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APPROXIMATELY INNER AUTOMORPHISMS ON INCLUSIONS OF TYPE III_λ-FACTORS

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For arbitrary inclusions of factors with finite index, we define a "fundamental homomorphism" which is a generalization of both the Connes-Takesaki fundamental homomorphism for properly infinite (single) factors and Loi's construction for inclusions of type II_1-factors.

It is shown that for nice inclusions of type III_λ-factors ($0 < \lambda < 1$), the kernel of the fundamental homomorphism coincides with the set of approximately inner automorphisms on the inclusion. To prove this, we first give a characterization of approximate innerness on type III_λ-inclusions in terms of Loi's and Connes-Takesaki's invariants.

1. Introduction. The importance of studying automorphisms on von Neumann algebras was highlighted through Connes' classification theory for type III-factors. Recently it has been suggested to generalize Connes' automorphism approach to subfactor theory (see e.g. [**Ka1**],[**L2**]).

In Connes' theory, an important class of automorphisms on a von Neumann algebra M is $\overline{\mathrm{Int}}(M)$, the closure — in u-topology, as usual — of $\mathrm{Int}(M)$ in $\mathrm{Aut}(M)$; members of this set are called *approximately inner*. Assume M is a hyperfinite factor. If M is of type I or II_1, then $\overline{\mathrm{Int}}(M) = \mathrm{Aut}(M)$, but if M is of type II_∞ or III, one has $\overline{\mathrm{Int}}(M) = \mathrm{Ker}(\mathrm{mod})$, where mod is the fundamental homomorphism of Connes and Takesaki (see [**CT**, IV.1]). For type III-factors, this was announced by Connes in 1975, and the first published proof was given recently in [**KST**]. The result had prominent applications long before a proof appeared, cf. [**KST**, §0]. As another recent development along these lines, we mention [**HS**], which will be crucial here.

In the case of an inclusion $M \supseteq N$ of factors, one consider the groups

$$\begin{aligned} \mathrm{Aut}(M,N) &= \{\alpha \in \mathrm{Aut}(M) \mid \alpha(N) = N\} \\ \mathrm{Int}(M,N) &= \{\mathrm{Ad}(u) \in \mathrm{Aut}(M) \mid u \in \mathcal{U}(N)\} \end{aligned}$$

and here, the closure $\overline{\mathrm{Int}}(M,N)$ of $\mathrm{Int}(M,N)$ contains what is called the *approximately inner* automorphisms of $M \supseteq N$. A characterization of these in the case where $M \supseteq N$ is a finite index inclusion of type II_1-factors with the generating property was given for the irreducible case by Loi [**L2**] and generalized by Kawahigashi [**Ka1**], cf. also [**L3**, §2] and §2 here.

In this paper, we treat primarily the situation where M and N are of type III_λ for some $\lambda \in]0,1[$. Our first result is essentially a combination of Connes' and Takesaki's approach to the single type III-factor case and Loi's method for the type II_1-inclusion case. Then we define a "fundamental homomorphism" for factor inclusions, which generalizes the constructions of both Connes-Takesaki and Loi, and use it to give another characterization of approximate innerness in the III_λ-case.

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2. Preliminaries on semifinite inclusions. Let $P \supseteq Q$ be an inclusion of type II_∞-factors with separable preduals and a common normal semifinite faithful trace τ. Fix a finite projection $e \in Q$. Then with $A = ePe, B = eQe$, we call $A \supseteq B$ the *associated* II_1-*inclusion* of $P \supseteq Q$. Taking a system $(e_{ij})_{i,j=1}^{\infty}$ of matrix-units in Q with $e_{11} = e$, we get a type I_∞-subfactor $F = \mathrm{span}(e_{ij})$ of Q such that $P \supseteq Q$ is isomorphic to $A \otimes F \supseteq B \otimes F$.

LEMMA 2.1. *If $\alpha \in \mathrm{Aut}(P,Q)$ satisfies $\tau \circ \alpha = \tau$, then there exists a unitary $u \in \mathcal{U}(Q)$ such that with $\beta = \mathrm{Ad}(u) \circ \alpha$, we have*

$\beta = \beta|_A \otimes 1$ *and* $\beta|_Q = \beta|_B \otimes 1$ *with respect to the above splitting of* $P \supseteq Q$.

Proof. Apply [**C2**, 3.11] to Q. □

$P \supseteq Q$ is called *strongly stable* if $(P \otimes R \supseteq Q \otimes R) \cong (P \supseteq Q)$ where R is the hyperfinite II_1-factor. Note that, by [**B**], $P \supseteq Q$ is strongly stable if $A \supseteq B$ has the generating property as defined in [**P1**, 4.1].

Now let ω be a free ultrafilter on $\mathbb{N}$ and recall from [**C1**] the standard notations for asymptotic centralizers and their automorphisms. As in [**L2**], $C_\omega(P, Q)$ will denote the set of ω-centralizing sequences for P with elements belonging to Q. We have the following analogue of [**L2**, 4.4] and [**Ka1**, 3.1].

LEMMA 2.2. *Assume* $P \supseteq Q$ *is strongly stable and of finite index. Let* $\theta \in \mathrm{Aut}(P, Q)$ *satisfy* $\tau \circ \theta = \lambda\tau$ *for some* $\lambda \neq 1$. *Then* $\theta_\omega|_{C_\omega(P,Q)}$ *is aperiodic.*

Proof. Since every power of θ satisfy the assumptions for θ, it suffices to find a sequence $(x_n) \in \ell^\infty(\mathbb{N}, Q)$ which is centralizing in P and satisfies $\theta(x_n) - x_n \not\longrightarrow 0$ $(\sigma s^*,\ n \to \omega)$. By [**L2**, 4.5] we may assume that

$$(P \supseteq Q, \theta) = (A \otimes R_{0,1} \supseteq B \otimes R_{0,1}, \theta_0 \otimes \theta_\lambda)$$

where $\theta_0 \in \mathrm{Aut}(A, B)$, $R_{0,1}$ is the hyperfinite II_∞-factor and θ_λ is the (up to conjugacy unique) automorphism of $R_{0,1}$ with $\mathrm{mod}(\theta_\lambda) = \lambda^{-1}$. Since $\theta_\lambda \notin \mathrm{Int}(R_{0,1}) = \mathrm{Ct}(R_{0,1})$ – cf. [**C1**, lemma 5] – we have a sequence $(y_n) \in (R_{0,1})_\omega$ such that

$$\theta_\lambda(y_n) - y_n \not\longrightarrow 0 \ (\sigma s^*,\ n \to \omega).$$

Thus putting $x_n = 1 \otimes y_n \in Q$ $(n \in \mathbb{N})$ produces the desired sequence. □

LEMMA 2.3. *If* P, Q, θ *are as in* (2.2), *then* $\theta_\omega|_{C_\omega(P,Q)}$ *is stable, i.e. for any* $u \in \mathcal{U}(C_\omega(P, Q))$ *there exist* $v \in \mathcal{U}(C_\omega(P, Q))$ *such that* $\theta_\omega(v) = uv$.

Proof. This follows from (2.2) and [**L2**, 4.2 2)]. □

We close this section with a brief review of Loi's characterization of approximately inner automorphisms on II_1-inclusions, including the improvements made by Kawahigashi, and with some notational modifications and trivial extensions. Let now $A \supseteq B$ be any inclusion of II_1-factors with finite index, and let $B \subseteq A \subseteq A_1 \subseteq A_2 \subseteq \cdots$ be the tower for $A \supseteq B$, with Jones projections $e_k \in A_k$. Assume the action of A to be standard with respect to its tracial state, and let J_A be a modular conjugation. We then get a tunnel $(B_k)_{k\geq 0}$ for $A \supseteq B$ by defining $B_k = J_A A'_{k+1} J_A$, with Jones projections $e_{-k} = J_A e_{k+2} J_A \in B_{k-1}$, $k \geq 0$. By an inner perturbation argument, we obtain a homomorphism $\Phi : \mathrm{Aut}(A,B) \to \mathcal{G}$, where $\mathcal{G}$ is the topological group of sequences $(\alpha^{(k)})_{k=0}^{\infty}$ of automorphisms $\alpha^{(k)} \in \mathrm{Aut}(B'_k \cap A)$ such that

- $\alpha^{(k)} \in \mathrm{Aut}(B'_j \cap A, B'_j \cap B)$, $j = 0, 1, ..., k$
- $\alpha^{(k)}|_{B'_{k-1} \cap A} = \alpha^{(k-1)}$
- $\alpha^{(k)}(e_{-j}) = e_{-j}$, $j = 0, 1, ..., k-1$

The multiplication and topology on $\mathcal{G}$ are defined "pointwise". From [**L2**] and [**Ka1**], we then have:

THEOREM 2.4. *With the above notation, Φ is continuous. If $A \supseteq B$ has the generating property, then* $\mathrm{Ker}(\Phi) = \overline{\mathrm{Int}}(A,B)$.

Φ can also be defined using the tower: each $\alpha \in \mathrm{Aut}(A,B)$ extends in a unique manner to $\alpha_k \in \mathrm{Aut}(A_k)$ satisfying $\alpha_k(e_j) = e_j$, $j = 1, ..., k$, cf. [**Ka1**, 1.5], [**L2**, 3.1]. If $\Phi(\alpha) = (\alpha^{(k)})_{k=0}^{\infty}$, we then have

$$\alpha^{(k)}(x) = J_A \alpha_{k+1}(J_A x J_A) J_A, \; x \in B'_k \cap A, \; k \geq 0.$$

Therefore, (2.4) implies

COROLLARY 2.5. *When $A \supseteq B$ has the generating property and$\alpha \in \mathrm{Aut}(A,B)$ satisfies $\alpha_k|_{A_k \cap B'} = 1, k \in \mathbb{N}$, then $\alpha \in \overline{\mathrm{Int}}(A,B)$.*

For a II_∞-inclusion $P \supseteq Q$ of finite index, we can still construct the continuous homomorphism $\Phi : \mathrm{Aut}(P,Q) \to \mathcal{G}$ as above (see [**L3**, §2] and [**Ka1**, 3.6]) and we still get $\overline{\mathrm{Int}}(P,Q) \subseteq \mathrm{Ker}(\Phi)$. The opposite inclusion follows in the II_1-case by the generating property, which is impossible in the II_∞-case; instead, we note the following version of (2.4):

COROLLARY 2.6. *If $P \supseteq Q$, $A \supseteq B$ and F are as in the beginning of this section and $A \supseteq B$ has finite index and the generating property, then $\Phi : \mathrm{Aut}(P,Q) \to \mathcal{G}$ as defined above satisfies* $\mathrm{Ker}(\Phi|_{\mathrm{Aut}(A,B)}) = \overline{\mathrm{Int}}(A,B)$.

A proper characterization of approximate innerness in the type II_∞-case is obtained in §4.

3. The discrete decomposition method. Fix $\lambda \in]0,1[$ and let $M \supseteq N$ be an inclusion of type III_λ-factors with separable preduals. We assume $M \supseteq N$ to be of finite index (cf. [**Ko**]) and denote by E the minimal conditional expectation of M onto N (cf. [**H**]).

Let ϕ be a λ-trace on N. We then assume that $\psi = \phi \circ E$ is a λ-trace on M; this is the case e.g. when $M \cap N'$ is a factor. As shown in [**L1**, §2.6], our assumption means that M and N have a *common discrete decomposition*; to fix notation, we repeat the details here. With $P = M_\psi$, $Q = N_\phi$, we get an inclusion $P \supseteq Q$ of II_∞-factors with common trace ψ. As in §2, we obtain the associated II_1-inclusion $A \supseteq B$ and a type I_∞-factor F such that

$$(P \supseteq Q) \cong (A \otimes F \supseteq B \otimes F).$$

It is also clear that if $u \in \mathcal{U}(N)$ satisfies $\phi \circ \mathrm{Ad}(u) = \lambda\phi$ then $\psi \circ \mathrm{Ad}(u) = \lambda\psi$. Hence if we define $\theta = \mathrm{Ad}(u) \in \mathrm{Aut}(P,Q)$, we have

$$(M \supseteq N) \cong (P \rtimes_\theta \mathbb{Z} \supseteq Q \rtimes_\theta \mathbb{Z})$$

which means that M and N have a common discrete decomposition.

We finally assume that $M \supseteq N$ is strongly amenable in the sense of Popa [**P2**]. One way to express this condition is to say that $A \supseteq B$ has the generating property. This is the case for instance when $M \supseteq N$ has finite depth, cf. [**L2**, 3.2], [**P1**, 4.9], [**O**]. Note also that the strong amenability assumption implies hyperfiniteness of all the factors introduced above.

THEOREM 3.1. *Let notations be as above. For $\alpha \in \mathrm{Aut}(M,N)$, the following statements are equivalent:*

(i) $\alpha \in \overline{\mathrm{Int}}(M,N)$

(ii) There is a unitary $u_0 \in \mathcal{U}(N)$ such that with $\beta = \mathrm{Ad}(u_0) \circ \alpha$, we have $\phi \circ \beta|_N = \phi$ and $\beta|_A \in \overline{\mathrm{Int}}(A,B)$.

(iii) There is a unitary $u_0 \in \mathcal{U}(N)$ *such that with* $\beta = \mathrm{Ad}(u_0) \circ \alpha$, *we have* $\mod (\beta|_N) = 1, \beta|_A \in \mathrm{Aut}(A, B)$ *and* $\Phi(\beta|_A) = 1$, *where* $\Phi : \mathrm{Aut}(A, B) \to \mathcal{G}$ *is as described in* §2.

The equivalence of (i) and (ii), applied to the single factor case $M = N$, yields the following result (also noted in [**HS**, 13.6(iv)]):

COROLLARY 3.2. *With notations as above,* $\alpha \in \mathrm{Aut}(N)$ *is approximately inner if and only if there is a unitary* $u_0 \in \mathcal{U}(N)$ *such that* $\phi \circ Ad(u_0) \circ \alpha = \phi$.

NOTE.The existence of such a unitary for any λ-trace is, of course, equivalent to its existence for some specific λ-trace.Using this and the characterizations of approximate innerness mentioned in §§1-2, the equivalence of (ii) and (iii) follows immediately. Thus we only need to prove (i) $\Leftrightarrow$ (ii).

Proof that (i) $\Rightarrow$ *(ii).* Let $\alpha \in \overline{\mathrm{Int}}(M, N)$. Since $\alpha|_N \in \overline{\mathrm{Int}}(N)$, we have $\phi \circ \alpha^{-1}|_N = \lambda^n \phi \circ \mathrm{Ad}(w)$ for some $n \in \mathbf{N}$ and some $w \in \mathcal{U}(N)$, according to [**CT**, IV.1:3;9]. So with $u_0 = u^n w$ one has $\phi \circ \mathrm{Ad}(u_0) = \phi \circ \alpha^{-1}|_N$, i.e. with the notations of [**CT**, IV.1.7] we have $\phi \circ \alpha^{-1}|_N \in W_\phi$. Also from [**CT**, IV.1.7] we get a Borel map $u : W_\phi \to \mathcal{U}(N)$ satisfying $\phi \circ \mathrm{Ad}(u(\chi)) = \chi,\ \chi \in W_\phi$. Thus $\phi \circ \mathrm{Ad}(u(\phi \circ \alpha^{-1}|_N)) = \phi \circ \alpha^{-1}|_N$. Let

$$h(\alpha) = \mathrm{Ad}(u(\phi \circ \alpha^{-1}|_N)) \circ \alpha,$$

then $\phi \circ h(\alpha)|_N = \phi$ and

$$\psi \circ h(\alpha) = \phi \circ E \circ h(\alpha) = \phi \circ h(\alpha) \circ E = \phi \circ E = \psi,$$

so $h(\alpha)|_P \in \mathrm{Aut}(P, Q)$. Putting the pieces together, essentially as in [**CT**, IV.1.9], we have defined a Borel map $h : \overline{\mathrm{Int}}(M, N) \to \overline{\mathrm{Int}}(M, N)$ satisfying $\phi \circ h(\alpha)|_N = \phi$ for all α, and we can define a Borel map $\Psi : \overline{\mathrm{Int}}(M, N) \to \mathrm{Aut}(P, Q)$ by $\Psi(\alpha) = h(\alpha)|_P, \alpha \in \overline{\mathrm{Int}}(M, N)$.

Note that if $v \in \mathcal{U}(N), \alpha = \mathrm{Ad}(v)$, and $v_0 = u(\phi \circ \mathrm{Ad}(v^*)|_N)$, then $h(\alpha) = \mathrm{Ad}(v_0 v)$ and $\phi \circ h(\alpha)|_N = \phi$ implies $v_0 v \in Q$; this means that we have $\Psi(\mathrm{Int}(M, N)) \subseteq \mathrm{Int}(P, Q)$.

Also note that if Π denotes the canonical projection of $\mathrm{Aut}(P, Q)$ onto $\mathrm{Out}(P, Q) = \mathrm{Aut}(P, Q) / \mathrm{Int}(P, Q)$, then $\Psi' = \Pi\Psi$ defines a homomorphism. Indeed, if $\alpha_1, \alpha_2 \in \overline{\mathrm{Int}}(M, N)$, $v_i = u(\phi \circ \alpha_i^{-1}|_N)$, $i =$

$1, 2$ and $v_{12} = u(\phi \circ \alpha_2^{-1}\alpha_1^{-1}|_N)$, then

$$\begin{aligned} h(\alpha_1)h(\alpha_2) &= \mathrm{Ad}(v_1\alpha_1(v_2))\alpha_1\alpha_2 \\ h(\alpha_1\alpha_2) &= \mathrm{Ad}(v_{12})\alpha_1\alpha_2 \end{aligned}$$

and

$$\phi \circ \mathrm{Ad}(v_1\alpha_1(v_2)) \circ \alpha_1\alpha_2|_N = \phi = \phi \circ \mathrm{Ad}(v_{12}) \circ \alpha_1\alpha_2|_N,$$

which imply $v_1\alpha_1(v_2)v_{12}^* \in Q$. This shows $\Psi(\alpha_1\alpha_2) = \Psi(\alpha_1)\Psi(\alpha_2)$ modulo $\mathrm{Int}(P, Q)$.

On the other hand, from §2, we have the continuous homomorphism $\Phi : \mathrm{Aut}(P, Q) \to \mathcal{G}$. Since $\Phi(\mathrm{Int}(P, Q)) = 1$, we get a homomorphism $\Phi' : \mathrm{Out}(P, Q) \to \mathcal{G}$ determined by $\Phi'\Pi = \Phi$. As $\Phi\Psi = \Phi'\Pi\Psi = \Phi'\Psi'$, we infer that $\Phi\Psi$ is a Borel homomorphism between Polish groups, and therefore is continuous. It follows that $\Phi\Psi(\overline{\mathrm{Int}}(M, N)) = \{1\}$.

By construction, $\Psi(\alpha)$ preserves the trace on $P \supseteq Q$, so (2.1) gives a unitary $v_0 \in \mathcal{U}(Q)$ such that $\beta = \mathrm{Ad}(v_0) \circ \Psi(\alpha)$ has $\beta|_P = \beta|_A \otimes 1$ and $\beta|_Q = \beta|_B \otimes 1$. Since $\Psi(\alpha) \in \mathrm{Ker}(\Phi)$ we get $\beta|_A \in \overline{\mathrm{Int}}(A, B)$ by (2.6). Also $\phi \circ \beta|_N = \phi$ since $v_0 \in Q$, and β is the perturbation of α by a unitary from N.

Proof that (ii) $\Rightarrow$ (i). Let $\alpha \in \mathrm{Aut}(M, N)$ and let $u_0 \in (N)$ be given such that with $\beta = \mathrm{Ad}(u_0) \circ \alpha$, we have $\beta|_A \in \overline{\mathrm{Int}}(A, B)$ and $\phi \circ \beta|_N = \phi$. From the last property, $[\sigma_t^\phi, \beta|_N] = 0, t \in \mathbb{R}$, so $\beta(Q) \subseteq Q$. Also $[\beta, E] = 0$ since E is the minimal expectation, so $\psi \circ \beta = \phi \circ E \circ \beta = \psi$, whence $\beta(P) \subseteq P$. Using (2.1) it is now easy to see that $\beta|_P \in \overline{\mathrm{Int}}(P, Q)$.

We have $\beta(u)u^* \in Q$ since $[\sigma_t^\phi, \beta|_N] = 0, t \in \mathbb{R}$, so by [**CT**, III.5.3] there exists a unitary $v \in \mathcal{U}(Q)$ with the property $v^*\theta(v) = \beta(u)u^*$, i.e. $v\beta(u)v^* = \theta(v)uv^* = uvu^*uv^* = u$. Let $\gamma = \mathrm{Ad}(v) \circ \beta$, then still $\gamma|_P \in \overline{\mathrm{Int}}(P, Q)$, but $\gamma(u) = u$ so that $[\gamma|_P, \theta] = 0$. Choose a sequence $(u_n) \subseteq \mathcal{U}(Q)$ and a free ultrafilter ω on $\mathbb{N}$ such that $\lim_{n\to\omega} \mathrm{Ad}(u_n) = \gamma|_P$ in $\mathrm{Aut}(P)$. Then

$$\begin{aligned} &\|\chi \circ \mathrm{Ad}(\theta(u_n)) \circ \theta - \chi \circ \gamma \circ \theta\| \\ &\qquad = \|\chi \circ \theta \circ \mathrm{Ad}_P(u_n) - \chi \circ \theta \circ \gamma|_P\| \longrightarrow 0 \end{aligned}$$

for all $\chi \in P_*$, i.e. $\lim_{n\to\omega} \mathrm{Ad}(\theta(u_n)) = \gamma|_P$, and therefore $(\theta(u_n^*)u_n) \subseteq \mathcal{U}(Q)$ is an ω-centralizing sequence in P. (2.3) now provides a sequence $(v_n) \subseteq \mathcal{U}(Q)$ which is centralizing in P and satisfies

$$\theta(u_n v_n) - u_n v_n \xrightarrow{\sigma s^*} 0.$$

Resuming, if we put $w_n = u_n v_n, n \in \mathbb{N}$, then $(w_n) \subseteq \mathcal{U}(Q)$, $\theta(w_n) - w_n \xrightarrow{\sigma s^*} 0$ and $\mathrm{Ad}(w_n) \to \gamma|_P$ in $\mathrm{Aut}(P)$. At this point, we can repeat the end of the proof of the type III_0-case in [**KST**, Theorem 1(i)] to conclude that $\mathrm{Ad}(w_n) \to \gamma$ in $\mathrm{Aut}(M)$. Thus $\gamma \in \overline{\mathrm{Int}}(M, Q)$ and in particular $\alpha \in \overline{\mathrm{Int}}(M, N)$. □

The equivalence of (i) and (iii) in (3.1) means that the topoligical property of approximate innerness is described by the algebraic invariants Φ and mod , defined in terms of the II_1-inclusion and the single type III-factor. One may wonder if it is possible to define a similar invariant "directly" on the type III-inclusion. We shall see in the next section that this can indeed be done, but the following application of (3.1) shows that simply repeating Loi's construction does not suffice:

EXAMPLE 3.3. Let $A \supseteq B$ be an irreducible inclusion of hyperfinite II_1-factors which has D_{2n} as principal graph for some $n \geq 2$. (The existence of such an object was claimed in [**O**] and proved in [**Ka2**].) Since $A \supseteq B$ has finite depth and hence the generating property, we can define $\sigma \in \mathrm{Aut}(A, B)$ by its action on the derived tower as follows: σ interchanges the last two vertices of the principal graph and leaves the other vertices fixed (cf. the last part of [**KL**]). Fix $\lambda \in]0,1[$ and let $\theta_\lambda \in \mathrm{Aut}(R_{0,1})$ be the automorphism of the hyperfinite II_∞-factor $R_{0,1}$ which scales the traces of $R_{0,1}$ by λ. Let $P = A \otimes R_{0,1}, Q = B \otimes R_{0,1}$ and $\theta = \sigma \otimes \theta_\lambda \in \mathrm{Aut}(P, Q)$. Then we have a type III_λ-inclusion $M \supseteq N$ given by $M = P \rtimes_\theta \mathbb{Z}$ and $N = Q \rtimes_\theta \mathbb{Z}$, the principal graph of which is A_{4n-3}. (This last – and important – fact was first noted in [**L2**, 6.5]; details can be found in [**KL**].) In particular, any automorphism acts trivially on the derived tower of $M \supseteq N$.

As a specific example, let $\alpha = \sigma \otimes 1 \in \mathrm{Aut}(P, Q)$; then $[\alpha, \theta] = 0$ whence α can be extended in the obvious way to $\tilde{\alpha} \in \mathrm{Aut}(M, N)$, cf. [**HS**, 13.2]. Thus *the action of $\tilde{\alpha}$ on the derived tower* (*defined in analogy with Loi's construction*) *is trivial, but $\tilde{\alpha}$ is not approximately inner.* For suppose it is. Choose a λ-trace ϕ on N such that $Q = N_\phi$. Then by our supposition and (3.1), there exists a unitary $u_0 \in \mathcal{U}(N)$

such that $\phi \circ \mathrm{Ad}(u_0) \circ \tilde{\alpha}|_N = \phi$, $\mathrm{Ad}(u_0) \in \mathrm{Aut}(A)$ and $\Phi(\mathrm{Ad}(u_0) \circ \tilde{\alpha}|_A) = 1$. Identifying Q with its image in N, we have $N = (Q \cup \{u\})''$ where u is the canonical generating unitary in the discrete crossed product. Since

$$\tilde{\alpha} \circ \sigma_t^{\phi}(u) = \lambda^{it} u = \sigma_t^{\phi} \circ \tilde{\alpha}(u),\ t \in \mathbb{R}$$
$$\tilde{\alpha} \circ \sigma_t^{\phi}(x) = \alpha(x) = \sigma_t^{\phi} \circ \tilde{\alpha}(x),\ x \in Q,\ t \in \mathbb{R}$$

we have $[\tilde{\alpha}|_N, \sigma_t^{\phi}] = 0$ and hence $\sigma_t^{\phi \circ \tilde{\alpha}|_N} = \sigma_t^{\phi}$, $t \in \mathbb{R}$. It follows that $(D\phi \circ \tilde{\alpha}|_N : D\phi)_t = \mu^{it} 1$, $t \in \mathbb{R}$, for some $\mu > 0$. But as $\tilde{\alpha}^2 = 1$, an application of the chain rule for Connes' cocycle derivatives now gives $\mu = 1$ whence $\phi \circ \mathrm{Ad}(u_0) = \phi \circ \tilde{\alpha}|_N = \phi$. Thus $u_0 \in N_\phi = Q$. Since the tower for $P \supseteq Q$ can be obtained from the tower of $A \supseteq B$ by simple tensoring, it is therefore clear that

$$\Phi(\tilde{\alpha}|_P) = \Phi(\mathrm{Ad}(u_0) \circ \tilde{\alpha}|_P) = 1,$$

with Φ here defined on $\mathrm{Aut}(P, Q)$ as explained in §2. But $\tilde{\alpha}|_P = \alpha = \sigma \otimes 1$ and we then obtain the contradiction that $\Phi(\sigma) = 1$. Therefore $\tilde{\alpha} \notin \overline{\mathrm{Int}}(M, N)$.

4. The continuous crossed product method. Let $M \supseteq N$ be an inclusion of σ-finite factors with finite index and minimal expectation $E : M \to N$. Let ϕ be a normal semifinite faithful weight on N and put $\psi = \phi \circ E$. Also define

$$\tilde{N} = N \rtimes_{\sigma^\phi} \mathbb{R}, \quad \tilde{M} = M \rtimes_{\sigma^\psi} \mathbb{R}.$$

Then $\tilde{M} \supseteq \tilde{N}$ is an inclusion of semifinite von Neumann algebras. Let $\lambda : \mathbb{R} \to \tilde{N} \subseteq \tilde{M}$ be the canonical unitary representation, and denote by π the usual injection of M in $\tilde{M}$. As in [**HS**, 12.1] we have for each $\alpha \in \mathrm{Aut}(M, N)$ an automorphism $\tilde{\alpha} \in \mathrm{Aut}(\tilde{M}, \tilde{N})$ given by

$$\tilde{\alpha}(\pi(x)) = \pi(\alpha(x)),\ x \in M$$
$$\tilde{\alpha}(\lambda(t)) = \pi((D\psi \circ \alpha^{-1} : D\psi)_t)\lambda(t),\ t \in \mathbb{R}.$$

Using Connes' unitary cocycle theorem, it is routine to check that the pair $(\tilde{M} \supseteq \tilde{N}, \tilde{\alpha})$ depends only on $(M \supseteq N, \alpha)$, i.e. it does not (up to isomorphism) depend on the choice of ϕ.

We shall need the following fact, which is a (partial) generalization of [**HS**, 12.2 (v)], though the proof given here is entirely different.

PROPOSITION 4.1. *If M and N are properly infinite with separable preduals, then the map $\alpha \mapsto \tilde{\alpha}|_{\tilde{M}\cap\tilde{N}'}$ is a continuous homomorphism.*

Proof. The homomorphism property is immediate from [**HS**, 12.1]. To prove continuity, choose ϕ to be a dominant weight on N; then ψ is dominant on M (see [**CT**, II.1.2]), and we obtain a "common continuous decomposition" of M and N, i.e. a continuous action θ of $\mathbb{R}$ on $M_\psi \supseteq N_\phi$ such that

$$(M \supseteq N, \sigma^\psi) \cong (M_\psi \rtimes_\theta \mathbb{R} \supseteq N_\phi \rtimes_\theta \mathbb{R}, \hat{\theta})$$

where $\hat{\theta}$ denotes the action dual to θ - and a trace τ on $M_\psi \supseteq N_\phi$ satisfying $\tau \circ \theta_s = e^{-s}\tau,\ s \in \mathbb{R}$ (see e.g. [**CT**, II.1.3],[**L1**, p.47]). Let $\lambda_0 : \mathbb{R} \to N_\phi \rtimes_\theta \mathbb{R} \subseteq M_\psi \rtimes_\theta \mathbb{R}$ be the canonical unitary representation.

Now let W_ϕ and $u : W_\phi \to \mathcal{U}(N)$ be as in [**CT**, IV.1.7]. Let $\alpha \in \mathrm{Aut}(M, N)$. Then $\phi \circ \alpha^{-1}|_N$ is also dominant on N and thus $\phi \circ \alpha^{-1}|_N \in W_\phi$ by [**CT**, II.1.1]. Put $\alpha' = \mathrm{Ad}(u(\phi \circ \alpha^{-1}|_N)) \circ \alpha$. Since $\phi \circ \alpha'|_N = \phi$, we can choose a unitary $v_\alpha \in \mathcal{U}(N_\phi)$ like in [**CT**, p.569], so that with $\alpha'' = \mathrm{Ad}(v_\alpha) \circ \alpha'$, we have

$$\alpha''(\lambda_0(s)) = \lambda_0(s), s \in \mathbb{R}.$$

It now follows from the arguments of [**HS**, §13], with obvious adjustments, that we have an isomorphism $I : \tilde{M} \cap \tilde{N}' \to M_\psi \cap N'_\phi$ such that

$$(\alpha'')^\sim|_{\tilde{M}\cap\tilde{N}'} = I^{-1}\alpha''|_{M_\psi\cap N'_\phi} I,\ \ \alpha \in \mathrm{Aut}(M, N).$$

Observe that $(\alpha'')^\sim|_{\tilde{M}\cap\tilde{N}'} = \tilde{\alpha}|_{\tilde{M}\cap\tilde{N}'}$ and that $\alpha''|_{M_\psi\cap N'_\phi} = \alpha'|_{M_\psi\cap N'_\phi}$ for all $\alpha \in \mathrm{Aut}(M, N)$. Moreover from the proof of [**CT**, IV.1.9], the map $\alpha \mapsto \alpha'$ is a Borel map, and therefore so is $\alpha \mapsto \tilde{\alpha}|_{\tilde{M}\cap\tilde{N}'}$. This together with the homomorphism property establish continuity. □

Now let $N = M_{-1} \subseteq M = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ be the tower for $(M \supseteq N, E)$ with Jones projections $e_k \in M_k, k \in \mathbb{N}$ (cf. [**Ko**]). As in §2, each $\alpha \in \mathrm{Aut}(M,N)$ has unique extensions α_k to M_k with $\alpha_k(e_k) = e_k$ and $\alpha_k|_{M_{k-1}} = \alpha_{k-1}$. Define $\psi_0 = \psi$ and (recursively) $\psi_k = \psi_{k-1} \circ E_k$, $k \in \mathbb{N}$, where $E_k : M_k \to M_{k-1}$ are the expectations arising from the tower construction (see [**Ko**, §5]). Put $\tilde{M}_k = M_k \rtimes_{\sigma^{\psi_k}} \mathbb{R}$, $k \in \mathbb{N}$, and extend each α_k to $\tilde{\alpha}_k \in \mathrm{Aut}(\tilde{M}_k, \tilde{M}_{k-1})$ as above. Also, we let $\alpha_0 = \alpha$ for notational convenience. Then $\left(\tilde{\alpha}_k|_{\tilde{M}_k \cap \tilde{N}'}\right)_{k=0}^{\infty}$ belongs to the topological group G defined as the set of sequences $\left(\beta^{(k)}\right)_{k=0}^{\infty}$ of automorphisms $\beta^{(k)} \in \mathrm{Aut}\left(\tilde{M}_k \cap \tilde{N}', \tilde{M}_{k-1} \cap \tilde{N}'\right)$ satisfying $\beta^{(k)}|_{\tilde{M}_{k-1} \cap \tilde{N}'} = \beta^{(k-1)}$, with multiplication and topology defined "pointwise".

DEFINITION 4.2. *The map* $\Upsilon : \mathrm{Aut}(M,N) \to G$ *given by*

$$\Upsilon(\alpha) = (\tilde{\alpha}_k|_{\tilde{M}_k \cap \tilde{N}'})_{k=0}^{\infty}, \quad \alpha \in \mathrm{Aut}(M,N)$$

is called the fundamental homomorphism *of the inclusion* $M \supseteq N$.

PROPOSITION 4.3.

(i) Υ *is a homomorphism*

(ii) *If* $M = N$ *is properly infinite with separable predual, then* Υ *is the classical Connes-Takesaki fundamental homomorphism*

(iii) *If* M *and* N *are* II_1*-factors and* E *coincides with the trace preserving conditional expectation (e.g. if* $M \cap N' = \mathbb{C}$*), then* Υ *is Loi's homomorphism* Φ *as described in §2.*

Proof. (i) and (ii) are obvious consequences of [**HS**, 12.1; 13.1].

(iii) follows by choosing ϕ to be a trace; we then see that

$$\tilde{M}_k \cap \tilde{N}' \cong (M_k \cap N') \otimes L^{\infty}(\mathbb{R}), \ \ k \geq 0$$

and that $\Upsilon(\alpha)$ corresponds to $(\alpha_k|_{M_k \cap N'} \otimes 1)_{k=0}^{\infty}$ under this isomorphism for all $\alpha \in \mathrm{Aut}(M,N)$. □

PROPOSITION 4.4. *In each of the following cases, the fundamental homomorphism is continuous:*

(i) M *and* N *are properly infinite factors with separable preduals*

(ii) *M and N are II_1-factors, and E is trace preserving.*

Proof. This follows in case (i) from (4.1) and in case (ii) from the proof of (4.3)(iii), since $\alpha \mapsto \alpha_k$ is continuous for all k. □

COROLLARY 4.5. *If M and N are properly infinite factors with separable preduals, then*

$$\overline{\mathrm{Int}}(M,N) \subseteq \mathrm{Ker}(\Upsilon).$$

Proof. Let $\pi_k : M_k \to \tilde{M}_k$ be the natural injections. If $\alpha = \mathrm{Ad}(u) \in \mathrm{Int}(M,N)$ for some $u \in \mathcal{U}(N)$, then by uniqueness $\alpha_k = \mathrm{Ad}(u)$, $k \geq 0$, and therefore $\tilde{\alpha}_k = \mathrm{Ad}(\pi_k(u)) \in \mathrm{Int}(\tilde{M}_k, \tilde{N})$, $k \geq 0$. Thus $\mathrm{Int}(M,N) \subseteq \mathrm{Ker}(\Upsilon)$, and we apply (4.4)(i). □

It is natural to expect equality to hold in (4.5) for strongly amenable inclusions of finite index. In fact, using (2.1), (2.5) and the analogue to the argument of (4.3)(iii), this is easily seen to be true when M and N are II_∞-factors with a common trace preserved by E. We now show that it is also true in the situation considered in the previous section:

THEOREM 4.6. *Let $\lambda \in]0,1[$. The set of approximately inner automorphisms on a strongly amenable, finite index inclusion $M \supseteq N$ of type III_λ-factors with a common discrete decomposition is equal to the kernel of the fundamental homomorphism of the inclusion:*

$$\overline{\mathrm{Int}}(M,N) = \mathrm{Ker}(\Upsilon).$$

Proof. For this proof, we have the assumptions from the beginning of §3, so in addition to the notation introduced in this section, we also use the notation given in §3 prior to the statement of (3.1). In particular, ϕ and ψ are now assumed to be λ-traces.

By (4.5), we only have to prove that $\mathrm{Ker}(\Upsilon) \subseteq \overline{\mathrm{Int}}(M,N)$. So assume $\alpha \in \mathrm{Aut}(M,N)$ has $\Upsilon(\alpha) = 1$. To show that $\alpha \in \overline{\mathrm{Int}}(M,N)$ we must by (3.1) and (3.2) show that for some $v \in \mathcal{U}(N)$, $\beta =$

$\mathrm{Ad}(v) \circ \alpha$ satisfies $\beta|_N \in \overline{\mathrm{Int}}(N)$ and $\beta|_A \in \overline{\mathrm{Int}}(A, B)$. Whatever v may be, the first follows from [**KST**, Theorem 1(i)] and [**HS**, 13.1] since $\Upsilon(\alpha) = 1$ implies $\tilde{\alpha}|_{Z(\tilde{N})} = 1$. By (3.2) we may thus assume that $\phi \circ \alpha|_N = \phi$, and it remains according to (2.5) to explain why, for some perturbation β of α as above, we have $\beta|_A \in \mathrm{Aut}(A, B)$ and $(\beta|_A)_k|_{A_k \cap B'} = 1,\ k \in \mathbb{N}$.

The (nontrivial) fact that $E \circ E_1 \circ \cdots \circ E_k$ is the minimal expectation for the inclusion $M_k \supseteq N$ was established in [**KL**], so $\phi \circ \alpha|_N = \phi$ entails $\psi_k \circ \alpha_k = \psi_k$ for all k. If we let P_k denote the centralizer $(M_k)_{\psi_k}$ for all $k \geq 0$, then as noted in [**L2**, §3], $Q \subseteq P \subseteq P_1 \subseteq P_2 \cdots$ is the tower for $P \supseteq Q$, with Jones projections $(e_k)_{k \geq 1}$ - the same as for the tower of $M \supseteq N$. By [**Ko**, 5.1], for each k we have $\sigma_t^{\psi_k}(e_k) = e_k,\ t \in \mathbb{R}$ and hence $\sigma_{t_0}^{\psi_k} = 1$, where $t_0 = -2\pi/\log\lambda$. Thus we have $M_k \cong P_k \rtimes_\theta \mathbb{Z}$ and $P_k \cong A_k \otimes F$ for all k, where (A_k) denotes the tower for $A \supseteq B$.

Let $P_0 = P \rtimes_\theta \mathbb{Z} \rtimes_{\sigma^\psi} \mathbb{R}/t_0\mathbb{Z}$, where $t_0 = -2\pi/\log\lambda$. Then P_0 can be viewed as the crossed product $M \rtimes_{\sigma^\psi} \mathbb{R}/t_0\mathbb{Z}$. We let π_0 be the canonical injection of M in P_0 and λ_0 be the canonical unitary representation of $\mathbb{R}/t_0\mathbb{Z}$ in P_0, so that $M \rtimes_{\sigma^\psi} \mathbb{R}/t_0\mathbb{Z} = (\pi_0(M) \cup \lambda_0(\mathbb{R}/t_0\mathbb{Z}))''$. For ease of notation, we extend λ_0 to $\mathbb{R}$ in the obvious way. By [**HS**, 5.6] we have an isomorphism I from $\tilde{M}$ onto $P_0 \otimes L^\infty(0, \log\lambda^{-1})$ which is given by

$$I(\pi(x)) = \pi_0(x) \otimes 1,\ x \in M$$
$$I(\lambda(t)) = \lambda_0(t) \otimes m(e^{it}),\ t \in \mathbb{R}$$

where $m(e^{it})\xi(s) = e^{its}\xi(s)$ for $t \in \mathbb{R},\quad s \in]0, \log\lambda^{-1}[$ and $\xi \in L^2(0, \log\lambda^{-1})$. As in the proof of (ii) $\Rightarrow$ (i) of (3.1), we may assume that $\alpha(u) = u$ by perturbing α by a unitary from Q. (Since $u \in \mathcal{U}(N)$, this perturbation also gives $\alpha_k(u) = u,\ k \geq 1$.) Thus we can extend $\alpha|_P$ to $(\alpha|_P)^- \in \mathrm{Aut}(P \rtimes_\theta \mathbb{Z})$ as in [**HS**, 13.2], and actually $(\alpha|_P)^- = \alpha$ under the identification of M with $P \rtimes_\theta \mathbb{Z}$ because $\alpha(u) = u$. Similarly, since $[\alpha, \sigma^\psi] = 0$, we have the extension $\bar{\alpha} = (\alpha|_P)^=$ of α to P_0, given by

$$\bar{\alpha}(\pi_0(x)) = \pi_0(\alpha(x)),\ x \in M$$
$$\bar{\alpha}(\lambda_0(t)) = \lambda_0(t),\ t \in \mathbb{R}/t_0\mathbb{Z}.$$

Since $\psi \circ \alpha = \psi$, we have $\tilde{\alpha}(\lambda(t)) = \lambda(t), t \in \mathbb{R}$, so

$$I\tilde{\alpha}I^{-1}(\pi_0(x) \otimes 1) = \pi_0(\alpha(x)) \otimes 1, \ x \in M$$
$$I\tilde{\alpha}I^{-1}(\lambda_0(t) \otimes m(e^{it})) = \lambda_0(t) \otimes m(e^{it}), \ t \in \mathbb{R}$$

and we now see that $I\tilde{\alpha}I^{-1} = \bar{\alpha} \otimes 1$.

On the other hand, since $[\alpha|_P, \theta] = 0$, we have the version [**HS**, 13.3] of the Takesaki duality theorem, i.e. an isomorphism $J_0 : P_0 \to P \otimes B(\ell^2(\mathbb{Z}))$ such that $J_0(\alpha|_P)^{=}J_0^{-1} = \alpha|_P \otimes 1$. Let

$$J = J_0 \otimes 1 : P_0 \otimes L^\infty(0, \log \lambda^{-1}) \to P \otimes B(\ell^2(\mathbb{Z})) \otimes L^\infty(0, \log \lambda^{-1})$$

then JI is an isomorphism of $\tilde{M}$ onto $P \otimes B(\ell^2(\mathbb{Z})) \otimes L^\infty(0, \log \lambda^{-1})$ which satisfies

$$JI\tilde{\alpha}(JI)^{-1} = (J_0 \otimes 1)(\bar{\alpha} \otimes 1)(J_0^{-1} \otimes 1) = \alpha|_P \otimes 1 \otimes 1$$

where we used the fact that $\bar{\alpha} = (\alpha|_P)^{=}$.

Now observe that all the arguments of the preceding two paragraphs go through with $(M_k, \alpha_k, \psi_k, P_k)$ and $(N, \alpha|_N, \phi, Q)$ in the place of (M, α, ψ, P), so in particular we have, for each k, an isomorphism

$$I_k : \tilde{M}_k \cap \tilde{N}' \to (P_k \cap Q') \otimes \mathbb{C} \otimes L^\infty(0, \log \lambda^{-1})$$

which carry $\tilde{\alpha}_k|_{\tilde{M}_k \cap \tilde{N}'}$ into $\alpha_k|_{P_k \cap Q'} \otimes 1 \otimes 1$, where $\alpha_k|_{P_k} \in \mathrm{Aut}(P_k, Q)$ by the adjustments of α made above. Thus $\tilde{\alpha}_k|_{\tilde{M}_k \cap \tilde{N}'} = 1$ implies – after the perturbation of α – that $\alpha_k|_{P_k \cap Q'} = 1$ for each k. From (2.1) we have a unitary $w \in \mathcal{U}(Q)$ such that $\beta = \mathrm{Ad}(w) \circ \alpha$ satisfies $\beta|_P = \beta|_A \otimes 1$ and $\beta|_Q = \beta|_B \otimes 1$. Since $e_k \in Q'$ it easy to see that $\mathrm{Ad}(w) \circ \alpha_k|_{P_k} = \beta_k|_{P_k}$ $(k \geq 0)$ – recall that the towers for $P \supseteq Q$ and $M \supseteq N$ have the same Jones projections – and so the isomorphism $P_k \cap Q' \to A_k \cap B' \otimes \mathbb{C}$ carries $\beta_k|_{P_k \cap Q'}$ into $(\beta|_A)_k|_{A_k \cap B'} \otimes 1$, whence $(\beta|_A)_k|_{A_k \cap B'} = 1$ as required. □

References

[B] D. Bisch, *On the existence of central sequences in subfactors*, Trans., Amer., Math., Soc., **321** (1990), 117-128.

[C1] A. Connes, *Outer conjugacy classes of automorphisms of factors*, Ann. Sci. Ecole Norm., Sup., **4:8** (1975), 383-420.

[C2] ———, *A factor not antiisomorphic to itself*, Ann. of Math., **101** (1975), 536-554.

[CT] A. Connes and M. Takesaki, *The flow of weights on factors of type III*, Tôhoku Math. J., **29** (1977), 473-575.

[H] F. Hiai, *Minimizing indices of conditional expectations onto a subfactor*, Publ. RIMS, **24** (1988), 673-678.

[HS] U. Haagerup and E. Størmer, *Equivalence of normal states on von Neumann algebras and the flow of weights*, Adv. Math., **83** (1990), 180-262.

[Ka1] Y. Kawahigashi, *Automorphisms commuting with a conditional expectation onto a subfactor with finite index*, J. Op. Theory, **28** (1992), 127-145.

[Ka2] ———, *On flatness of Ocneanu's connections on the Dynkin diagrams and classification of subfactors*, , preprint (1990).

[KST] Y. Kawahigashi, C. Sutherland and M. Takesaki, *The Structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian groups*, Acta Math., **169** (1992), 105-130.

[Ko] H. Kosaki, *Extension of Jones' theory on index to arbitrary factors*, J. Funct. Anal., **66** (1986), 123-140.

[KL] H. Kosaki and R. Longo, *A remark on the minimal index of subfactors*, J. Funct. Anal., **107** (1992), 458-470.

[L1] P. H. Loi, *On the theory of index and type* III *factors*, Thesis, Penn. State Univ., (1988).

[L2] ———, *On automorphisms of Subfactors*, preprint (1990).

[L3] ———, *On the derived tower of certain inclusions of type* III_λ*-factors of index* 4, (to appear in Pacific. J. Math.).

[O] A. Ocneanu, *Quantized group string algebras and Galois theory for algebras*, in: *Operator algebras and applications*, vol. 2, London Math. Soc. lect. notes series, **136** (1988), 119-172.

[P1] S. Popa, *Classification of subfactors: the reduction to commuting squares*, Invent. Math., **101** (1990), 19-43.

[P2] ———, *Classification of amenable subfactors of type* II, Acta Math., **172** (1994), 252-345.

Note added in proof. After the completion and circulation of this paper as a preprint, much progress has been made on the subject matter. In [**2**], the author proves that the fundamental homomorphism as defined above is a complete cocycle conjugacy invariant for centrally free actions of discrete amenable groups, thus providing a further analogy between this invariant and the Connes-Takesaki module. Our invariant is further studied in [**1**], [**3**] and [**4**], and the fact – contained in §4 of the present paper – that the approximately inner automorphisms on a strongly amenable inclusion of type II_∞ are just the kernel of the fundamental homomorphism on that inclusion, turns out to be essential in the classification **[5]** of strongly amenable subfactors of type III_o.

References

[1] H. Kosaki, *Relative T-set for an inclusion of factors*, in preparation.

[2] C. Winsløw, *Strongly free actions on subfactors*, Internat. J. Math.,**4** (1993), 675-688.

[3] ———, *Automorphisms on an inclusion of factors*, in *Subfactors* (ed. H. Araki et. al.), World Scientific 1994.

[4] ———, *The flow on weights in subfactor theory*, to appear in Publ. R.I.M.S.

[5] ———, *Classification of strongly amenable subfactors of type* III_0, preprint 1994.

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