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A NOTE ON MEISTERS AND OLECH'S PROOF OF THE GLOBAL ASYMPTOTIC STABILITY JACOBIAN CONJECTURE

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Let $f: \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-vector field with $f(0) = 0$. For $p \in \mathbb{R}^n$ let $Jf(p)$ denote its Jacobian matrix evaluated at p. Then it is a well-known result, due to Lyapunov, that the origin is a locally asymptotic rest point of the non-linear autonomous system of ordinary differential equations $\dot{x} = f(x)$ if the origin is a locally asymptotic rest point of the linearized system $\dot{y} = Jf(0)y$ (or equivalently if all eigenvalues of the matrix $Jf(0)$ have negative real parts).

In 1960 it was conjectured by Markus and Yamabe that the origin is a globally asymptotic rest point $\dot{x} = f(x)$ if for each $p \in \mathbb{R}^n$ the orgin is a locally asymptotic rest point of the linearized system $\dot{y} = Jf(p)y$. Until now this conjecture is still open. However in 1988 Meisters and Olech proved this conjecture for two-dimensional polynomial vector fields $f: \mathbb{R}^2 \to \mathbb{R}^2$. The proof is an immediate consequence of earlier results of Olech, (1963) and the proposition below. The main result of this paper (Theorem 1) generalizes the proposition to polynomial maps $F: k^n \to k^n$ having the property that $\det JF(x) \neq 0$ for all $x \in k^n$ (k is a field of characteristic zero).

PROPOSITION. *If $F: \mathbb{R}^2 \to \mathbb{R}^2$ is a polynomial map such that $\det JF(x) \neq 0$ for all $x \in \mathbb{R}^2$, then there exists a positive integer N such that the number of elements in each fiber $F^{-1}(x)$ ($x \in \mathbb{R}^2$) is bounded by N.*

The proof of this proposition given by Meisters and Olech uses topological methods. In this note we generalize this result to polynomial maps $F: k^n \to k^n$ with the property that $\det JF(x) \neq 0$ for all $x \in k^n$ (k is a field of characteristic zero). Our proof is purely algebraic and uses some well-known techniques from the theory of $\mathscr{D}$-modules. For the reader's convenience we have included a section reviewing some results concerning $\mathscr{D}$-modules.

1. The Main Theorem. Throughout this paper we have the following notations: k is a field of characteristic zero and $F: k^n \to k^n$ is a polynomial map ($n \geq 1$) i.e. F is given by coordinate functions F_i

which are elements of the polynomial ring $k[X] := k[X_1, \dots, X_n]$. The determinant of the Jacobian matrix $JF := (\partial F_i / \partial X_j)$ we denote by Δ. So $\Delta \in k[X]$. For $a \in k[X]$, $\deg a$ denotes the (total) degree of a. Finally $\deg F := \max \deg F_i$. Now we can formulate the main result of this note:

THEOREM 1. *If $F\colon k^n \to k^n$ is a polynomial map with the property that* $\det JF(x) \neq 0$ *for all $x \in k^n$, then there exists a positive integer N such that for each $x \in k^n$ the number of elements in the fiber $F^{-1}(x)$ is bounded by N.*

The proof of this theorem uses some well-known techniques from the theory of $\mathscr{D}$-modules (due to I. N. Bernstein, [**1**]). A review of some of the results concerning A_n-modules is given in §2.

1.1. *The A_n-module structure on $k[X][\Delta^{-1}]$.* From now on $F = (F_1, \dots, F_n)$ is a polynomial map from k^n to k^n such that $\Delta(x) \neq 0$ for all $x \in k^n$. In particular we have $\Delta \neq 0$ so the elements $F_1, \dots, F_n$ are algebraically independent over k by [**6**], satz 61. So $k[F] := k[F_1, \dots, F_n]$ is a subring of $k[X]$ isomorphic to $k[X]$. First we define derivations on the localization $k[X][\Delta^{-1}]$, denoted by $\partial/\partial F_i$, which satisfy

$$(1.2) \qquad \frac{\partial}{\partial F_i}(F_j) = \delta_{ij}, \qquad \text{all } 1 \le i, j \le n.$$

Therefore set $\partial/\partial F_i = \sum_k a_{ik}(\partial/\partial X_k)$, and we try to find elements $a_{ik} \in k[X][\Delta^{-1}]$ such that (1.2) is satisfied. In matrix notation (1.2) is equivalent to

$$(1.3) \qquad (a_{ik})(JF)^T = I_n.$$

Since $\det(JF)^T = \det JF = \Delta \neq 0$ we can solve the a_{ik} uniquely in $k[X][\Delta^{-1}]$. In fact by Cramer's rule we find

$$(1.4) \quad \Delta a_{ik} \in k[X] \quad \text{and} \quad \deg \Delta a_{ik} \le (n-1)\deg F, \quad \text{all } i, k.$$

Now we claim that the k-derivations $\partial/\partial F_i$ commute pairwise on $k[X][\Delta^{-1}]$. Therefore let $\tau := [\partial/\partial F_i, \partial/\partial F_j]$ be the commutator of $\partial/\partial F_i$ and $\partial/\partial F_j$. Then τ is a k-derivation on $k[X][\Delta^{-1}]$ and it is zero on $k[F]$ (since $\tau(F_p) = 0$ for all p). Consequently, the unique extension of τ to the completion $k[[F]]$ is also zero. However by the local inversion theorem ([7], §4, no. 5. Proposition 5) $k[[F]] = k[[X]]$ (for this last statement we assumed that $F(0) = 0$, which is a harmless assumption since $\partial/\partial F_i = \partial/\partial(F_i + \lambda)$ for all $\lambda \in k$). So τ is zero on $k[[X]]$ and hence on the subring $k[X][\Delta^{-1}]$ $(\Delta(0) \neq 0$, so $\Delta^{-1} \in k[[X]])$, which proves the claim.

The results above enable us to endow $k[X][\Delta^{-1}]$ with a left $A_n = k[Y_1, \dots, Y_n, \partial_1, \dots, \partial_n]$-module structure, as follows: Define

$$Y_i \cdot g := F_i g, \quad \partial_i \cdot g = \frac{\partial g}{\partial F_i} \quad \text{for all } 1 \le i \le n, \quad \text{all } g \in k[X][\Delta^{-1}].$$

The left A_n-module associated to F in this way we denote by $M(F)$.

LEMMA 1.5. *$M(F)$ possesses an $(n, e(F))$-filtration, where $e(F) = 2^n(2n \deg F + 1)^n$.*

Proof. Put $d := \deg F$. For each $v \in \mathbb{Z}$, $v \ge 0$ we define

$$\Gamma_v := \{q\Delta^{-2v} \in k[X][\Delta^{-1}] \mid \deg q \le 2v(2nd+1)\}.$$

By definition $\dim_k \Gamma_v$ is the dimension of the k-vector space of all polynomials in $k[X]$ of degree $\le 2v(2nd+1)$, which implies

$$\dim_k \Gamma_v \le \frac{2^n(2nd+1)^n}{n!} v^n + \mathscr{O}(v^{n-1}).$$

So it suffices to prove that $\{\Gamma_n\}$ is a filtration on $M(F)$. We first show that $\partial_i \Gamma_v \subset \Gamma_{v+1}$ (the inclusion $x_i \Gamma_v \subset \Gamma_{v+1}$ is proved in a similar way). So let $g = q\Delta^{-2v} \in \Gamma_v$. Then

$$\partial_i g = \frac{\partial q}{\partial F_i} \Delta^{-2v} + q(-2v)\Delta^{-2v-1} \frac{\partial \Delta}{\partial F_i}.$$

By (1.4) we know

$$\frac{\partial}{\partial F_i} = \frac{1}{\Delta} \sum_k \Delta a_{ik} \frac{\partial}{\partial X_k} \quad \text{and}$$

$$\Delta a_{ik} \in k[X] \quad \text{with } \deg \Delta a_{ik} \le (n-1)d.$$

So

$$\partial_i g = \left(\Delta \sum_k \Delta a_{ik} \frac{\partial q}{\partial X_k} + (-2v) q \sum_k \Delta a_{ik} \frac{\partial \Delta}{\partial X_k} \right) \Delta^{-2(v+1)}.$$

Using $\deg \Delta \le nd$ and $\deg \Delta a_{ik} \le (n-1)d$ we conclude that $\partial_i g \in \Gamma_{v+1}$. Finally we show that $\bigcup \Gamma_v = M(F)$. So let $q\Delta^{-r} \in k[X][\Delta^{-1}]$ with $\deg q = s$ and $r \ge 0$. Let $v \ge \max(r, s)$. Then

$$q\Delta^{-r} = q(\Delta^{2v-r})\Delta^{-2v} \quad \text{and}$$

$$\deg q\Delta^{2v-r} \le s + (2v-r)nd \le s + 2vnd \le 2v(2nd+1)$$

since $v \ge s$. So $q\Delta^{-r} \in \Gamma_v$, which completes the proof. $\square$

Proof of Theorem 1. (i) Let $x \in k^n$. Then the number of elements in the fiber $F^{-1}(x)$ is equal to the number of zeros of the ideal $(F_1 - x_1, \ldots, F_n - x_n)$. Therefore we consider the polynomial map $F - x$ and form its left A_n-module $M(x) := M(F - x)$. (Observe that $\det J(F - x) = \det JF = \Delta$ has no zeros in k^n.) By Lemma 1.5 $M(x)$ possesses an $(n, e(x))$-filtration, where

$$e(x) = 2^n(2n \deg(F - x) + 1)^n = 2^n(2n \deg F + 1)^n.$$

So by Corollary 2.4 $M(x)/\sum_i(F_i - x_i)M(x)$ is a finite dimensional k-vector space with dimension bounded by $N_0 := 2^n(2n \deg F + 1)^n$, which is independent of x! So

$$\dim_k k[X][\Delta^{-1}]\Big/\sum_i(F_i - x_i)k[X][\Delta^{-1}] \leq N_0 \quad \text{for all } x \in k^n.$$

Consequently the residue classes of $1, X_1, X_1^2, \ldots, X_1^{N_0}$ must be linearly dependent over k. So there exists a non-zero polynomial $g(X_1) \in k[X_1]$ of degree $\leq N_0$ and a positive integer ρ such that $\Delta^\rho g(X_1) \in \sum k[X](F_i - x_i)$.

(ii) Now let $p = (p_1, \ldots, p_n) \in k^n$ such that $F(p) = x$; i.e. $F_i(p) = x_i$ for all i. Then $\Delta(p)^\rho g(p_1) = 0$. Since Δ has no zeros on k^n it follows that $g(p_1) = 0$. So there are at most N_0 possibilities for the first coordinate of p (since $\deg g \leq N_0$). Arguing in a similar way for the other coordinates of p we conclude that the number of $p \in k^n$ with $F(p) = x$ is bounded by $N := N_0^n$. □

Comment. It was kindly pointed out to me by Professor J. Bochnak that for some special fields k such as $\mathbb{R}$, $\mathbb{C}$, real closed or algebraically closed fields, Theorem 1 is a consequence of the following result.

THEOREM 1.6. *Let $F: \mathbb{R}^n \to \mathbb{R}^p$ be a polynomial map of degree d such that $F^{-1}(x)$ is finite for each $x \in \mathbb{R}^p$. Then the number of elements in each fiber $F^{-1}(x)$ is bounded by $d(2d - 1)^{n-1}$.*

This theorem is a very special case of Theorem 11.5.2 (p. 243) of **[8]**. To see that Theorem 1.6 implies Theorem 1 one only needs to observe that the condition $\det JF(x) \neq 0$ for all $x \in \mathbb{R}^n$ implies that each fiber $F^{-1}(x)$ is discrete (by the implicit function theorem) and that obviously $F^{-1}(x)$ is an algebraic subset of $\mathbb{R}^n$ and hence has a finite number of connected components. So $F^{-1}(x)$ is finite.

2. A review of some results concerning A_n-modules. All results of this section come from I. N. Bernstein's work in **[1]** and can also be found in Chapter I of **[2]**.

Let $A_n := k[Y_1, \dots, Y_n, \partial_1, \dots, \partial_n]$ be the nth *Weyl-algebra*, i.e. the k-algebra with relations $[Y_i, Y_j] = [\partial_i, \partial_j] = 0$ and $[\partial_i, Y_j] = \delta_{ij}$ for all $1 \le i, j \le n$. It is a filtered ring with filtration $\{T_v\}_{v=0}^{\infty}$ where T_v is the k-vector space generated by the monomials $Y^\alpha \partial^\beta$ with $|\alpha| + |\beta| \le v$ (with the usual multi-index notation). Let M be a left A_n-module. A *filtration* Γ on M is an increasing sequence $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots$ of finite dimensional k-subspaces of M such that $\bigcup T_v = M$ and $T_k \Gamma_v \subset \Gamma_{v+k}$ for all $k, v \ge 0$. Such a filtration is called *good* if there exist $m_1, \dots, m_s \in M$ and $n_1, \dots, n_s \in \mathbb{Z}$ such that $\Gamma_v = \sum T_{v-n_i} m_i$ for all $v \ge 0$ (by definition $T_{-v} = 0$ for all $v \ge 1$). One readily verifies that an A_n-module possesses a good filtration if and only if it is finitely generated. Furthermore we have

PROPOSITION 2.1 ([2], *Corollary* 3.3, *Chapter* I). *If Γ is a good filtration on a finitely generated left A_n-module M, then there exist an integer $d \ge 0$ and rational numbers $a_0, \dots, a_d$ such that*

$$\dim_k \Gamma_v = a_d v^d + a_{d-1} v^{d-1} + \cdots + a_0, \quad \textit{for all large } v.$$

Furthermore $d!a_d$ is an integer ≥ 1.

The crucial point is that the integers d and $d!a_d$ are independent of the choice of the good filtration; they form two important invariants of the A_n-module M, called the *dimension* and the *multiplicity* of M, denoted $d(M)$, resp. $e(M)$. The fundamental Bernstein inequality asserts that $d(M) \ge n$ for every non-zero A_n-module M of finite type! The non-zero A_n-modules of finite type having the minimal dimension n are called *holonomic* A_n-modules. They play a very important role in the theory of $\mathscr{D}$-modules. A useful fact is that a holonomic A_n-module with multiplicity $e(M)$ has a finite length, bounded by $e(M)$.

To decide if a given A_n-module is holonomic, there exists a very powerful criterion. Before we describe it we introduce some terminology. Let M be a left A_n-module, not necessary of finite type. A filtration Γ on M is called a (d, e)-*filtration* if $\dim_k \Gamma_n \le \frac{e}{d!} v^d + \mathscr{O}(v^{d-1})$ where $d \ge 0$ and $e \ge 1$ are integers. Observe that if M is holonomic it possesses an (n, e)-filtration (namely take any good filtration on M and apply Proposition 2.1). However the converse also holds i.e.

THEOREM 2.2 ([2], *Theorem* 5.4, *Chapter* I). *Let M be an arbitrary A_n-module* (*so we don't assume M to be of finite type*). *If M possesses* *an* (n, e)*-filtration for some integer* $e \geq 1$*, then* M *is holonomic (and hence of finite type). Furthermore* $e(M) \leq e$.

Now consider the multiplication $Y_n: M \to M$. Then

$$\operatorname{coker} Y_n := M/Y_nM$$

can be given the structure of a left $A_{n-1} = k[Y_1, \ldots, Y_{n-1}\partial_1, \ldots, \partial_{n-1}]$-module by putting $\partial_i(m + Y_nM) := \partial_i m + Y_nM$. If $n = 1$ we put $A_0 := k$.

THEOREM 2.3 ([2], *Theorem* 6.2, *Chapter* I). *Let* M *be an* A_n*-module with an* (n, e)*-filtration. Then* M/Y_nM *is an* A_{n-1}*-module with an* $(n-1, e)$*-filtration. If* $n = 1$ *it means that* M/Y_nM *is a* k*-vector space of dimension* $\leq e$.

By applying this result n-times we arrive at

COROLLARY 2.4. *Let* M *be an* A_n*-module with an* (n, e)*-filtration. Then* $M/\sum_i Y_iM$ *is a finite dimensional* k*-vector space with dimension bounded by* e.

REFERENCES

[1] I. N. Bernstein, *The analytic continuation of generalized functions with respect to a parameter*, Funktsional. Anal. i Prilozhen., **6** (1972), 26–40. English transl. in Functional Anal. Appl., **6** (1972), 273–285.
[2] J.-E. Björk, *Rings of Differential Operators*, Vol. 21, North-Holland Mathematical Library (1979).
[3] L. Markus and H. Yamabe, *Global stability criteria for differential systems*, Osaka Math. J., **12** (2) (1960), 305–317.
[4] G. Meisters and C. Olech, *Solution of the Global Asymptotic Stability Jacobian Conjecture for the Polynomial Case*, Analyse Mathématique et Applications, Gauthier-Villars, Paris (1988), 373–381.
[5] C. Olech, *On the global stability of an autonomous system on the plane*, Contributions to Differential Equations, **1** (1963), 389–400.
[6] O. Perron, *Algebra* I, Die Grundlagen, Walter de Gruyter & Co., Berlin 1951.
[7] N. Bourbaki, *Algèbre Commutative*, Chapter 3, Hermann Paris.
[8] J. Bochnak, M. Coste and M. -F. Roy, *Géometrie algébrique réelle*, Ergebnisse der Math. u.i. Grenzgebieten, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

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