \\ -I & I\end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & V\end{pmatrix}\begin{pmatrix} I & -I \\ I & I\end{pmatrix}; U, V \in \mathscr{U}(n;\mathbf{F})\right\}.\end{aligned}$$

REMARK 1.7. Consider the three-dimensional hyperbolic space

$$\mathscr{H} = \left\{z = \sum_{j=1}^{3} z_j e_j; z_j \in \mathbf{R}, z_3 > 0\right\}$$

investigated in [**6**]. Clearly $\mathscr{H}$ becomes a real submanifold of

$$e_3 \cdot \mathscr{H}(1;\mathbf{H}) = \left\{z = \sum_{j=1}^{4} z_j e_j; z_j \in \mathbf{R}, z_3 > 0\right\}.$$

In view of (0.3) one easily verifies that the group

$$\Sigma = \begin{pmatrix} e_3 & 0 \\ 0 & 1\end{pmatrix} \mathrm{MSp}(1;\mathbf{H}) \begin{pmatrix} e_3 & 0 \\ 0 & 1\end{pmatrix}^{-1}$$

contains $\mathrm{SL}(2;\mathbf{C})$ as a subgroup. Now one can show that

$$\{M \in \Sigma; M\langle\mathscr{H}\rangle = \mathscr{H}\} = \mathrm{SL}(2;\mathbf{C}) \cup (e_3 I)\cdot \mathrm{SL}(2;\mathbf{C}).$$

The right-hand side proves to be a group by virtue of $(e_3 I)\cdot M \cdot (e_3 I)^{-1} = \overline{M}$ for $M \in \mathrm{Mat}(2;\mathbf{C})$. Moreover, note that $z = z_1 e_1 + z_2 e_2 + z_3 e_3 \in \mathscr{H}$ implies

$$(e_3 I)\langle z\rangle = z_1 e_1 - z_2 e_2 + z_3 e_3.$$

2. The half-space as a symmetric space. One can proceed in the same way, as C. L. Siegel [**29**] did in the classical situation, in order to turn the half-space into a symmetric space.

Given $Z, W \in \operatorname{Mat}(n;\mathbf{F})$, $Z = (z_{kl})$, $z_{kl} = \sum_{j=1}^{r} z_{kl}^{(j)} e_j$, $z_{kl}^{(j)} \in \mathbf{R}$, set $\tau(Z, W) := \frac{1}{2}\operatorname{trace}(Z\overline{W}' + W\overline{Z}')$ and let dZ denote the matrix of differentials

$$dZ = \left(\sum_{j=1}^{r} dz_{kl}^{(j)} e_j\right)_{1 \le k,l \le n}.$$

Now consider the quadratic differential form

$$ds^2 := \tau(Y^{-1} dZ Y^{-1}, dZ),$$

whenever $Z = X + Y \in \mathscr{H}(n;\mathbf{F})$. The case $\mathbf{F} = \mathbf{C}$ of the following assertion is due to H. Braun **[3]**.

LEMMA 2.1. *The quadratic differential form ds^2 is positive definite in $\mathscr{H}(n;\mathbf{F})$ and invariant under the maps $Z \mapsto M\langle Z\rangle$, $M \in \operatorname{MSp}(n;\mathbf{F})$, as well as $Z \mapsto \overline{Z}'$.*

Proof. $\tau(A, B) = \tau(\overline{A}', \overline{B}')$ yields the invariance under $Z \mapsto \overline{Z}'$. Let $M \in \operatorname{MSp}(n;\mathbf{F})$, $Z \in \mathscr{H}(n;\mathbf{F})$ and set $Z_1 = M\langle Z\rangle$. Then (1.1) and Proposition 1.4 lead to

$$dZ_1 = \overline{\tilde{M}\{\overline{Z}'\}}^{\prime -1} dZ (M\{Z\})^{-1}.$$

Next $Y_1 = (M\{Z\}) Y^{-1} \overline{M\{Z\}}' = (\tilde{M}\{\overline{Z}'\}) Y^{-1} \overline{\tilde{M}\{\overline{Z}'\}}'$ follows from Theorem 1.3 and Proposition 1.4. Finally, the use of **[16]**, IV.1.1, yields

$$\tau(Y_1^{-1} dZ_1 Y_1^{-1}, dZ_1) = \tau(Y^{-1} dZ Y^{-1}, dZ).$$

ds^2 is obviously positive definite in the point $Z = I$. Since $\operatorname{MSp}(n;\mathbf{F})$ acts transitively, the assertion follows. □

In Helgason's notation **[8]** we obtain

THEOREM 2.2. *$\mathscr{H}(n;\mathbf{F})$ endowed with the metric ds^2 is a Riemannian globally symmetric space of the noncompact type, which is irreducible except for the cases $\mathbf{F} = \mathbf{R}$, $n = 1, 2$.*

Proof. The map $Z \mapsto Q\langle Z\rangle = Z^{-1}$ becomes an involutive isometry, which possesses I as an isolated fixed point.

With the aid of Proposition 1.6 we determine the associated Lie algebras, namely

$$\begin{aligned}\operatorname{Lie} \operatorname{MSp}(n;\mathbf{F}) &= \{M \in \operatorname{Mat}(2n;\mathbf{F}); \overline{M}'Q + QM = 0\} \\ &= \left\{\begin{pmatrix} A & B \\ C & -\overline{A}' \end{pmatrix}; A \in \operatorname{Mat}(n;\mathbf{F}),\ B, C \in \operatorname{Alt}(n;\mathbf{F})\right\},\end{aligned}$$

$$\operatorname{Lie} \operatorname{Stab}(I) = \operatorname{Lie} \operatorname{MSp}(n;\mathbf{F}) \cap \operatorname{Alt}(2n;\mathbf{F}).$$

Now one easily checks

$$\begin{pmatrix} I & I \\ -I & I \end{pmatrix} \text{Lie MSp}(n;\mathbf{F}) \begin{pmatrix} I & I \\ -I & I \end{pmatrix}^{-1} = \begin{cases} \mathfrak{so}(n,n) & \text{if } \mathbf{F}=\mathbf{R}, \\ \mathfrak{u}(n,n) & \text{if } \mathbf{F}=\mathbf{C}, \end{cases}$$

$$\begin{pmatrix} I & I \\ -I & I \end{pmatrix} \text{Lie Stab}(I) \begin{pmatrix} I & I \\ -I & I \end{pmatrix}^{-1} = \begin{cases} \mathfrak{so}(n)\times\mathfrak{so}(n) & \text{if } \mathbf{F}=\mathbf{R}, \\ \mathfrak{u}(n)\times\ \mathfrak{u}(n) & \text{if } \mathbf{F}=\mathbf{C}, \end{cases}$$

(cf. **[8]**, p. 341). In the case $\mathbf{F} = \mathbf{H}$ a similar map yields an isomorphism between Lie MSp$(n;\mathbf{H})$ and $\mathfrak{sp}(n,n)$ as well as between Lie Stab(I) and $\mathfrak{sp}(n)\times\mathfrak{sp}(n)$. Now the assertion follows from Helgason's classification (cf. **[8]**, IX,§4). □

REMARK 2.3. (a) $\mathscr{H}(n;\mathbf{F})$ corresponds to BDI for $\mathbf{F} = \mathbf{R}$, to AIII for $\mathbf{F} = \mathbf{C}$ and to CII for $\mathbf{F} = \mathbf{H}$ in Helgason's classification (cf. **[8]**, p. 354), where in every case $p = q = n$. Note that the spaces $\mathscr{H}(n;\mathbf{R})$, $n \neq 2$, and $\mathscr{H}(n;\mathbf{H})$, $n \geq 1$, fail to be Hermitian (cf. **[8]**, p. 354).

(b) In view of **[8]**, p. 353, (x), the space $\mathscr{H}(2;\mathbf{R})$ is isomorphic to the direct product of two copies of the upper half-plane $\mathscr{H} = \{z = x+iy \in \mathbf{C}; y > 0\}$ in $\mathbf{C}$. Each $Z \in \mathscr{H}(2;\mathbf{R})$ is uniquely representable as

$$Z = xJ + Y = \begin{pmatrix} y_1 & y+x \\ y-x & y_2 \end{pmatrix}.$$

Now define the map

$$\chi_2\colon \mathscr{H}(2;\mathbf{R}) \to \mathscr{H}\times\mathscr{H}, \quad Z \mapsto (x + i\sqrt{\det Y}, \frac{1}{y_1}(-y + i\sqrt{\det Y})).$$

Clearly χ_2 becomes a bijection. If $\chi_2(Z) = (z,w)$ and $U \in \mathrm{GL}(2;\mathbf{R})$ one easily verifies

$$\begin{aligned} &\chi_2(Z+J) = (z+1, w), \\ &\chi_2(U'ZU) = \begin{cases} (\det U\cdot z, U^{-1}\langle w\rangle) & \text{if } \det U > 0, \\ (\det U\cdot \bar{z}, U^{-1}\langle \bar{w}\rangle) & \text{if } \det U < 0, \end{cases} \\ &\chi_2(Z^{-1}) = \left(-\frac{1}{z}, -\frac{1}{w}\right), \\ &\chi_2((Q\times I)\langle Z\rangle) = (w,z), \quad \text{where } Q = Q^{(2)}, \quad I = I^{(2)}. \end{aligned}$$

(c) In view of **[8]**, p. 352, (iv), the space $\mathscr{H}(3;\mathbf{R})$ is isomorphic to the space $\mathrm{SPos}(4;\mathbf{R}) = \mathrm{Pos}(4;\mathbf{R}) \cap \mathrm{SL}(4;\mathbf{R})$ (cf. **[32]**). Given $x = (x_1, x_2, x_3)' \in \mathbf{R}^3$ we define

$$\mathrm{ad}\, x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix} \in \mathrm{Alt}(3;\mathbf{R}),$$

which comes from the vector product (cf. **[15]**, p. 205). Now set

$$\chi_3 : \mathscr{H}(3;\mathbf{R}) \to \mathrm{SPos}(4;\mathbf{R}),$$
$$\mathrm{ad}\,x + Y \mapsto (\det Y)^{-1/2} \begin{pmatrix} Y & 0 \\ 0 & \det Y \end{pmatrix} \left[\begin{pmatrix} I & x \\ 0 & 1 \end{pmatrix} \right].$$

Given $s \in \mathbf{R}^3$, $U \in \mathrm{GL}(3;\mathbf{R})$ one easily verifies

$$\chi_3(Z + \mathrm{ad}\,s) = \chi_3(Z) \left[\begin{pmatrix} I & s \\ 0 & 1 \end{pmatrix} \right],$$
$$\chi_3(U'ZU) = \chi_3(Z)[U^*], \quad \text{where } U^* = |\det U|^{-1/2} \begin{pmatrix} U & 0 \\ 0 & \det U \end{pmatrix},$$
$$\chi_3(Z^{-1}) = (\chi_3(Z))^{-1}.$$

Now we are going to describe the associated invariant volume element and the Laplace-Beltrami-operator, which was determined by H. Maaß **[21]** in the case of the Siegel half-space. Therefore define the vector

$$d\mathfrak{z} = (dz_{11}^{(1)}, \ldots, dz_{11}^{(r)}, dz_{12}^{(1)}, \ldots, dz_{1n}^{(r)}, dz_{21}^{(1)}, \ldots, dz_{nn}^{(r)})'$$

of the length rn^2. Given $Y \in \mathrm{Pos}(n;\mathbf{F})$ there exists $S_Y \in \mathrm{Pos}(rn^2;\mathbf{R})$ satisfying

$$ds^2 = \tau(Y^{-1}dZY^{-1}, dZ) = S_Y[d\mathfrak{z}] \tag{2.1}$$

in view of Lemma 2.1.

PROPOSITION 2.4. *The volume element*

$$dv = (\det Y)^{-rn} \prod_{k=1}^{n} \prod_{l=1}^{n} \prod_{j=1}^{r} dz_{kl}^{(j)}$$

of $\mathscr{H}(n;\mathbf{F})$ *is invariant under the modified symplectic transformations* $Z \mapsto M\langle Z\rangle$, $M \in \mathrm{MSp}(n;\mathbf{F})$, *as well as* $Z \mapsto \overline{Z}'$.

Proof. Define $d := \det S_Y$; then $dv = d^{1/2} \prod_{k,l,j} dz_{kl}^{(j)}$ has the desired invariance property due to Lemma 2.1. One calculates $d = (\det Y)^{-2rn}$. □

We compute the effect of differential operators on determinants.

PROPOSITION 2.5. *Let $Y \in \operatorname{Pos}(n;\mathbf{F})$, $Y^{-1} = (\tilde{y}_{kl})$ and $s \in \mathbf{C}$. Given $1 \le k, l \le n$, $1 \le j \le r$, one has*

$$\frac{\partial}{\partial z_{kl}^{(j)}}(\det Y)^s = s(\det Y)^s \tilde{y}_{kl}^{(j)}.$$

Proof. Due to the method of completing squares (cf. **[16]**, I.3.2), we may confine ourselves to the case $n = 2$. Then an explicit calculation completes the proof. □

In order to get an explicit description of the Laplace-Beltrami-operator, let $\partial/\partial Z$ denote the matrix differential operator

$$\frac{\partial}{\partial z} = \left(\sum_{j=1}^{r} \frac{\partial}{\partial z_{kl}^{(j)}} e_j\right)_{1 \le k,l \le n}.$$

THEOREM 2.6. *The Laplace-Beltrami-operator Δ is invariant under the maps $Z \mapsto M\langle Z\rangle$, $M \in \operatorname{MSp}(n;\mathbf{F})$, as well as $Z \mapsto \overline{Z}'$ and is given by*

$$\Delta = \tau\left(Y\frac{\partial}{\partial Z}Y\frac{\partial}{\partial Z}\right) - \left(\frac{1}{2}r(n+1) - 1\right)\tau\left(Y\frac{\partial}{\partial Z}\right).$$

Proof. The invariance follows from Lemma 2.1 and **[8]**, X.2.1. Using (2.1) an elementary but lengthy calculation yields $(S_Y)^{-1} = S_{Y^{-1}}$. Then the definition of Δ leads to

$$\Delta = \sum_{\substack{1 \le j,k,l,m \le n \\ 1 \le \nu,\mu \le r}} (\det Y)^{rn} \frac{\partial}{\partial z_{kl}^{(\nu)}} \operatorname{Re}(y_{jk} e_\nu y_{lm} \bar{e}_\mu)(\det Y)^{-rn} \frac{\partial}{\partial z_{jm}^{(\mu)}}.$$

Now one can use Proposition 2.5 and another lengthy calculation shows that Δ has the form given above. □

Theorem 2.6 combined with Proposition 2.5 yields

COROLLARY 2.7. *Let $Z \in \mathscr{H}(n;\mathbf{F})$, $M \in \operatorname{MSp}(n;\mathbf{F})$ and $s \in \mathbf{C}$. Then one has*

$$\Delta(\det Y_M)^s = ns\left(s + 1 - \frac{1}{2}r(n+1)\right)(\det Y_M)^s.$$

REMARK 2.8. One can proceed in the same way as C. L. Siegel **[29]**, resp. H. Klingen **[12]**, in order to derive normal forms for pairs of points under modified symplectic transformations. As a result one obtains that the geodesics in $\mathscr{H}(n;\mathbf{F})$ are given by the images of the curves

$$Z(u) = \begin{pmatrix} e^{up_1} & & 0 \\ & \ddots & \\ 0 & & e^{up_n} \end{pmatrix}$$

under the transformations $Z \mapsto M\langle Z\rangle$, $M \in \mathrm{MSp}(n;\mathbf{F})$. Here $p_1, \ldots, p_n$ satisfy $0 \leq p_1 \leq \cdots \leq p_n$ as well as $\sum_{k=1}^n p_k^2 = 1$ and u runs through the interval $[0, \rho]$, where ρ denotes the geodesic distance of the points. On the other hand the geodesics in $\mathscr{H}(n;\mathbf{F})$ coincide with the solutions of the differential equation

$$\ddot{Z} = \dot{Z}Y^{-1}\dot{Z}.$$

Thus in the relations

$$\mathscr{H}(n;\mathbf{R}) \subset \mathscr{H}(n;\mathbf{C}) \subset \mathscr{H}(n;\mathbf{H})$$

every half-space becomes a totally geodesic submanifold of the following one.

3. The modified modular group. We proceed in the same way as in **[16]**. Thus we obtain integral elements by the choice of a special order $\mathscr{O} = \mathscr{O}(\mathbf{F})$, namely

$$\mathscr{O}(\mathbf{R}) = \mathbf{Z}, \quad \mathscr{O}(\mathbf{C}) = \mathbf{Z}e_1 = \mathbf{Z}e_2, \quad \mathscr{O}(\mathbf{H}) = \mathbf{Z}e_0 + \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3,$$

where $e_0 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. Here $\mathscr{O}(\mathbf{C})$ of course denotes the Gaussian integers and $\mathscr{O}(\mathbf{H})$ the quaternions of Hurwitz (cf. **[9]** or **[5]**, §91). Then the set of integral modified symplectic matrices

$$\Gamma(n;\mathscr{O}) := \mathrm{MSp}(n;\mathbf{F}) \cap \mathrm{Mat}(2n;\mathscr{O})$$

becomes a subgroup of $\mathrm{MSp}(n;\mathbf{F})$, which acts discontinuously on the half-space $\mathscr{H}(n;\mathbf{F})$.

Definition. $\Gamma(n;\mathscr{O})$ is called the *modified modular group of degree* n.

Clearly, we include the trivial case

$$\Gamma(1;\mathbf{Z}) = \{\pm I, \pm Q\} \tag{3.1}$$

in view of (1.3). In the case $\mathbf{F} = \mathbf{C}$ (0.2) implies that

$$\begin{pmatrix} e_2 I & 0 \\ 0 & I \end{pmatrix} \Gamma(n;\mathbf{Z}e_1 + \mathbf{Z}e_2) \begin{pmatrix} e_2 I & 0 \\ 0 & I \end{pmatrix}^{-1} \tag{3.2}$$

equals the Hermitian modular group with respect to the Gaussian number field (cf. [3]).

Let $\mathrm{Alt}(n;\mathcal{O})$ denote the lattice of all integral skew-Hermitian $n\times n$ matrices. $\mathrm{GL}(n;\mathcal{O})$ stands for the group of units in the ring $\mathrm{Mat}(n;\mathcal{O})$. Thus (1.5) yields

$$(3.3)\quad \Gamma(n;\mathcal{O})_\infty := \mathrm{MSp}(n;\mathbf{F})_\infty \cap \mathrm{Mat}(2n;\mathcal{O}) = \left\{ \begin{pmatrix} \overline{U}' & \overline{U}'S \\ 0 & U^{-1} \end{pmatrix}; U\in \mathrm{GL}(n;\mathcal{O}), S\in \mathrm{Alt}(n;\mathcal{O}) \right\}.$$

Set $N(a) := a\bar{a} \in \mathbf{R}$ for $a\in\mathbf{F}$. Hence one easily verifies the property:

(3.4) Given $a\in \mathrm{Alt}(1;\mathbf{F})$ then $g\in \mathrm{Alt}(1;\mathcal{O})$ exists such that $N(a-g)<1$.

Hence the Euclidean algorithm is valid in $\mathcal{O}$ as well as in $\mathrm{Alt}(1;\mathcal{O})$. Thus we can derive a result of L. Kronecker **[18]**—often cited as Witt's Theorem **[33]**—on the generators of the modified modular group. The proofs in **[16]**, II.2.2 and II.2.3, can be adapted by the use of (1.1) and (3.4) in order to obtain

THEOREM 3.1. *The modified modular group* $\Gamma(n;\mathcal{O})$ *is generated by the matrices*

$$Q^{(2)}\times I,\quad \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},\quad S\in \mathrm{Alt}(n;\mathcal{O}),\quad \begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1} \end{pmatrix},\quad U\in \mathrm{GL}(n;\mathcal{O}).$$

The same arguments that were applied in the proof of Lemma 1.1b yield that $\Gamma(n;\mathcal{O})$ can also be generated by the matrices

$$Q,\quad \begin{pmatrix} I & S \\ 0 & I \end{pmatrix},\quad S\in \mathrm{Alt}(n;\mathcal{O}),\quad \begin{pmatrix} \overline{U}' & 0 \\ 0 & U^{-1} \end{pmatrix},\quad U\in \mathrm{GL}(n;\mathcal{O}),$$

except for the case $\mathcal{O}=\mathbf{Z}$, n even.

Combining this with (1.8) it becomes clear that the group Δ_n^* considered by H. Maaß in **[23]** equals $\Gamma(n;\mathbf{Z})$, whenever n is odd, and $\Gamma(n;\mathbf{Z})\cap \mathrm{SL}(2n;\mathbf{Z})$, whenever n is even.

Now we are going to determine a suitable fundamental domain. Therefore let $\mathcal{C}(n;\mathcal{O})$ denote the fundamental parallelotope of the lattice $\mathrm{Alt}(n;\mathcal{O})$ in $\mathrm{Alt}(n;\mathbf{F})$, which consists of the matrices $X=(x_{kl})\in \mathrm{Alt}(n;\mathbf{F})$ such that

$$x_{kl} = \sum_{j=1}^{r} x_{kl}^{(j)} e_j,\qquad -\frac{1}{2}\le x_{kl}^{(j)} \le \frac{1}{2},\ 1\le k\le l\le n,\ 1\le j\le r,$$

where $x_{kl}^{(1)} \geq 0$ in the case $\mathbf{F} = \mathbf{H}$. Moreover, $\mathscr{R}(n;\mathbf{F})$ stands for the set of reduced matrices in $\mathrm{Pos}(n;\mathbf{F})$ (cf. **[16]**, p. 29). Now let $\mathscr{F}(n;\mathscr{O})$ consist of all matrices $Z = X + Y \in \mathscr{H}(n;\mathbf{F})$, which satisfy

(i) $X \in \mathscr{C}(n;\mathscr{O})$,

(ii) $Y \in \mathscr{R}(n;\mathbf{F})$,

(iii) $|\det M\{Z\}| \geq 1$, i.e. $\det Y_M \leq \det Y$, for all $M \in \Gamma(n;\mathscr{O})$.

Clearly, one has

$$\mathscr{F}(1;\mathbf{Z}) = \{y \in \mathbf{R}; y \geq 1\}, \tag{3.5}$$

$$i\mathscr{F}(n;\mathbf{Z}e_1 + \mathbf{Z}e_2) = \mathscr{F}(n;\mathbf{C}), \tag{3.6}$$

where $\mathscr{F}(n;\mathbf{C})$ denotes the fundamental domain in **[3]** resp. **[16]**, p. 58.

At first we derive some properties of the domain $\mathscr{F}(n;\mathscr{O})$.

PROPOSITION 3.2. *There exists a constant $\rho = \rho(n;\mathbf{F})$ such that $Y \geq \rho I$ holds for all $Z = X + Y \in \mathscr{F}(n;\mathscr{O})$.*

Proof. $1 \leq |\det(Q^{(2)} \times I)\{Z\}|^2 = N(z_{11}) = y_{11}^2 + N(x_{11})$ holds in view of (iii). The definition of $\mathscr{C}(n;\mathscr{O})$ yields $N(x_{11}) \leq \frac{3}{4}$, hence $y_{11} \geq \frac{1}{2}$. Now **[16]**, I.4.7 and I.5.1, combined with (ii) imply $Y \geq \frac{1}{2}\beta I$, where β only depends on n. □

Let dv again denote the invariant volume element (cf. Proposition 2.4). One can apply nearly the same arguments, which were used for the proof of **[16]**, II.3.2, II.3.9, in order to obtain

LEMMA 3.3. (a) *$\lambda I \in \mathscr{F}(n;\mathscr{O})$ for all $\lambda \geq 1$.*

(b) *Given $Z = X+Y \in \mathscr{F}(n;\mathscr{O})$, then $Z_\lambda := X+\lambda Y \in \mathscr{F}(n;\mathscr{O})$ holds for $\lambda \geq 1$.*

(c) *$\mathscr{F}(n;\mathscr{O})$ is arcwise connected.*

(d) $\mathrm{vol}(\mathscr{F}(n;\mathscr{O})) := \int_{\mathscr{F}(n;\mathscr{O})} dv < \infty$ *except for $n = 1$, $\mathscr{O} = \mathbf{Z}$.*

Hence the domain $\mathscr{F}(n;\mathscr{O})$ fails to be compact. Given $\alpha > 0$ the subset $\mathscr{E}(n;\mathbf{F})[\alpha]$ of $\mathrm{Pos}(n;\mathbf{F})$ consists of the matrices

$$\begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix} \left[\begin{pmatrix} 1 & & b_{kl} \\ & \ddots & \\ 0 & & 1 \end{pmatrix}\right],$$

where $0 < d_j < \alpha d_{j+1}$ for $1 \leq j < n$ and $N(b_{kl}) < \alpha^2$ for $1 \leq k < l \leq n$ (cf. **[16]**, p. 33). Then we define the Siegel set

$$\mathscr{S}(n;\mathbf{F})[\alpha] := \{Z \in \mathscr{H}(n;\mathbf{F}); N(x_{kl}) < \alpha^2, Y \in \mathscr{E}(n;\mathbf{F})[\alpha], 1 < \alpha y_{11}\},$$

confer [7], p. 90, in the case of the Siegel half-space. Recall the definition of κ from Theorem 1.5 and consider the matrices

$$V_0 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix} \in \mathrm{GL}(n;\mathscr{O}) \quad \text{and} \quad W_0 = \begin{pmatrix} V_0 & 0 \\ 0 & I \end{pmatrix} \in \mathrm{GL}(2n;\mathscr{O}).$$

LEMMA 3.4. (a) *There exists* $\alpha = \alpha(n;\mathbf{F}) > 0$ *such that*

$$\mathscr{F}(n;\mathscr{O}) \subset \mathscr{S}(n;\mathbf{F})[\alpha].$$

(b) *Given a compact subset* $\mathscr{C}$ *in* $\mathscr{H}(n;\mathbf{F})$, *there exists* $\beta = \beta(\mathscr{C}) > 0$ *satisfying*

$$\mathscr{C} \subset \mathscr{S}(n;\mathbf{F})[\beta].$$

(c) *Given* $\gamma > 0$ *one can find* $\delta > 0$ *such that*

$$\kappa(\mathscr{S}(n;\mathbf{F})[\gamma])[W_0] \subset \mathscr{E}(2n;\mathbf{F})[\delta].$$

(d) *Let* $\gamma > 0$, *then there are only finitely many* $M \in \Gamma(n;\mathscr{O})$ *satisfying*

$$M\langle\mathscr{S}(n;\mathbf{F})[\gamma]\rangle \cap \mathscr{S}(n;\mathbf{F})[\gamma] \neq \emptyset.$$

Proof. (a) and (b) The proof is settled in analogy with [**16**], II. 3.6, where Proposition 3.2 is applied.

(c) Proceed in the same way as in [**16**], II.3.7.

(d) The assertion follows from part (c) combined with [**16**], I.4.10. □

We take the definition of a fundamental domain from [**16**], p. 6.

THEOREM 3.5. $\mathscr{F}(n;\mathscr{O})$ *is a fundamental domain of* $\mathscr{H}(n;\mathbf{F})$ *with respect to the action of* $\Gamma(n;\mathscr{O})$ *except for* $\mathbf{F} = \mathbf{H}$, $n = 1$. *The domain* $\mathscr{F}(n;\mathscr{O})$ *is arcwise connected and closed in* $\mathrm{Mat}(n;\mathbf{F})$. *Moreover* $\mathrm{vol}(\mathscr{F}(n;\mathscr{O})) < \infty$ *holds except for* $\mathbf{F} = \mathbf{R}$, $n = 1$.

Proof. Given $Z \in \mathscr{H}(n;\mathbf{F})$ we can show in the same way as in [**16**], II.3.3, that there exists $M \in \Gamma(n;\mathscr{O})$ satisfying

$$\det Y_K \leq \det Y_M \quad \text{for all } K \in \Gamma(n;\mathscr{O}).$$

We may replace M by KM, where $K \in \Gamma(n;\mathscr{O})_\infty$, in order to map Z into $\mathscr{F}(n;\mathscr{O})$ by a modified modular transformation.

In view of the definition $\mathscr{F}(n;\mathscr{O})$ is relatively closed in $\mathscr{H}(n;\mathbf{F})$. Now $\mathscr{F}(n;\mathscr{O})$ proves to be closed in $\mathrm{Mat}(n;\mathbf{F})$ according to Proposition 3.2. By virtue of

$$\bigcup_M M\langle\mathscr{F}(n;\mathscr{O})\rangle = \mathscr{H}(n;\mathbf{F}),$$

where M runs through $\Gamma(n;\mathcal{O})$, clearly $\mathcal{F}(n;\mathcal{O})$ contains interior points.

Let $M \in \Gamma(n;\mathcal{O})$ and $Z \in \mathcal{F}(n;\mathcal{O})$ such that Z and $W := M\langle Z\rangle$ are interior points of $\mathcal{F}(n;\mathcal{O})$. We obtain $(M\{Z\})^{-1} = M^{-1}\{W\}$ from Theorem 1.3. Thus $|\det M\{Z\}| = |\det M^{-1}\{W\}| = 1$ follows. Since Z and W are interior points, we conclude $C = 0$. Then (3.3) implies

$$W = Z[U] + S$$

for appropriate $U \in \mathrm{GL}(n;\mathcal{O})$ and $S \in \mathrm{Alt}(n;\mathcal{O})$. Since Y is an interior point of $\mathcal{R}(n;\mathbf{F})$, whenever $Z = X + Y$, we conclude $U = \varepsilon I$, where ε is a unit in $\mathcal{O}$ and belongs to the center of $\mathbf{F}$, if $n > 1$. Finally we obtain $S = 0$, because X lies in the open kernel of $\mathcal{C}(n;\mathcal{O})$.

The remaining assertions follow from Lemma 3.3 and 3.4. □

In the case $\mathbf{F} = \mathbf{H}$, $n = 1$ we observe that the matrices $M = \varepsilon I^{(2)}$, where $\varepsilon \in \mathcal{E} = \{g \in \mathcal{O}; N(g) = 1\}$, induce the identity map on $\mathrm{Pos}(1;\mathbf{H}) = \mathbf{R}^+$. Using **[16]**, I.1.3, and the considerations above, we obtain a fundamental domain $\mathcal{F}^*$ of $\mathcal{H}(1;\mathbf{H})$ with respect to the action of $\Gamma(1;\mathcal{O})$, where

$$\mathcal{F}^* = \left\{ z = x + y \in \mathcal{F}(1;\mathcal{O}); x = \sum_{j=2}^{4} x_j e_j, x_2 \geq x_3 \geq 0, x_2 \geq |x_4| \right\}.$$

But we can simplify the condition (iii) and gain

COROLLARY 3.6. *A fundamental domain of $\mathcal{H}(1;\mathbf{H})$ with respect to the action of $\Gamma(1;\mathcal{O})$ is given by*

$$\mathcal{F}^* = \left\{ z = \sum_{j=1}^{4} z_j e_j \in \mathbf{H}; z_1 > 0, \frac{1}{2} \geq z_2 \geq z_3 \geq 0, z_2 \geq |z_4|, N(z) \geq 1 \right\}.$$

Moreover, besides the obvious cases $n = 1$, $\mathbf{F} = \mathbf{R}, \mathbf{C}$ (cf. (3.5), (3.6)) the domain $\mathcal{F}(2;\mathbf{Z})$ can be described easily.

EXAMPLE 3.7. The fundamental domain $\mathcal{F}(2;\mathbf{Z})$ consists of the matrices

$$Z = \begin{pmatrix} y_1 & y + x \\ y - x & y_2 \end{pmatrix} \in \mathrm{Mat}(2;\mathbf{R}),$$

where

$$1 \leq y_1 \leq y_2, \quad 0 \leq 2y \leq y_1, \quad -\frac{1}{2} \leq x \leq \frac{1}{2},$$
$$\det Z = y_1 y_2 - y^2 + x^2 \geq 1.$$

REMARK 3.8. Let us replace $\Gamma(n;\mathbf{Z})$ by $\Gamma^*(n;\mathbf{Z}) := \Gamma(n;\mathbf{Z}) \cap \mathrm{SL}(2n;\mathbf{Z})$. In the corresponding fundamental domain $\mathscr{F}^*(n;\mathbf{Z})$ the condition (iii) is only valid for $M \in \Gamma^*(n;\mathbf{Z})$. However $\mathscr{F}^*(n;\mathbf{Z})$ possesses more than one cusp. As an example observe that

$$\mathscr{F}^*(1;\mathbf{Z}) = \mathscr{H}(1;\mathbf{R}) = \mathbf{R}^+,$$

$$\mathscr{F}^*(2;\mathbf{Z}) = \left\{ Z = \begin{pmatrix} y_1 & y+x \\ y-x & y_2 \end{pmatrix} \in \mathscr{H}(2;\mathbf{R}); \begin{array}{l} 0 \le 2y \le y_1 \le y_2, -\frac{1}{2} \le x \le \frac{1}{2}, \\ \det Z \ge 1 \end{array} \right\}.$$

In general the diagonal matrix $[\frac{1}{\lambda}, \lambda, \dots, \lambda]$ belongs to $\mathscr{F}^*(n;\mathbf{Z})$, whenever $\lambda \ge 1$.

In this special case we can compute the volume of the fundamental domain explicitly.

PROPOSITION 3.9. $\mathrm{vol}(\mathscr{F}(2;\mathbf{Z})) = \pi^2/9$.

Proof. In view of Example 3.7 and Remark 3.8 one has

$$\mathrm{vol}(\mathscr{F}(2;\mathbf{Z})) = \frac{1}{4}\int_{\mathscr{D}} d\nu,$$

where

$$\mathscr{D} = \left\{ Z = \begin{pmatrix} y_1 & y+x \\ y-x & y_2 \end{pmatrix} \in \mathscr{H}(2;\mathbf{R}); \; 0 \le |2y| \le y_1 \le y_2, |x| \le \frac{1}{2}, \det Z \ge 1 \right\}.$$

Remark 2.3 yields

$$\chi_2(\mathscr{D}) = \mathscr{F} \times \mathscr{F}, \quad \mathscr{F} = \{x + iy \in \mathbf{C}; y > 0, |x| \le \tfrac{1}{2}, |z| \ge 1\}.$$

Change of variables leads to

$$\mathrm{vol}(\mathscr{F}(2;\mathbf{Z})) = \left(\int_{\mathscr{F}} y^{-2}\, dx\, dy \right)^2 = \frac{\pi^2}{9}. \qquad \square$$

4. Eisenstein-series. We are going to define non-analytic Eisenstein-series in analogy with the classical case, cf. **[19]**, **[20]**. Special attention is devoted to the behavior of convergence, which is investigated after the model of Eisenstein-series on the Siegel half-space.

DEFINITION. Given $\varepsilon > 0$ the set

$$\mathcal{VS}_\varepsilon(n;\mathbf{F}) := \{Z = X + Y \in \mathcal{H}(n;\mathbf{F}); Y \geq \varepsilon I, \varepsilon^{-2} I \geq \overline{X}' X\}$$

is called a *vertical strip of height ε.*

Using (1.9), (1.10), (1.12) as well as the definition of a vertical strip $\mathcal{V}_\varepsilon(n;\mathbf{F})$ in $H(n;\mathbf{F})$ (cf. **[16]**, p. 148), we obtain

$$\mathcal{VS}_\varepsilon(n;\mathbf{R}) \subset \mathcal{VS}_\varepsilon(n;\mathbf{C}) \subset \mathcal{VS}_\varepsilon(n;\mathbf{H}), \tag{4.1}$$

$$i\mathcal{VS}_\varepsilon(n;\mathbf{C}) = \mathcal{V}_\varepsilon(n;\mathbf{C}), \tag{4.2}$$

$$\{i\check{Z}; Z \in \mathcal{VS}_\varepsilon(n;\mathbf{H})\} \subset \mathcal{V}_\varepsilon(2n;\mathbf{C}). \tag{4.3}$$

PROPOSITION 4.1. *Given $\varepsilon > 0$ there exists $c = c(n;\varepsilon) > 0$ such that*

$$|\det M\{Z\}| \geq c|\det M\{I\}|$$

holds for all $Z \in \mathcal{VS}_\varepsilon(n;\mathbf{F})$ and $M \in \mathrm{MSp}(n;\mathbf{F})$.

Proof. In view of (4.1) and (1.12) we may restrict to the case $\mathbf{F} = \mathbf{H}$. Now apply (4.3), (1.11) and **[16]**, V.2.5. □

Analogous arguments using **[16]**, V.2.7, and Theorem 1.3 yield

PROPOSITION 4.2. *Given a compact subset $\mathcal{C}$ in $\mathcal{H}(n;\mathbf{F})$ there exists a constant $c = c(\mathcal{C})$ such that all $Z = X + Y$, $W = U + V \in \mathcal{C}$ and $M \in \mathrm{MSp}(n;\mathbf{F})$ satisfy*

$$\det Y_M \leq c \cdot \det V_M.$$

We use the abbreviations

$$\Gamma_n := \Gamma(n;\mathcal{O}) \quad \text{and} \quad \Gamma_n^\infty := \Gamma(n;\mathcal{O})_\infty.$$

LEMMA 4.3. *Let $\varepsilon \in \mathbf{R}$, $\varepsilon > 0$ and $k \in \mathbf{R}$, $k > r(n+1) - 2$. Then the series*

$$\sum_{M:\,\Gamma_n^\infty\backslash\Gamma_n} |\det M\{Z\}|^{-k}$$

converges uniformly for $Z \in \mathcal{VS}_\varepsilon(n;\mathbf{F})$.

Proof. In view of (3.3) the definition does not depend on the choice of the representatives. Hence let $\mathcal{R}$ denote a fixed set of representatives. According to Proposition 4.1 the series is uniformly majorized by

$$\sum_{M\in\mathcal{R}} |\det M\{I\}|^{-k}.$$

Observe that $|\det M\{I\}|^{-2} = \det Y$, whenever $M\langle I\rangle = X + Y$. Let dv denote the invariant volume element quoted in Proposition 2.4. Moreover set

$$\mathscr{C} = \{Z = X + Y \in \mathscr{F}(n;\mathscr{O}); \det Y \leq c\}$$

for sufficiently large $c > 1$. Then $\mathscr{C}$ becomes a compact subset with positive volume. Hence the series is majorized by

$$G_k := \sum_{M \in \mathscr{R}} \int_{M\langle\mathscr{C}\rangle} (\det Y)^{k/2}\, dv$$

in view of Proposition 4.2. Let l denote the number of neighbors of $\mathscr{F}(n;\mathscr{O})$ and set $\mathscr{U} = \bigcup_{M \in \mathscr{R}} M\langle\mathscr{C}\rangle$. Thus we obtain

$$G_k \leq l \int_{\mathscr{U}} (\det Y)^{k/2}\, dv.$$

Now $\mathscr{U}$ is contained in a fundamental domain of $\mathscr{H}(n;\mathbf{F})$ with respect to the action of $\Gamma(n;\mathscr{O})_\infty$. Every $Z = X + Y \in \mathscr{U}$ satisfies $\det Y \leq c$ in virtue of $\mathscr{C} \subset \mathscr{F}(n;\mathscr{O})$. According to (3.3) it suffices to check the convergence of the integral

$$\int_{\substack{X \in \mathscr{C}(n;\mathscr{O}), Y \in \mathscr{R}(n;\mathbf{F}) \\ \det Y \leq c}} (\det Y)^{k/2}\, dv.$$

In view of $dv = 2^{rn(n-1)/2}(\det Y)^{-rn}\, dX\, dY$ it suffices to estimate the integral

$$\int_{Y \in \mathscr{R}(n;\mathbf{F}), \det Y \leq c} (\det Y)^{k/2-rn}\, dY.$$

According to **[16]**, I.5.10, this integral exists, whenever $k > r(n+1) - 2$. □

Thus we can easily derive

THEOREM 4.4. *The series*

$$E_n^{\mathbf{F}}(Z, s) := \sum_{M:\, \Gamma_n^\infty \backslash \Gamma_n} (\det Y_M)^s$$

converges absolutely and uniformly, whenever Z belongs to a compact subset of $\mathscr{H}(n;\mathbf{F})$ and $s \in \mathbf{C}$ satisfies $\operatorname{Re}(s) \geq k$, $k > \frac{1}{2}r(n+1) - 1$. *Given $Z \in \mathscr{H}(n;\mathbf{F})$ the function*

$$\left\{s \in \mathbf{C}; \operatorname{Re}(s) > \frac{1}{2}r(n+1) - 1\right\} \to \mathbf{C}, \quad s \mapsto E_n^{\mathbf{F}}(Z, s),$$

becomes holomorphic. Let $s \in \mathbf{C}$, $\operatorname{Re}(s) > \frac{1}{2}r(n+1)-1$, *be fixed. Then*

$$E_n^{\mathbf{F}}(M\langle Z\rangle, s) = E_n^{\mathbf{F}}(\overline{Z}', s) = E_n^{\mathbf{F}}(Z, s) \tag{4.4}$$

holds for all $Z \in \mathscr{H}(n;\mathbf{F})$ *and* $M \in \Gamma(n;\mathscr{O})$. *Given* $\varepsilon > 0$ *there exists* $c > 0$ *such that*

$$|E_n^{\mathbf{F}}(Z, s)| \leq c(\det Y)^{\operatorname{Re}(s)} \tag{4.5}$$

holds for all $Z \in \mathscr{H}(n;\mathbf{F})$ *satisfying* $Y \geq \varepsilon I$.

Proof. The definition does not depend on the choice of the representatives in view of (3.3). Using $\det Y_M = (\det Y)\cdot|\det M\{Z\}|^{-2}$ the properties of convergence follow from the previous lemma.

The uniform convergence implies that the function $s \mapsto E_n^{\mathbf{F}}(Z, s)$ becomes holomorphic. If K then also KM, where $M \in \Gamma(n;\mathscr{O})$, resp. $\tilde{K}$ (cf. Proposition 1.4), run through sets of representatives of $\Gamma_n^\infty\backslash\Gamma_n$. Hence (4.4) follows by a rearrangement. In order to prove (4.5), we may assume $Z \in \mathscr{VS}_\varepsilon(n;\mathbf{F})$ in virtue of $E_n^{\mathbf{F}}(Z+S, s) = E_n^{\mathbf{F}}(Z, s)$ for $S \in \operatorname{Alt}(n;\mathscr{O})$. Then Lemma 4.3 completes the proof. □

DEFINITION. $E_n^{\mathbf{F}}(Z, s)$ is called *Eisenstein-series in* Z *and* s.

In virtue of (3.1) the case $\mathbf{F} = \mathbf{R}$, $n = 1$ becomes trivial, namely

$$E_1^{\mathbf{R}}(y, s) = y^s + y^{-s}, \quad \text{whenever } y \in \mathscr{H}(1;\mathbf{R}) = \mathbf{R}^+. \tag{4.6}$$

Consider the classical non-analytic Eisenstein-series

$$E(z, s) = \frac{1}{2} \sum_{(c,d)\in\mathbf{Z}^2 \text{coprime}} \left(\frac{y}{|cz+d|^2}\right)^s, \tag{4.7}$$

where $s \in \mathbf{C}$, $\operatorname{Re}(s) > 1$, $z = x + iy \in \mathbf{C}$, $y > 0$ (cf. **[19]**, **[20]**). Then (3.2) and **[16]**, II.2.6, imply

$$E_1^{\mathbf{C}}(z, s) = E(iz, s), \qquad z \in \mathscr{H}(1;\mathbf{C}). \tag{4.8}$$

Consider the Laplace-Beltrami-operator Δ in Theorem 2.6. Corollary 2.7 immediately leads to

COROLLARY 4.5. *The Eisenstein-series is an eigenfunction of the Laplace-Beltrami-operator. More precisely, if* $s \in \mathbf{C}$, $\operatorname{Re}(s) > \frac{1}{2}r(n+1) - 1$, *then*

$$\Delta E_n^{\mathbf{F}}(Z, s) = ns(s - \tfrac{1}{2}r(n+1) + 1)E_n^{\mathbf{F}}(Z, s).$$

According to the classical procedure by H. Braun [2], we can show that the abscissa of absolute convergence is given by $\mathrm{Re}(s) = \frac{1}{2}r(n+1) - 1$ except for the trivial case (4.6), of course. Therefore some preliminaries are necessary.

A matrix $G \in \mathrm{Mat}(n, m; \mathscr{O})$, where $m \geq n$ (resp. $n \geq m$), is called *primitive* if there exists $U \in \mathrm{GL}(m; \mathscr{O})$ such that $U = \binom{G}{*}$ (resp. $U \in \mathrm{GL}(n; \mathscr{O})$ such that $U = (G, *)$). Clearly if $m \geq n$

(4.9) G is primitive if and only if $H \in \mathrm{Mat}(m, n; \mathscr{O})$ exists such that $GH = I$.

In the cases $\mathscr{O} = \mathbf{Z}$, $\mathbf{Z}e_1 + \mathbf{Z}e_2$ the matrix G proves to be primitive if and only if the n-rowed subdeterminants of G are coprime.

Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{MSp}(n; \mathbf{F})$ then (C, D) is called the second row of M.

PROPOSITION 4.6. *The second rows of the matrices in $\Gamma(n; \mathscr{O})$ coincide with the primitive pairs $(C, D) \in \mathrm{Mat}(n, 2n; \mathscr{O})$ satisfying $C\overline{D}' + D\overline{C}' = 0$.*

Proof. If M belongs to $\Gamma(n; \mathscr{O})$, apply (1.1) and use $\Gamma(n; \mathscr{O}) \subset \mathrm{GL}(2n; \mathscr{O})$. Conversely, let such a pair (C, D) be given. According to (4.9) $F, G \in \mathrm{Mat}(n; \mathscr{O})$ exist such that $CF + DG = I$. Now set

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A := \overline{G}' - \overline{F}'GC, \quad B := \overline{F}' - \overline{F}'GD$$

and verify $M \in \Gamma(n; \mathscr{O})$. □

Next we consider $\Gamma(1; \mathscr{O}(\mathbf{H}))$ and compute the number of d's, whenever an odd c is given.

PROPOSITION 4.7. *Let $c \in \mathscr{O}(\mathbf{H})$ such that $N(c)$ is odd and set $l := \max\{m \in \mathbf{N}; \frac{1}{m}c \in \mathscr{O}\}$. Then there exist $l \cdot N(c)$ cosets $d + c\mathrm{Alt}(1; \mathscr{O})$ such that $c\bar{d} + d\bar{c} = 0$.*

Proof. We can replace c by εc, $\varepsilon \in \mathscr{E} = \{g \in \mathscr{O}; N(g) = 1\}$, and may assume $c = \sum_{j=1}^{4} c_j e_j$, $c_j \in \mathbf{Z}$. Thus $l = \text{g.c.d.}(c_1, c_2, c_3, c_4)$ holds. Let $q = N(c)$, then there are exactly lq^3 tuples $(d_1, d_2, d_3, d_4)'$ in $\mathbf{Z}^4 \bmod q$ such that

$$c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4 \equiv 0 \mod q$$

holds. Hence there are lq^3 cosets $d_j + q\mathscr{O}$ such that $2\,\mathrm{Re}(d_j\bar{c}) \equiv 0 \bmod q$. Observe that each coset $c\mathscr{O}$ decomposes into q^2 cosets $d + q\mathscr{O}$

(cf. **[17]**). After renumbering we therefore may assume that

$$\bigcup_{j=1}^{lq}(d_j + c\mathcal{O}) = \bigcup_{j=1}^{lq^3}(d_j + q\mathcal{O}).$$

Since q is odd, we can choose the representatives such that $\mathrm{Re}(d_j\bar{c}) = 0$ holds for $1 \le j \le lq$. Hence $d_j + c\mathrm{Alt}(1;\mathcal{O})$, $1 \le j \le lq$, are the cosets with the desired property. □

Next it is necessary to compute an integral. The same arguments, which were used by H. Braun in **[2]**, **[3]** resp. in **[16]**, V.1.2, yield

LEMMA 4.8. *In the case* $\mathbf{F} = \mathbf{R}$ *let* $n > 1$, $s \in \mathbf{C}$, $\mathrm{Re}(s) > n - 3/2$. *If* $\mathbf{F} = \mathbf{C}, \mathbf{H}$, *let* $n \ge 1$, $s \in \mathbf{C}$, $\mathrm{Re}(s) > rn - 1$. *Given* $Z = X + Y \in \mathscr{H}(n;\mathbf{F})$ *the integral*

$$\eta_s(Z) := \int_{\mathrm{Alt}(n;\mathbf{F})} |\det(Z+T)|^{-s}\, dT$$

exists and satisfies

$$\eta_s(Z) = (\det Y)^{r(n+1)/2-1-s}\eta^{\mathbf{F}}_{s,n}, \tag{4.10}$$

where

$$\eta^{\mathbf{F}}_{s,n} = \pi^{rn(n+1)/4-n/2}\prod_{j=1}^{n}\frac{\Gamma(s+1-\frac{1}{2}r(n+j))}{\Gamma(s+1-rj)}\frac{\Gamma(\frac{1}{2}(s+1-rj))}{\Gamma(\frac{1}{2}(s+r-rj))}.$$

Note that in the case $\mathbf{F} = \mathbf{R}$, i.e. $r = 1$, several factors on the right-hand side can be reduced such that the reduced product even exists for $\mathrm{Re}(s) > n - 3/2$. Here $\Gamma(s)$ denotes the gamma-function, since confusion with the modular group is not possible.

The existence of the integral implies the convergence of a series.

COROLLARY 4.9. *Let* $k \in \mathbf{R}$ *and* $k > n - 3/2, n > 1$ *for* $\mathbf{F} = \mathbf{R}$ *resp.* $k > rn - 1$, $n \ge 1$ *for* $\mathbf{F} = \mathbf{C}, \mathbf{H}$. *Given* $\varepsilon > 0$ *there exists* $c > 0$ *such that*

$$c^{-k}\eta_k(Z) \le \sum_{T\in\mathrm{Alt}(n;\mathcal{O})} |\det(Z+T)|^{-k} \le c^k\eta_k(Z)$$

holds for all $Z = X + Y \in \mathscr{H}(n;\mathbf{F})$ *satisfying* $Y \ge \varepsilon I$.

Proof. The assertion follows from an estimation between $|\det(Z+T)|^{-k}$ and

$$\int_{\mathscr{C}(n;\mathcal{O})} |\det(Z+T+H)|^{-k}\, dH.$$

This estimation can be derived by (1.10), (1.11), (1.12) and **[16]**, V.1.4. □

Now we follow H. Braun **[2]** in order to determine the abscissa of convergence of the Eisenstein-series. Hereby the result on real Eisenstein-series quoted by H. Maaß **[23]** can even be strengthened.

THEOREM 4.10. *Let* $n > 1$ *for* $\mathbf{F} = \mathbf{R}$ *and* $n \geq 1$ *for* $\mathbf{F} = \mathbf{C}, \mathbf{H}$. *Then the Eisenstein-series* $E_n^{\mathbf{F}}(Z, s)$ *does not converge absolutely, whenever* $\mathrm{Re}(s) = \frac{1}{2}r(n+1) - 1$.

Proof. According to Proposition 4.2 it suffices to show that the series

$$E_n^{\mathbf{F}}(I, k) = \sum_{M:\, \Gamma_n^\infty \backslash \Gamma_n} |\det M\{I\}|^{-2k}, \qquad k = \tfrac{1}{2}r(n+1) - 1,$$

diverges. Therefore we take second rows (C, D) of matrices $M \in \Gamma(n; \mathscr{O})$ such that the cosets $\Gamma_n^\infty M\left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right)$, $S \in \mathrm{Alt}(n; \mathscr{O})$, are mutually disjoint. In view of

$$\begin{aligned} E_n^{\mathbf{F}}(I, k) &\geq \sum_{M\left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right)} |\det M\{I\}|^{-2k} \\ &= \sum_{C,D,S} |\det C|^{-2k} |\det(I + C^{-1}D + S)|^{-2k} \end{aligned}$$

and Corollary 4.9 it suffices to estimate

$$E_k := \sum_{C,D} |\det C|^{-2k}.$$

In the case $\mathbf{F} = \mathbf{R}$, $n \geq 2$ choose

$$C = \begin{pmatrix} cI^{(2)} & 0 \\ G & I \end{pmatrix}, \quad D = \begin{pmatrix} dJ & -dJG' \\ 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $c \in \mathbf{N}$, d, $1 \leq d \leq c$, is relatively prime to c and G runs through a set of representatives of $\mathrm{Mat}(n-2, 2; \mathbf{Z})/c\mathrm{Mat}(n-2, 2; \mathbf{Z})$, which consists of c^{2n-4} elements. (C, D) has the desired property. If φ denotes Euler's φ-function, we obtain $k = \frac{1}{2}(n-1)$ and

$$E_k = \sum_{c,d} c^{-2} = \sum_{c=1}^{\infty} \varphi(c) c^{-2}.$$

But this series diverges.

In the case $\mathbf{F} = \mathbf{C}$ apply **[3]**, Theorem II.

In the case $\mathbf{F} = \mathbf{H}$ let c run through a system of representatives of

$$\mathscr{E}\backslash\{x \in \mathscr{O}; N(x) = p\},$$

where $\mathscr{E} = \{g \in \mathscr{O}; N(g) = 1\}$ and p runs through all odd primes. For every prime p we have $p+1$ possibilities for c according to **[9]**. Given c choose $d_1, \dots, d_p$ according to Proposition 4.7 and assume $d_p = 0$. Hence we may suppose $p \nmid N(d_j)$ for $1 \le j < p$. Set $x = (c_2, \dots, c_n)'$ and let each c_j run through a set of representatives of $\mathscr{O}/\mathscr{O}c$, which consists of $N(c)^2 = p^2$ elements (cf. **[17]**). Now set

$$C = \begin{pmatrix} c & 0 \\ x & I \end{pmatrix}, \quad D = \begin{pmatrix} d & -d\bar{x}' \\ 0 & 0 \end{pmatrix}, \quad d = d_j, \ 1 \le j < p,$$

and observe that (C, D) has the desired property. Now we obtain $k = 2n+1$ and

$$E_k = \sum_{p > 2 \text{ prime}} (p-1)(p+1)p^{-3}.$$

This series diverges. □

Just as in the case of Siegel modular forms we can define a modified ϕ-operator. Given a function $f\colon \mathscr{H}(n; \mathbf{F}) \to \mathbf{C}$ and $s \in \mathbf{C}$, we set

$$f\,|_s\phi\colon \mathscr{H}(n-1; \mathbf{F}) \to \mathbf{C}, \qquad Z \mapsto \lim_{\lambda \to \infty} \lambda^{-s} f\left(\begin{pmatrix} Z & 0 \\ 0 & \lambda \end{pmatrix}\right),$$

if this limit exists. $f\,|_s\phi$ has to be regarded as a constant, if $n = 1$. Then ϕ is called the modified Siegel ϕ-operator.

Finally we show that the modified Siegel ϕ-operator can be applied to Eisenstein-series just as in the classical case.

THEOREM 4.11. *Given $s \in \mathbf{C}$, $\mathrm{Re}(s) > \frac{1}{2}r(n+1) - 1$, then one has*

$$E_n^{\mathbf{F}}(\cdot, s)\,|_s\phi = E_{n-1}^{\mathbf{F}}(\cdot, s) \quad \textit{for} \ \ n \ge 2,$$
$$E_1^{\mathbf{F}}(\cdot, s)\,|_s\phi = 1.$$

Proof. According to Lemma 4.3 the limit may be distributed through the infinite series. The case $n = 1$ becomes clear in view of

$$\lim_{\lambda \to \infty} |M\{\lambda\}|^{-2} = \lim_{\lambda \to \infty} N(c\lambda + d)^{-1} = \begin{cases} N(d)^{-1} & \text{if } c = 0, \\ 0 & \text{if } c \ne 0. \end{cases}$$

Let $n \ge 2$ and let Γ_n^* denote the set of matrices $M \in \Gamma_n$ such that the elements $m_{2n,j}$, $1 \le j < 2n$, vanish. Γ_n^* proves to be a subgroup and one easily verifies that the map

$$\Gamma_{n-1}^{\infty}\backslash\Gamma_{n-1} \to (\Gamma_n^* \cap \Gamma_n^{\infty})\backslash\Gamma_n^*, \quad \Gamma_{n-1}^{\infty}M \mapsto (\Gamma_n^* \cap \Gamma_n^{\infty})(M \times I^{(2)}),$$

becomes a bijection. Let $Z_\lambda := \left(\begin{smallmatrix} Z & 0 \\ 0 & \lambda \end{smallmatrix}\right)$. Given $M \in \Gamma_n^*$ then $|\det M\{Z_\lambda\}|$ does not depend on λ. Hence we obtain

$$\sum_{M:\ (\Gamma_n^* \cap \Gamma_n^\infty)\backslash \Gamma_n^*} (\det Y)^s |\det M\{Z_\lambda\}|^{-2s} = E_{n-1}^{\mathbf{F}}(Z, s).$$

Given $M \in \Gamma(n;\mathscr{O})$ such that $\Gamma_n^\infty M \cap \Gamma_n^* = \emptyset$ one checks that $\lim_{\lambda\to\infty} |M\{Z_\lambda\}| = \infty$ holds. □

The isomorphisms χ_2 and χ_3 in Remark 2.3 between symmetric spaces correspond to identities between the associated Eisenstein-series. Therefore the Eisenstein-series (4.7) and Eisenstein-series for $GL(4;\mathbf{Z})$, which were investigated by A. Terras **[31]**, appear. Note that the action of $\Gamma(3;\mathbf{Z})_\infty$ corresponds to the action of the parabolic subgroup $P_{3,1}$ of $GL(4;\mathbf{Z})$ via χ_3. Consider the attached Eisenstein-series of the second type in **[31]**

$$E_{s,0}(Y) := \sum_{P:\ \mathrm{Pr}(4,3,\mathbf{Z})/GL(3;\mathbf{Z})} (\det Y[P])^{-s},$$

where $Y \in \mathrm{SPos}(4;\mathbf{R})$ and $\mathrm{Pr}(4,3,\mathbf{Z})$ denotes the set of primitive 4×3 matrices over $\mathbf{Z}$. Thus an explicit computation yields

LEMMA 4.12. (a) *Given*

$$Z = xJ + Y = \begin{pmatrix} y_1 & y+x \\ y-x & y_2 \end{pmatrix} \in \mathscr{H}(2;\mathbf{R})$$

and $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > \frac{1}{2}$ *one has*

$$E_2^{\mathbf{R}}(Z,s) = E(x + i\sqrt{\det Y}, 2s) + E\left(\frac{1}{y_1}(-y + i\sqrt{\det Y}), 2s\right).$$

(b) *Given* $Z \in \mathscr{H}(3;\mathbf{R})$ *and* $s \in \mathbf{C}$ *with* $\mathrm{Re}(s) > 1$ *one has*

$$E_3^{\mathbf{R}}(Z,s) = E_{2s,0}(\chi_3(Z)) + E_{2s,O}(\chi_3(Z)^{-1}).$$

5. Fourier-expansion of Eisenstein-series. The Fourier-expansion of non-analytic Eisenstein-series on the Siegel half-space was investigated by H. Maaß **[22]**, §18. G. Shimura **[27]** dealt with the case $\mathbf{F} = \mathbf{C}$, if we regard (0.2) and (1.9). Some of the following results on real Eisenstein-series were already obtained by H. Maaß **[23]**.

Throughout this paragraph let $s \in \mathbf{C}$ be fixed such that $\mathrm{Re}(s) > \frac{1}{2}r(n+1) - 1$ holds. In order to describe the Fourier-development, we have to determine the dual lattice. Therefore set

$$\mathscr{O}^\#(\mathbf{F}) = \mathscr{O}(\mathbf{F}), \quad \mathbf{F} = \mathbf{R}, \mathbf{C},$$

$$\mathscr{O}^\#(\mathbf{H}) = \mathbf{Z}2e_1 + \mathbf{Z}(e_1 + e_2) + \mathbf{Z}(e_1 + e_3) + \mathbf{Z}(e_1 + e_4)$$

(cf. **[16]**, p. 12). Using the definition of τ in §2 we derive

$$\begin{aligned}\mathrm{Alt}^{\tau}(n;\mathcal{O}) &:= \{T \in \mathrm{Alt}(n;\mathbf{F}); \tau(T,S) \in \mathbf{Z} \text{ for all } S \in \mathrm{Alt}(n;\mathcal{O})\}\\ &= \{T = (t_{kl}) \in \mathrm{Alt}(n;\mathbf{F}); t_{kk} \in \mathcal{O}, 2t_{kl} \in \mathcal{O}^{\#} \text{ for } k \neq l\}.\end{aligned}$$

Since the Eisenstein-series is invariant under the transformations $Z \mapsto Z+S$, $S \in \mathrm{Alt}(n;\mathcal{O})$, we obtain

$$E_n^{\mathbf{F}}(Z,s) = \sum_{T \in \mathrm{Alt}^{\tau}(n;\mathcal{O})} c(Y;T)e^{2\pi i \tau(X,T)}, \qquad Z = X + Y \in \mathcal{H}(n;\mathbf{F}).$$

The use of $E_n^{\mathbf{F}}(Z[U],s) = E_n^{\mathbf{F}}(\overline{Z}',s) = E_n^{\mathbf{F}}(Z,s)$ according to (4.4) as well as the uniqueness of the Fourier-coefficients yield

$$c(Y[U];T) = c(Y;T[\overline{U}']), \quad c(Y;T) = c(Y;-T)$$

for all $U \in \mathrm{GL}(n;\mathcal{O})$.

It is convenient to decompose the Eisenstein-series into $n+1$ partial series. Given $0 \leq j \leq n$ we set

$$E_{n,j}^{\mathbf{F}}(Z,s) = \sum_{\substack{M:\, \Gamma_n^{\infty}\backslash\Gamma_n \\ \operatorname{rank} C = j}} (\det Y_M)^s.$$

The definition leads to the obvious relations

$$E_n^{\mathbf{F}}(Z,s) = \sum_{j=0}^{n} E_{n,j}^{\mathbf{F}}(Z,s), \tag{5.1}$$

$$E_{n,0}^{\mathbf{F}}(Z,s) = (\det Y)^s. \tag{5.2}$$

Set $\mathrm{Pr}(n,m;\mathcal{O}) := \{G \in \mathrm{Mat}(n,m;\mathcal{O}); G \text{ primitive}\}$. Following H. Maaß **[22]**, §11, the same arguments yield

LEMMA 5.1. *Given $0 < j < n$ let P run through a set of representatives of $\mathrm{Pr}(n,j;\mathcal{O})/\mathrm{GL}(j;\mathcal{O})$. Each P is completed to a matrix $U = (P,*) \in \mathrm{GL}(n;\mathcal{O})$ in exactly one way. Let M_1 run through the subset of representatives of $\Gamma_j^{\infty}\backslash\Gamma_j$, where $|\det C_1| \neq 0$. Then $(M_1 \times I)\left(\begin{smallmatrix}\overline{U}' & 0\\ 0 & U^{-1}\end{smallmatrix}\right)$ runs through the subset of representatives of $\Gamma_n^{\infty}\backslash\Gamma_n$, where $\operatorname{rank} C = j$.*

Thus we easily compute

COROLLARY 5.2. *Given $0 < j < n$ one has*

$$E_{n,j}^{\mathbf{F}}(Z,s) = \sum_{P:\, \mathrm{Pr}(n,j;\mathcal{O})/\mathrm{GL}(j;\mathcal{O})} (\det Y)^s(\det Y[P])^{-s}E_{j,j}^{\mathbf{F}}(Z[P],s).$$

Given $S \in \mathrm{Pos}(n;\mathbf{R}), 0 < j < n$, and $\omega \in \mathbf{C}$ satisfying $\mathrm{Re}(\omega) > \frac{1}{2}n$, we can define the Dirichlet-series

$$\zeta_j(S,\omega) := \sum_{P:\ \mathrm{Pr}(n,j;\mathbf{Z})/\mathrm{GL}(j;\mathbf{Z})} (\det S[P])^{-\omega}.$$

A related series was investigated by M. Koecher **[13]**. $\zeta_1(S,\omega)$ proves to be the quotient of the corresponding Epstein-zeta-function over the Riemann-zeta-function $2\zeta(2\omega)$. In view of (5.1), (5.2), (4.6) and Corollary 5.2 we gain

$$E^{\mathbf{R}}_{n,1}(Z,s) = (\det Y)^s \zeta_1(Y,2s), \tag{5.3}$$

whenever $n \geq 2$.

In view of the corollary the problem is reduced to the investigation of $E^{\mathbf{F}}_{n,n}(Z,s)$. Set $\mathbf{F}_{\mathbf{Q}} = \mathbf{Q}e_1 + \cdots + \mathbf{Q}e_r$. The matrices in $\mathrm{Mat}(n;\mathbf{F}_{\mathbf{Q}})$ are called rational.

LEMMA 5.3. *Let* $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ *run through the subset of representatives of* $\Gamma^\infty_n \backslash \Gamma_n$, *where* $\mathrm{rank}\, C = n$. *Then each* $R \in \mathrm{Alt}(n;\mathbf{F}_{\mathbf{Q}})$ *is represented in the form* $R = C^{-1}D$ *exactly once. Moreover*

$$\nu(R) = |\det C|$$

becomes well-defined and satisfies

$$\nu(R+S) = \nu(R) \quad \textit{for} \quad S \in \mathrm{Alt}(n;\mathcal{O}).$$

If $\mathcal{O} = \mathbf{Z}$, $\mathbf{Z}e_1 + \mathbf{Z}e_2$, *then* $\nu(R)$ *coincides with the absolute value of the product of the denominators of the reduced elementary divisors of* R.

Proof. Given $R \in \mathrm{Alt}(n;\mathbf{F}_{\mathbf{Q}})$ choose $U, V \in \mathrm{GL}(n;\mathcal{O})$ such that

$$URV = [q_1,\dots,q_n], \qquad q_j \in \mathbf{F}_{\mathbf{Q}}, \quad q_{j+1} \in \mathcal{O}q_j,$$

according to **[16]**, I.2.3. Each q_j possesses a representation $q_j = c_j^{-1}d_j$, $c_j \neq 0$, $c_j, d_j \in \mathcal{O}$, where c_j and d_j are relatively left-prime. Define $C_0 = [c_1,\dots,c_n]$, $D_0 = [d_1,\dots,d_n]$, then (C_0, D_0) becomes primitive (cf. **[16]**, I.1.11). Hence $(C,D) := (C_0U, D_0V^{-1})$ proves to be primitive and satisfies $\mathrm{rank}\, C = n$ as well as

$$C^{-1}D = U^{-1}[q_1,\dots,q_n]V^{-1} = R.$$

Now (C,D) turns out to be the second row of a matrix in $\Gamma(n;\mathcal{O})$ according to Proposition 4.6. If $\mathcal{O} = \mathbf{Z}$, $\mathbf{Z}e_1 + \mathbf{Z}e_2$, moreover $|\det C|$ equals the absolute value of the product of the denominators of the reduced elementary divisors of R.

Clearly, the representation $R = C^{-1}D$ and $|\det C|$ do not depend on the choice of the representative in the coset $\Gamma_n^\infty M$ in view of (3.3). Now suppose that $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ and $M_1 = \left(\begin{smallmatrix} A_1 & B_1 \\ C_1 & D_1 \end{smallmatrix}\right)$ belong to $\Gamma(n;\mathcal{O})$ and fulfill $\operatorname{rank} C = \operatorname{rank} C_1 = n$ as well as $C^{-1}D = C_1^{-1}D_1 = R$. Then $\overline{R}' = -R$ yields $C\overline{D}_1' + D\overline{C}_1' = 0$. Hence (1.2) implies $MM_1^{-1} \in \Gamma_n^\infty$, i.e. $\Gamma_n^\infty M = \Gamma_n^\infty M_1$. Replacing M by $M\left(\begin{smallmatrix} I & S \\ 0 & I \end{smallmatrix}\right), S \in \operatorname{Alt}(n;\mathcal{O})$, yields $\nu(R+S) = \nu(R)$. □

In the case $\mathcal{O} = \mathbf{Z}$ we obtain information about the elementary divisor normal form of the C-block in a matrix $M \in \Gamma(n;\mathbf{Z})$.

COROLLARY 5.4. *Given $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma(n;\mathbf{Z})$ then the elementary divisor matrix of C has the form*

$$[c_1, c_1, c_2, c_2, \dots, c_m, c_m, 0, \dots, 0], \qquad \text{if } \operatorname{rank} C = 2m,$$
$$[1, c_1, c_1, c_2, c_2, \dots, c_m, c_m, 0, \dots, 0], \qquad \text{if } \operatorname{rank} C = 2m+1,$$

where $c_1, \dots, c_m \in \mathbf{N}$ such that $c_j \mid c_{j+1}$.

Proof. We may assume $\operatorname{rank} C = n$. Then a combination of **[25]**, Theorem IV.1, with Lemma 5.3 yields the assertion. □

Replacing M by a product of M and Q a corresponding result is true for each other block of the matrix $M \in \Gamma(n;\mathbf{Z})$.

Furthermore, Lemma 5.3 immediately yields

$$E_{n,n}^{\mathbf{F}}(Z,s) = (\det Y)^s \sum_{R \in \operatorname{Alt}(n;\mathbf{F}_\mathbf{Q})} \nu(R)^{-2s} |\det(Z+R)|^{-2s}. \tag{5.4}$$

In view of $\nu(R+S) = \nu(R)$ for $S \in \operatorname{Alt}(n;\mathcal{O})$, the partial series $E_{n,j}^{\mathbf{F}}(Z,s)$ possesses a Fourier-expansion, too. Let $R \bmod 1$ indicate that R runs through a set of representatives of $\operatorname{Alt}(n;\mathbf{F}_\mathbf{Q})/\operatorname{Alt}(n;\mathcal{O})$. Given $T \in \operatorname{Alt}^\tau(n;\mathcal{O})$ and $Y \in \operatorname{Pos}(n;\mathbf{F})$, we define

$$\alpha_s(T) := \sum_{R \bmod 1} \nu(R)^{-2s} e^{2\pi i \tau(R,T)},$$
$$\beta_s(Y;T) := \int_{\operatorname{Alt}(n;\mathbf{F})} |\det(Y+X)|^{-2s} e^{-2\pi i \tau(X,T)}\, dX.$$

Given $U \in \operatorname{GL}(n;\mathcal{O})$ we immediately obtain

$$\alpha_s(T[U]) = \alpha_s(-T) = \alpha_s(T), \tag{5.5}$$
$$\beta_s(Y;T[U]) = \beta_s(Y[\overline{U}'];T), \quad \beta_s(Y;T) = \beta_s(Y;-T).$$

Hence Lemma 5.3 and the definition of the Fourier-coefficients imply

LEMMA 5.5.

$$E^{\mathbf{F}}_{n,n}(Z,s) = (\operatorname{vol}\mathscr{C}(n;\mathscr{O}))^{-1} \sum_{T\in\mathrm{Alt}^{\tau}(n;\mathscr{O})} (\det Y)^{s}\alpha_{s}(T)\beta_{s}(Y;T)e^{2\pi i\tau(X,T)}.$$

Combining this result with (5.1) and Corollary 5.2, we gain

COROLLARY 5.6.

$$E^{\mathbf{F}}_{n}(Z,s) = (\det Y)^{s} + (\det Y)^{s} \times \sum_{j=1}^{n} c_j^{-1} \sum_{P} \sum_{T\in\mathrm{Alt}^{\tau}(j;\mathscr{O})} \alpha_{s}(T)\beta_{s}(Y[P];T)e^{2\pi i\tau(X,T[\overline{P}'])},$$

where $c_j = \operatorname{vol}\mathscr{C}(j;\mathscr{O})$ *and* $P\colon \mathrm{Pr}(n,j;\mathscr{O})/\mathrm{GL}(j;\mathscr{O})$.

As a consequence we observe that in the Fourier-expansion of $E^{\mathbf{F}}_{n,j}(Z,s)$ all the coefficients of matrices $T\in\mathrm{Alt}^{\tau}(n;\mathscr{O})$ vanish, whenever $\operatorname{rank} T > j$.

Lemma 4.8 yields

$$\beta_{s}(Y;0) = (\det Y)^{r(n+1)/2-1-2s}\eta^{\mathbf{F}}_{2s,n}. \tag{5.6}$$

REMARK 5.7. It is possible to reduce the computation of $\beta_s(Y;T)$ to the case $|\det T| \neq 0$ by aid of (5.5). Therefore let

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Alt}^{\tau}(n;\mathscr{O}),\quad Y = \begin{pmatrix} Y_1 & * \\ * & * \end{pmatrix} \in \mathrm{Pos}(n;\mathbf{F}),$$
$$T_1 = T_1^{(m)},\quad Y_1 = Y_1^{(m)}.$$

Then one obtains

$$\beta_s(Y;T) = \beta_{s-r(n-m)/2}(Y_1;T_1)(\det Y)^{r(n+1)/2-2s} \cdot (\det Y_1)^{2s+1+r(m-1-2n)/2}\eta^{\mathbf{F}}_{2s,n-m}\pi^{rm(n-m)/2} \cdot \prod_{j=1}^{n-m} \frac{\Gamma(2s+1-\frac{1}{2}r(n+j))}{\Gamma(2s+1-\frac{1}{2}r(n-m+j))}.$$

In general the evaluation of the integral $\beta_s(Y;T)$ leads to generalized confluent hypergeometric functions, where the case $\mathbf{F}=\mathbf{C}$ was treated by G. Shimura **[26]**. On the other hand it might be possible to investigate $\alpha_s(T)$ in analogy with Y. Kitaoka's procedure **[11]** in the case of the Siegel half-space. But it seems to be plausible that the Fourier-coefficients of the Eisenstein-series can only be expressed by well-known functions, whenever the degree n is "sufficiently small".

Therefore let us consider the case $n = 1$. Now $\mathbf{F} = \mathbf{R}$ becomes trivial in view of (4.6). Dealing with $\mathbf{F} = \mathbf{C}$ we observe the connection (4.8) with the classical Eisenstein-series and obtain the Fourier-expansion from **[19]**, p. 46, or **[20]**.

In order to deal with the case $\mathbf{F} = \mathbf{H}$, it is more convenient to introduce the subring $\Lambda := \mathbf{Z}e_1 + \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ of $\mathscr{O}(\mathbf{H})$. Given $0 \neq c \in \Lambda$ define the greatest rational divisor of c in Λ by

$$\rho(c) := \max\{l \in \mathbf{N}; l^{-1}c \in \Lambda\}$$

and set $\rho(0) := 0$. Note that $\mathrm{Alt}(1;\mathscr{O}) = \mathrm{Alt}^{\tau}(1;\mathscr{O}) = \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4 \subset \Lambda$.

Given $S \in \mathrm{Pos}(n;\mathbf{R})$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > \frac{1}{2}n$, the Epstein-zeta-function associated with S is defined by

$$\zeta(S;s) := \sum_{0 \neq g \in \mathbf{Z}^n} (S[g])^{-s}.$$

Especially one has for $I = I^{(4)}$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 2$

$$\zeta(I;s) = \sum_{0 \neq c \in \Lambda} N(c)^{-s} = 8(1 - 2^{2-2s})\zeta(s)\zeta(s-1),$$

where ζ denotes the Riemann-zeta-function. Given $t, t^* \in \mathrm{Alt}(1;\mathscr{O})$ the Fourier-expansion involves the function

$$\sigma_s(t,t^*) := \sum_{\substack{0 \neq c \in \Lambda \\ ct = t^*c}} N(c)^{-s}.$$

Clearly $\sigma_s(t,t^*) = 0$ unless $N(t) = N(t^*)$. The structure of $\sigma_s(t,t^*)$ is elucidated by

PROPOSITION 5.8. *Let $t, t^* \in \mathrm{Alt}(1;\mathscr{O})$ with $N(t) = N(t^*) \neq 0$ and $s \in \mathbf{C}$ with $\mathrm{Re}(s) > 1$. Then there exists $S \in \mathrm{Pos}(2;\mathbf{Z})$ such that*

$$\sigma_s(t,t^*) = \zeta(S;s) \quad \textit{and} \quad \det S = \frac{4N(t)}{[\gcd(\rho(t+t^*), \rho(t-t^*))]^2}$$

Proof. Let

$$t = \sum_{j=2}^{4} t_j e_j, \quad t^* = \sum_{j=2}^{4} t_j^* e_j.$$

Then $c = \sum_{j=1}^{4} c_j e_j$ satisfies $ct = t^*c$ if and only if $(c_1, c_2, c_3, c_4)'$ belongs to the kernel of the matrix

$$\begin{pmatrix} t_2 - t_2^* & 0 & t_4 + t_4^* & -t_3 - t_3^* \\ 0 & t_2 - t_2^* & t_3 - t_3^* & t_4 - t_4^* \\ t_4 - t_4^* & t_3 + t_3^* & -t_2 - t_2^* & 0 \\ -t_3 + t_3^* & t_4 + t_4^* & 0 & -t_2 - t_2^* \end{pmatrix},$$

which has the rank 2. Hence $\sigma_s(t,t) = \zeta(S;s)$ holds for an appropriate $S \in \mathrm{Pos}(2;\mathbf{Z})$. If $t_2 \neq t_2^*$ the kernel over $\mathbf{Q}$ is spanned by $a = (t_4+t_4^*, t_3-t_3^*, -t_2+t_2^*, 0)'$ and $b = (t_3+t_3^*, -t_4+t_4^*, 0, t_2-t_2^*)'$. Hence we have

$$\det S = \frac{\det(G'G)}{[\delta_2(G)]^2}, \quad G = (a,b) \in \mathrm{Mat}(4,2;\mathbf{Z}),$$

where $\delta_2(G)$ denotes the second determinantal divisor of G (cf. **[25]**, p. 25). An elementary computation yields $\det(G'G) = 4(t_2-t_2^*)^2 N(t)$ and $\delta_2(G) = (t_2-t_2^*)\gcd(\rho(t+t^*), \rho(t-t^*))$. In the case $t_2 = t_2^*$ analogous arguments complete the proof. □

If K_s denotes the modified Bessel-function, the Fourier-expansion is given by

THEOREM 5.9.

$$E_1^{\mathbf{H}}(z,s) = \sum_{t \in \mathbf{Z}e_2+\mathbf{Z}e_3+\mathbf{Z}e_4} c(y;t) e^{2\pi i \mathrm{Re}(\overline{x}t)},$$

where $z = x + y \in \mathscr{H}(1;\mathbf{H})$ *and with* $I = I^{(4)}$

$$c(y;0) = y^s + \pi^{3/2} \frac{\Gamma(s-3/2)\zeta(I;s-1)\zeta(2s-3)}{\Gamma(s)\zeta(I;s)\zeta(2s-2)} y^{3-s},$$

$$c(y;t) = 2\pi^s \frac{\sum_{l \mid \rho(t)} l^{3-2s} \sum_{t^* \in \mathrm{Alt}(1;\mathscr{O})} \sigma_{s-1}(t, t+2lt^*)}{\Gamma(s)\zeta(I;s)\zeta(2s-2)} \cdot |t|^{s-3/2} y^{3/2} K_{s-3/2}(2\pi |t| y)$$

for $0 \neq t \in \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$.

Proof. At first (5.6) yields

$$\beta_s(y;0) = \pi^{3/2} \frac{\Gamma(s-3/2)}{\Gamma(s)} y^{3-2s}.$$

Given $0 \neq t \in \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ we use an orthogonal transformation and apply **[24]**, p. 85, in the following calculation

$$\begin{aligned}
\beta_s(y;t) &= \int_{\mathrm{Alt}(1;\mathbf{H})} |y+x|^{-2s} e^{-2\pi i \mathrm{Re}(\overline{x}t)}\, dx \\
&= y^{3-2s} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (1 + x_1^2 + x_2^2 + x_3^2)^{-s} e^{-2\pi i y |t| x_1}\, dx_1\, dx_2\, dx_3 \\
&= 2\pi^s \frac{1}{\Gamma(s)} y^{3/2-s} |t|^{s-3/2} K_{s-3/2}(2\pi |t| y).
\end{aligned}$$

Next observe that the representatives of $\Gamma_1^\infty\backslash\Gamma_1$ may be chosen in $\mathrm{Mat}(2;\Lambda)$. Given $0 \neq c \in \Lambda$ let $\mathscr{R}(c)$ denote a set of representatives of the cosets $d + c\mathrm{Alt}(1;\mathscr{O})$, $d \in \Lambda$, satisfying $c\bar{d} + d\bar{c} = 0$. In analogy with Proposition 4.7 one can show that $\mathscr{R}(c)$ consists of $\rho(c)N(c)$ elements. Moreover we use the abbreviation

$$\gamma(c,t) := \sum_{d \in \mathscr{R}(c)} e^{2\pi i \mathrm{Re}(c^{-1}d\bar{t})}$$

for $t \in \mathbf{Z}e_2 + \mathbf{Z}e_3 + \mathbf{Z}e_4$ and obtain

$$\begin{aligned}\alpha_s(t) &= \sum_{\omega \in \mathbf{Q}e_2 + \mathbf{Q}e_3 + \mathbf{Q}e_4 \bmod 1} \nu(\omega)^{-2s} e^{2\pi i \mathrm{Re}(\bar{\omega}t)} \\ &= \frac{1}{\zeta(I;s)} \sum_{0 \neq c \in \Lambda} N(c)^{-s} \gamma(c,t),\end{aligned}$$

where $I = I^{(4)}$. Especially we have

$$\alpha_s(0) = \frac{1}{\zeta(I;s)} \sum_{0 \neq c \in \Lambda} \rho(c) N(c)^{1-s} = \frac{\zeta(I;s-1)\zeta(2s-3)}{\zeta(I;s)\zeta(2s-2)}.$$

Now let $t \neq 0$. A standard argument (cf. [6], 4.5) shows that

$$(*) \qquad \gamma(c,t) = \begin{cases} \rho(c)N(c) & \text{if } \mathrm{Re}(c^{-1}d\bar{t}) \in \mathbf{Z} \text{ for all } d \in \mathscr{R}(c), \\ 0 & \text{otherwise.} \end{cases}$$

Given $c = c_2 c_1$, where $c_1, c_2 \in \Lambda$, $N(c_2) = 2^m$, $m \in \mathbf{N}_0$, $N(c_1)$ odd, we gain

$$\gamma(c,t) = \gamma(c_2,t)\gamma(c_1,t).$$

Using the isomorphism between $\Lambda/l\Lambda$ and $\mathrm{Mat}(2;\mathbf{Z}/l\mathbf{Z})$ for odd $l \in \mathbf{N}$ (cf. [9], Vorlesung 8, resp. [17]) and a direct computation for c_2, one can show that $\mathrm{Re}(c^{-1}d\bar{t}) \in \mathbf{Z}$ holds for all $d \in \mathscr{R}(c)$ if and only if

$$\rho(c) | \rho(t) \quad \text{and} \quad ctc^{-1} \in t + 2\rho(c)\mathrm{Alt}(1;\mathscr{O}).$$

Thus we calculate

$$\begin{aligned}\alpha_s(t) &= \frac{1}{\zeta(I;s)} \sum_{l \in \mathbf{N}, l | \rho(t)} \sum_{t^* \in \mathrm{Alt}(1;\mathscr{O})} l^{3-2s} \sum_{\substack{0 \neq c \in \Lambda, \rho(c) = 1 \\ c\frac{1}{l}t = (\frac{1}{l}t + 2t^*)c}} N(c)^{1-s} \\ &= \frac{1}{\zeta(I;s)\zeta(2s-2)} \sum_{l | \rho(t)} l^{3-2s} \sum_{t^* \in \mathrm{Alt}(1;\mathscr{O})} \sigma_{s-1}(t, t + 2lt^*).\end{aligned}$$

Hence the assertion follows from Lemma 5.5. □

Note that the sum over t^* in the formula above is finite.

In the case $\mathbf{F} = \mathbf{R}$ we are able to give the Fourier-expansions explicitly for $n = 2, 3$. Given $t \in \mathbf{N}$ and $s \in \mathbf{C}$ let

$$\sigma_s(t) := \sum_{l \in \mathbf{N}, l | t} l^s$$

denote the divisor sum. Then the application of Remark 2.3 and **[19]**, p. 46, resp. **[20]** leads to

COROLLARY 5.10. *One has*

$$E_2^{\mathbf{R}}(Z, s) = \sum_{t \in \mathbf{Z}} c(Y; t) e^{2\pi i x t}, \qquad Z = xJ + Y \in \mathscr{H}(2; \mathbf{R})$$

where

$$\begin{aligned} c(Y; O) &= (\det Y)^s + (\det Y)^s \zeta_1(Y, 2s) \\ &\quad + \sqrt{\pi} \frac{\Gamma(2s - 1/2)}{\Gamma(2s)} \cdot \frac{\zeta(4s-1)}{\zeta(4s)} (\det Y)^{1/2 - s}, \\ c(Y; t) &= 2\pi^{2s} |t|^{2s - 1/2} \frac{\sigma_{1-4s}(|t|)}{\Gamma(2s)\zeta(4s)} (\det Y)^{1/4} K_{2s-1/2}(2\pi |t| \sqrt{\det Y}) \end{aligned}$$

for $0 \neq t \in \mathbf{Z}$.

Note that the Fourier-coefficients $c(Y; t)$ for $t \neq 0$ only depend on $\det Y$ and s.

Let $n \geq 3$ and fix a set of representatives P: $\mathrm{Pr}(n, 2; \mathbf{Z})/\mathrm{GL}(2; \mathbf{Z})$. Then each $T \in \mathrm{Alt}^\tau(n; \mathbf{Z})$ with $\operatorname{rank} T = 2$ possesses a unique representation

$$T = \tfrac{1}{2} t J[P'], \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where $0 \neq t \in \mathbf{Z}$ and where $\varepsilon(2T) = |t|$ is the greatest common divisor of the entries of $2T \in \mathrm{Alt}(n; \mathbf{Z})$. Now observe that

$$t^2 \cdot \det(Y[P]) = 2\tau(T'YT, Y)$$

holds. Hence we can combine the Corollaries 5.2 and 5.10 in order to gain

$$\begin{aligned} (5.7) \quad E_{n,2}^{\mathbf{R}}(Z, s) &= \sqrt{\pi} \frac{\Gamma(2s - 1/2)}{\Gamma(2s)} \frac{\zeta(4s-1)}{\zeta(4s)} (\det Y)^s \zeta_2\left(Y, 2s - \frac{1}{2}\right) \\ &\quad + \sum_{\substack{T \in \mathrm{Alt}^\tau(n; \mathbf{Z}) \\ \operatorname{rank} T = 2}} 2\pi^{2s} \frac{\sigma_{4s-1}(\varepsilon(2T))}{\Gamma(2s)\zeta(4s)} (\det Y)^s (2\tau(T'YT, Y))^{\frac{1}{4} - s} \\ &\quad \cdot K_{2s-1/2}(2\pi \sqrt{2\tau(T'YT, Y)}). \end{aligned}$$

Now let $n = 3$. We compute

$$\beta_s(Y;0) = (\det Y)^{1-2s}\pi^{3/2}\frac{\Gamma(2s-3/2)}{\Gamma(2s)}$$

in view of (5.6) and Lemma 4.8. Let $0 \neq T \in \mathrm{Alt}^\tau(3;\mathbf{Z})$ and $Y \in \mathrm{Pos}(3;\mathbf{R})$. We choose $V \in \mathrm{GL}(3;\mathbf{R})$ such that $Y = V'V$. Change of variables yields

$$\begin{aligned}\beta_s(Y;T) &= \int_{\mathrm{Alt}(3;\mathbf{R})} (\det(Y+X))^{-2s} e^{-2\pi i \tau(X,T)}\,dX\\
&= (\det Y)^{-2s}\int_{\mathrm{Alt}(3;\mathbf{R})} (\det(I+X[V^{-1}]))^{-2s} e^{-2\pi i \tau(X,T)}\,dX\\
&= (\det Y)^{1-2s}\int_{\mathrm{Alt}(3;\mathbf{R})} (\det(I+X))^{-2s} e^{-2\pi i \tau(X,T[V'])}\,dX\\
&= (\det Y)^{1-2s}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (1+x_1^2+x_2^2+x_3^2) e^{-2\pi i \omega x_1}\,dx_1\,dx_2\,dx_3\end{aligned}$$

by the use of an orthogonal transformation, where

$$\omega = (2\tau(T[V'], T[V']))^{1/2} = (2\tau(T'YT, Y))^{1/2}.$$

The same calculations as in the proof of Theorem 5.9 show that

$$\begin{aligned}\beta_s(Y;T) = 2\pi^{2s}\frac{1}{\Gamma(2s)}(2\tau(T'YT,Y))^{s-3/4}(\det Y)^{1-2s}\\
\cdot K_{2s-3/2}(2\pi\sqrt{2\tau(T'YT,Y)}\,).\end{aligned}$$

Given $0 \neq R \in \mathrm{Alt}(3;\mathbf{Q})$ note that $\nu(R) = l^2$, where $l \in \mathbf{N}$, if and only if $R = l^{-1}T$, where $T \in \mathrm{Alt}(3;\mathbf{Z})$ and $\varepsilon(T) = 1$. Denoting the number of elements of a set $\mathcal{S}$ by $\#\mathcal{S}$, we calculate

$$\begin{aligned}\alpha_s(0) &= \sum_{R \bmod 1} \nu(R)^{-2s}\\
&= \sum_{l=1}^{\infty} l^{-4s}\cdot\#\{g \in \mathbf{Z}^3;\, 1 \le g_j \le l, \text{g.c.d. } g = 1\}\\
&= \frac{\zeta(4s-3)}{\zeta(4s)}.\end{aligned}$$

Given $0 \neq T \in \mathrm{Alt}^\tau(3;\mathbf{Z})$ we may restrict to the case

$$T = \frac{1}{2}\begin{pmatrix} 0 & t & 0\\ -t & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad t = \varepsilon(2T),$$

in view of (5.5). Hence we calculate

$$\begin{aligned}\alpha_s(T) &= \sum_{R \bmod 1} \nu(R)^{-2s} e^{2\pi i\tau(R,T)} \\ &= \frac{1}{\zeta(4s)} \sum_{l=1}^{\infty} \sum_{j=1}^{3} \sum_{q_j=1}^{l} l^{-4s} e^{2\pi i t q_1/l} \\ &= \frac{1}{\zeta(4s)} \sigma_{3-4s}(t).\end{aligned}$$

A combination of (5.2), (5.3), (5.7) and Lemma 5.5 yields the final

COROLLARY 5.11.

$$E_3^{\mathbf{R}}(Z,s) = \sum_{T \in \mathrm{Alt}^{\tau}(3;\mathbf{Z})} c(Y;T) e^{2\pi i\tau(X,T)}, \qquad Z = X + Y \in \mathscr{H}(3;\mathbf{R}),$$

where

$$\begin{aligned}c(Y;0) &= (\det Y)^s + (\det Y)^s \zeta_1(Y,2s) \\ &\quad + \sqrt{\pi}\frac{\Gamma(2s-1/2)}{\Gamma(2s)}\frac{\zeta(4s-1)}{\zeta(4s)}(\det Y)^s \zeta_2(Y,2s-1/2) \\ &\quad + \pi^{\frac{3}{2}}\frac{\Gamma(2s-3/2)}{\Gamma(2s)}\frac{\zeta(4s-3)}{\zeta(4s)}(\det Y)^{1-s}, \\ c(Y;T) &= 2\pi^{2s}\frac{\sigma_{4s-1}(\varepsilon(2T))}{\Gamma(2s)\zeta(4s)}(\det Y)^s (2\tau(T'YT,Y))^{1/4-s} \\ &\qquad \times K_{2s-1/2}(2\pi\sqrt{2\tau(T'YT,Y)}) \\ &\quad + 2\pi^{2s}\frac{\sigma_{3-4s}(\varepsilon(2T))}{\Gamma(2s)\zeta(4s)}(\det Y)^{1-s}(2\tau(T'YT,Y))^{s-3/4} \\ &\qquad \times K_{2s-3/2}(2\pi\sqrt{2\tau(T'YT,Y)})\end{aligned}$$

for $T \neq 0$.

REFERENCES

[1] A. A. Albert, *Structure of Algebras*, Amer. Math. Soc. Publications, Providence 1961.

[2] H. Braun, *Konvergenz verallgemeinerter Eisensteinscher Reihen*, Math. Z., **44** (1939), 387–397.

[3] ———, *Hermitian Modular Functions* I, Ann. of Math., II. Ser., **50** (1949), 827–855. II, Ann. of Math., II. Ser., **51** (1950), 92–104. III, Ann. of Math., II. Ser., **53** (1951), 143–160.

[4] U. Christian, *Siegelsche Modulfunktionen*, 2. Auflage, Vorlesungsausarbeitung, Göttingen 1981.

[5] L. E. Dickson, *Algebras and their Arithmetics*, Dover Publications, New York 1960.

[6] J. Elstrodt, F. Grunewald, J. Mennicke, *Eisenstein series on three-dimensional hyperbolic space and imaginary quadratic number fields*, J. Reine Angew. Math., **360** (1985), 160–213.

[7] E. Freitag, *Siegelsche Modulfunktionen*, Grundl. Math. Wiss. **254**, Springer-Verlag, Berlin-Heidelberg-New York 1983.

[8] S. Helgason, *Differential Geometry and Symmetric Spaces*, Academic Press, New York-San Francisco-London 1962.

[9] A. Hurwitz, *Vorlesungen über die Zahlentheorie der Quaternionen*, Springer-Verlag, Berlin 1919.

[10] E. Kähler, *Die Poincaré-Gruppe*, Mathematica, Festschrift Ernst Mohr, 117–144, Berlin 1985.

[11] Y. Kitaoka, *Dirichlet series in the theory of Siegel modular forms*, Nagoya Math. J., **95** (1984), 73–84.

[12] H. Klingen, *Diskontinuierliche Gruppen in symmetrischen Räumen* I, Math. Ann., **129** (1955), 345–369.

[13] M. Koecher, *Über Dirichlet-Reihen mit Funktionalgleichung*, J. Reine Angew. Math., **192** (1953), 1–23.

[14] ———, *An elementary approach to bounded symmetric domains*, Lecture Notes, Rice 1969.

[15] ———, *Lineare Algebra und analytische Geometrie*, 2. Auflage, Grundwissen Math., **2**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1985.

[16] A. Krieg, *Modular Forms on Half-Spaces of Quaternions*, Lecture Notes in Math. **1143**, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo 1985.

[17] ———, *The elementary divisor theory over the Hurwitz order of integral quaternions*, Linear Multilinear Algebra, **21** (1987), 325–344.

[18] L. Kronecker, *Über bilineare Formen*, J. Reine Angew. Math., **68** (1868), 273–285 (= Werke I, 143–162).

[19] T. Kubota, *Elementary Theory of Eisenstein Series*, Kodansha, Tokyo 1973.

[20] H. Maaß , *Automorphe Funktionen von mehreren Veränderlichen und Dirichletsche Reihen*, Abh. Math. Sem. Univ. Hamburg, **16** (1949), 72–100.

[21] ———, *Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen*, Math. Ann., **126** (1953), 44–68.

[22] ———, *Siegel's Modular Forms and Dirichlet Series*, Lecture Notes in Math. **216**, Springer-Verlag, Berlin-Heidelberg-New York 1971.

[23] ———, *Dirichletsche Reihen und Modulformen zweiten Grades*, Acta Arith., **24** (1973), 225–238.

[24] W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for special functions of mathematical physics*, Grundl. Math. Wiss., **52**, Springer-Verlag, New York 1966.

[25] M. Newman, *Integral Matrices*, Academic Press, New York-London 1972.

[26] G. Shimura, *Confluent hypergeometric functions on tube domains*, Math. Ann., **260** (1982), 269–302.

[27] ———, *On Eisenstein series*, Duke Math. J., **50** (1983), 417–476.

[28] C. L. Siegel, *Einheiten quadratischer Formen*, Abh. Math. Sem. Hansische Univ., **13** (1940), 209–239 (= Ges. Abh.II, 138–168).

[29] ———, *Symplectic Geometry*, Amer. J. Math., **65** (1943), 1–86 (= Ges. Abh.II, 274–359).

[30] ———, *Die Modulgruppe in einer einfachen involutorischen Algebra*, Festschrift Akad. Wiss. Göttingen 1951, 157–167 (= Ges. Abh.III, 143–153).

[31] A. Terras, *On automorphic forms for the general linear group,* Rocky Mountain J. Math., **12** (1982), 123–143.
[32] ———, *Harmonic Analysis on Symmetric Spaces and Applications*, II, to appear in Springer-Verlag.
[33] E. Witt, *Eine Identität zwischen Modulformen zweiten Grades*, Abh. Math. Sem. Hansische Univ., **14** (1941), 323–337.
[34] J. Wolf, *Spaces of Constant Curvature*, 4th edition, Publish or Perish, Berkeley 1977.

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