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AN UNKNOTTING LEMMA FOR SYSTEMS OF ARCS IN $F \times I$

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A criterion for the unknottedness of a system of arcs in the cartesian product of a closed surface and the unit interval is given.

0. Introduction. Unknottedness lemmas have a long and venerable history. The canonical example is Papakyriakopoulos' criterion for an embedded circle in S^3 to be unknotted [**P**]. In the early seventies Feustel and Brown [**Fe**, **B**] developed unknottedness criteria for systems of proper arcs in the cartesian product of a closed surface F and the unit interval. In this case unknotted means that there is an isotopy of $F \times I$ so that the arc system is of the form $\{p_1, \dots, p_n\} \times I$. More recently the author gave an unknottedness criterion for a proper arc in $F \times I$ that was useful in the study of minimal surfaces in the three torus. Specifically a proper arc k in $F \times I$ is unknotted if and only if the closure of the complement of a regular neighborhood of k in $F \times I$ is a handlebody. (Since speaking of the closure of the complement of a regular neighborhood of a set is rather clumsy we will abreviate by just referring to the complement of the set. It will be clear from the context when we actually mean the closure of the complement of a regular neighborhood of the set.) Finally Gordon [**G**] proved an unknottedness lemma for systems of proper arcs in $S^2 \times I$. A system of arcs in $S^2 \times I$ is unknotted if and only if the complement of any nonempty combination of the arcs in the system is a handlebody. In this paper we generalize Gordon's result by showing that a system of arcs in $F \times I$ where F is a closed surface of positive genus is unknotted if and only if the complement of every nonempty combination of the arcs is a handlebody.

The structure of the proof is as follows. It is little trouble to show that if K is a system of vertical arcs in $F \times I$ then the complement of any nonempty combination of the arcs in K is a handlebody; hence we will only concern ourselves with proving the converse. Our proof will be by induction. Lemma 1.1 of [**F**] handles the case of one arc. Since the inductive step is rather involved we will first prove the lemma when the system consists of two arcs. We then use a result appearing in **[G]** to show that the general inductive step can be carried out in a fashion similar to the case when the system consists of two arcs. We are indebted to Prof. Gordon for suggesting the lemma and useful conversations that led to the proof. We would also like to thank Bill Menasco for helping us clarify the proof.

1. The case of two arcs. In **[F]**, the following lemma appears:

LEMMA 1.1 **[F]**. *Let F be a closed surface of positive genus. A proper arc k in $F \times I$ is unknotted if and only if the complement of k is a handlebody.*

We now show how to use Lemma 1.1 to prove the lemma when there are two arcs in the system.

LEMMA 1.2. *Let F be a closed surface of positive genus. Let k_1, k_2 be proper arcs in $F \times I$ and let $N(k_1)$ and $N(k_2)$ be small regular neighborhoods. Suppose that $H_1 = {}^-(F \times I - N(k_1))$, $H_2 = {}^-(F \times I - N(k_2))$, and $H_{12} = {}^-(F \times I - N(k_1) \cup N(k_2))$ are all handlebodies. Then there is an isotopy of $F \times I$ so that k_1 and k_2 are vertical.*

Proof. Let A be a vertical incompressible nonseparating annulus in $F \times I$. By Lemma 1.1 of **[F]** after an isotopy we may assume that $k_1 \subset A$. Let c_i be a nontrivial simple closed curve on the frontier of $N(k_i)$. Since H_{12} and H_2 are handlebodies there exists a proper disk D in H_{12} such that $\partial D \cap c_1$ consists of a single point of intersection. We may assume that among all disks of this sort D minimizes the number of points of $\partial D \cap c_2$.

We can place coordinates on $N(k_i)$ as follows. Let E be the unit disk in the complex plane. Then $N(k_i)$ is $E \times I$ where $k_i = \{0\} \times I$. Since D minimizes $\partial D \cap c_i$ we may assume that $\partial D \cap N(k_1)$ is of the form $\{p\} \times I$ where p is a point on the unit circle and $\partial D \cap N(k_2)$ is of the form $\bigcup\{p_j\} \times I$ where the p_j lie on the unit circle. We may complete D to a singular surface $\overline{M}$ by letting $\overline{M}$ be the union of D and the fins of the form $r(p) \times I$ and $r(p_j) \times I$ where r ranges from 0 to 1. (The symbol $r(p)$ means the product of r and p viewed as complex numbers.) Let $\overline{D}$ be a disk and let $f \colon \overline{D} \twoheadrightarrow \overline{M}$ be a map that is a homeomorphism of $\operatorname{int} \overline{D}$ onto $\operatorname{int} \overline{M}$, and so that the boundary of $\overline{D}$ can be partitioned into segments a_i, b_i and k_1, where k_1 is an arc mapped homeomorphically onto k_1 in $F \times I$, each arc b_i is mapped homeomorphically onto k_2 and each a_i is mapped homeomorphically onto an embedded arc in $\partial(F \times I)$ (see Figure 1). If there is but one arc of type b_i in $\partial\overline{D}$ then we can make k_1 and k_2 simultaneously vertical. Hence we will assume that there is more than one arc of type b_i. It should be noted that since c_1 is homologous to c_2 on ∂H_{12}, there are an odd number of arcs b_i. Finally since the arcs b_i are identified to one another in $F \times I$, if $f^{-1}(A)$ intersects one b_i then it intersects all of them.

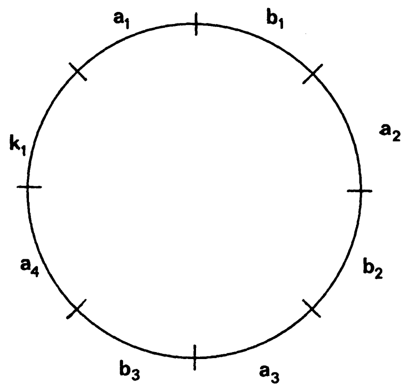

FIGURE 1

Our goal is to isotope k_2 so that it is disjoint from A. To this end make $\overline{M}$ transverse to A and so that (i) $\partial k_2 \cap A = \varnothing$, (ii) $f^{-1}(N(k_1) \cap A) = k_1$, (iii) then remove simple closed curves of intersection from $f^{-1}(A)$. We will now study the outermost arcs in $\overline{D}$ of $f^{-1}(A)$. If the set of such arcs is nonempty, there exist at least two outer most disks, and therefore we can choose one, say D', such that $D' \cap k_1 = \varnothing$. There are then five possibilities for the endpoints of the corresponding outermost arc κ.

Type 1. The endpoints of κ lie in distinct arcs a_i and a_j.

Type 2. Both endpoints of κ lie in the same a_i.

Type 3. Both endpoints of κ lie in the same b_i.

Type 4. One endpoint in b_i and one endpoint in a_j.

Type 5. The endpoints of κ lie in distinct intervals b_i and b_j.

We will now show how to pull k_2 off of A by analyzing outermost arcs in $f^{-1}(A)$. We define the complexity of the disk $\overline{D}$ to be the ordered pair consisting of the number of points in $k_2 \cap A$, followed by the number of points in $f^{-1}(A) \cap \partial\overline{D}$ and order them lexicographically. The following analysis will show that we can isotope $\overline{M}$ so that the disk $\overline{D}$ has complexity $(0, n)$. This implies that we can pull k_2 off of A.

Case 1. All outermost arcs are of type 1. In this case all the arcs in $f^{-1}(A)$ are of type 1. To see this note that since there are at least 2 outermost arcs, one of the associated outermost disks D' must miss k_1. Since a_i and a_j are distinct intervals we have that D' must intersect some b_i. This means that there are no arcs having their endpoints in any b_i. If an arc has both its endpoints in the same a_i, then there must be an outermost arc of this type. Thus there are no arcs of type 2. This shows that all arcs in $f^{-1}(A)$ are of type 1. Hence $k_2 \cap A = \varnothing$ and the complexity of the disk is of the form $(0, n)$; hence we are done.

Case 2. There is an outermost arc κ of type 2. The arc κ cuts a disk E' out of A. The union of E' and D' is a disk whose boundary lies in $\partial F \times I$. Since $F \times I$ is boundary irreducible we have that $E' \cup D'$ is the frontier of a ball embedded in $F \times I$. Use this ball to isotope A so as to replace E' with a pushoff of D', leaving that part of A that is away from E' fixed. This reduces the number of points in $f^{-1}(A) \cap \partial\overline{D}$ without increasing the number of points in $k_2 \cap A$; thus the complexity has decreased. (See Figure 2.)

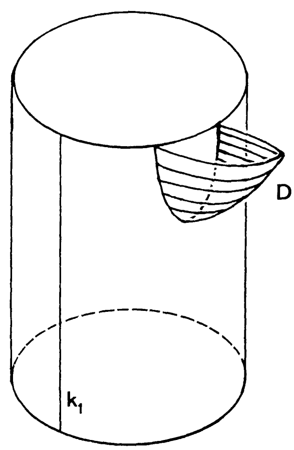

FIGURE 2

Case 3. There is an outermost arc κ of type 3. Use the outermost disk D' to isotope away two points of intersection of k_2 with A. Then remove trivial simple closed curves intersection between $\overline{M}$ and A. (See Figure 3.) This reduces the number of points of $f^{-1}(A) \cap \partial\overline{D}$, and the number of points of $k_2 \cap A$.

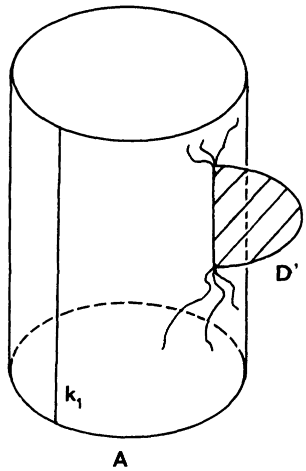

FIGURE 3

Case 4. There is an outermost arc κ of type 4. The hypotheses guarantee that the arcs a_i and b_j containing the endpoints of κ are adjacent to one another on $\partial\overline{D}$. Use the outermost disk D' to isotope away the point of intersection of k_2 and A corresponding to the endpoint of κ lying in b_j. (See Figure 4.) Then remove the trivial simple closed curves of intersection. This decreases the number of points of $k_2 \cap A$.

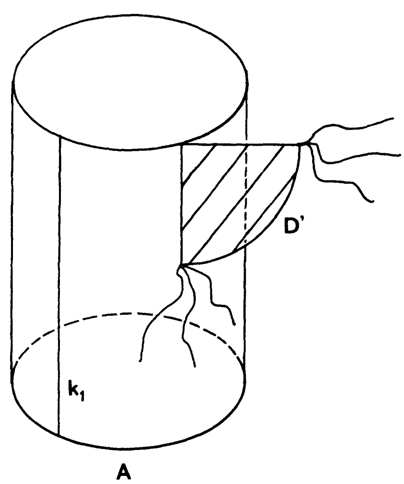

FIGURE 4

Case 5. There is an outermost arc κ of type 5. Because κ is outermost the arcs b_i and b_j must be adjacent to some a_r on $\partial\overline{D}$. The outermost disk D' cut out by κ has boundary consisting of four arcs, an arc d that is mapped into A, two arcs e_1 and e_2 that are mapped to subarcs of k_2, and an arc f that is mapped into $\partial(F \times I)$. Since the two arcs e_1 and e_2 are identified we have that d and f are mapped to simple closed curves. Since k_1 is a nontrivial arc on A and the image of d misses k_1 we have that d is mapped to a trivial simple closed curve on A. Let D'' be the disk bounded by the image of d on A. The image of D' along with D'' form a disk whose boundary lies in $\partial(F \times I)$. (See Figure 5.) Since the boundary of $F \times I$ is incompressible the image of the curve f bounds a disk on $\partial(F \times I)$. This allows us to isotope away two points of intersection between D and c_2, thus contradicting our choice of D.

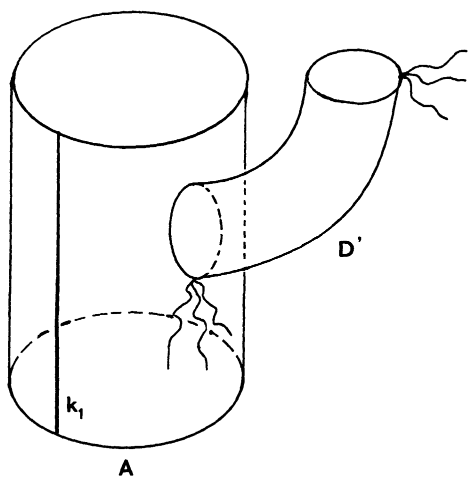

FIGURE 5

Now that $k_2 \cap A = \varnothing$, choose a family of vertical disks each having its boundary in $A \cup \partial(F \times I)$ so that the disks miss $A \cap \overline{M}$, and they miss the boundary of k_2, and along with A they form a hierarchy for $F \times I$. Using moves as above we can isotope k_2 so that it misses A and the vertical disks. Cutting along the hierarchy we get $D^2 \times I$ so that when you identify some disks in $\partial D^2 \times I$ you recover $F \times I$. Since k_2 is disjoint from the hierarchy we can view it as a proper arc in $D^2 \times I$. The hierarchy can be viewed as lying in H_2. If we cut along the surfaces in the hierarchy at each stage we will still have a handlebody. Consequently $D^2 \times I - N(k_2)$ is a solid torus. Since H_2 is a handlebody we can see that k_2 runs from $D^2 \times \{0\}$ to $D^2 \times \{1\}$. From this we conclude that k_2 is isotopic to a vertical arc in $D^2 \times I$, and hence in $F \times I$. $\square$

2. The general case. We would now like to give some definitions from [G] and state the major result of that paper. A family $\{C_1, \dots, C_m\}$ of disjoint simple loops in the boundary of a handlebody X is called primitive if there exist disjoint disks $\{D_1, \dots, D_m\}$ properly embedded in X so that all intersections between the D_i and C_j are transverse in ∂X, and the number of points in $D_i \cap C_j$ is equal to Kronecker's delta $\delta(i, j)$.

THEOREM 1 [G]. *Let $\{C_1, \dots, C_m\}$ be a system of disjoint simple closed curves on the boundary of a handlebody, such that the result of adding two handles along any subcollection of the C_i is still a handlebody. Then $\{C_1, \dots, C_m\}$ is primitive.*

We will now use this to prove our unknotting lemma.

LEMMA 2. *Let $\{k_1, \dots, k_n\}$ be a system of proper arcs in $F \times I$. If the closure of the complement of every nonempty combination of k_i is a handlebody then there is an isotopy of $F \times I$ so that the arcs k_i are vertical.*

Proof. The proof is by induction. Assume that we have shown the lemma for all collections of arcs having fewer than n elements. Let $k_1, \dots, k_n$ be a collection of n arcs satisfying the hypotheses and having n elements. Let A be an incompressible vertical annulus. By the inductive hypothesis we may assume that the curves $k_1, \dots, k_{n-1}$ have been isotoped so that they are vertical arcs lying on A. Choose small regular neighborhoods $N(k_i)$ of the arcs k_i. Let c_i be a nontrivial simple closed curve on the frontier of $N(K_i)$ for each i. By Gordon's result the system of curves $c_1, \dots, c_{n-1}$ is a primitive system of curves on the boundary of the handlebody $H_n = {}^-(F \times I - N(k_1) - \cdots - N(k_n))$. Hence there is a system of properly embedded disks $D_1, \dots, D_{n-1}$ in H_n so that ∂D_i and c_j intersect in $\delta(i, j)$ points of transverse intersection. In specific the curve ∂D_1 intersects c_1 in a single point, and misses c_2 through c_{n-1}. Homological considerations allow us to conclude that ∂D_1 has odd intersection number with c_n. Let D be the disk having the least number of points of intersection with c_n having these properties. Now proceed with D as we did in Lemma 1, first pulling k_n off of A and then a vertical hierarchy for the complement of A. The proof is now finished by the standard argument in the solid torus.

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