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VELOCITY MAPS IN VON NEUMANN ALGEBRAS

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When A is a C^*-algebra, a function $d : A_{sa} \to A_{sa}$ is said to be a velocity map if, for each commutative subalgebra B of A_{sa}, $d : B \to A_{sa}$ is a derivation.

Let A be a norm closed ideal, or quotient, in a von Neumann algebra without Type I_2 part and let $P(A)$ be the set of projections in A. It is shown that if $d : P(A) \to A$ is a bounded function such that $d(ef) = ed(f) + d(e)f$ whenever $ef = fe$, then d extends uniquely to a derivation of A. Hence every velocity map of A_{sa} bounded on the unit ball extends to a derivation of A.

Let A_{sa} denote the self-adjoint part of a C^*-algebra A. A function $d : A_{sa} \to A_{sa}$ is said to be a *velocity map* if

(i) $d(\lambda x) = \lambda d(x)$, for all $\lambda \in \mathbb{R}$, $x \in A_{sa}$

(ii) $d(x + y) = d(x) + d(y)$, for all $x, y \in A_{sa}$ with $xy = yx$

(iii) $d(xy) = xd(y) + d(x)y$, for all $x, y \in A_{sa}$ with $xy = yx$.

Velocity map are motivated by their relevance to quantum dynamics. Briefly, A_{sa} is identified with the bounded observables (of a physical system) the rate of change of which is measured by the velocity maps on A_{sa}; only when two observables commute are they simultaneously measurable and is their product again an observable. See [**4**] for more details.

The point of interest and natural question is: When do velocity maps extend to derivations?

It transpires that *linear* velocity maps always extend to derivations (see Proposition 1). If linearity is not assumed then we are still able to obtain positive results for a large class of C^*-algebras that includes Calkin algebras and all von Neumann algebras without Type I_2 part. In fact we prove something more.

Given a C^*-algebra, A, we let $P(A)$ denote the set of projections of A. Our main result is the following.

Let M be a von Neumann algebra without Type I_2 part and let A be a norm closed ideal of M or a quotient of a norm closed ideal of M. Let $d : P(A) \to A$ be any bounded function satisfying

$$d(ef) = d(e)f + ed(f), \text{ whenever } ef = fe.$$

Then d extends uniquely to a derivation of A.

We will show by example that the exclusion of Type I_2 algebras from the above statement is *necessary.*

By an ingenious direct constructive argument Rajarama Bhat [**4**, Theorem 3.5] proved that every linear velocity map on $B(H)_{sa}$ extends to a derivation of $B(H)$ when H is separable. We need to show that this extension theorem holds for all C^*-algebras. In fact, we have:

Proposition 1. *Let B be any C^*-algebra and let $d : B_{sa} \to B$ be a (real) linear map such that*

$$d(xy) = d(x)y + x\, d(y), \text{ whenever } xy = yx.$$

Then d extends to a derivation of B.

Proof. Recall that the Jordan product $a \circ b$ of $a, b \in B$ is given by $a \circ b = \frac{1}{2}(ab + ba)$. So, $d(a^2) = 2d(a) \circ a$, for each $a \in B_{sa}$.

Given $x, y \in B_{sa}$ we have

$$\begin{aligned} d(x \circ y) &= \frac{1}{2}[d((x+y)^2) - d(x^2) - d(y^2)] \\ &= (d(x+y)) \circ (x+y) - d(x) \circ x - d(y) \circ y \\ &= d(x) \circ y + d(y) \circ x. \end{aligned}$$

By a simple calculation we see that

$$\overline{d}(x \circ y) = \overline{d}(x) \circ y + x \circ \overline{d}(y), \text{ for all } x, y \in B,$$

where $\overline{d} : B \to B$ is the complex linear extension of d. In other words, $\overline{d}$ is a Jordan derivation of B and hence, by a result of Sinclair [**6**], is a derivation of B as required. □

In all that follows M is a von Neumann algebra without Type I_2 direct summand and I is a norm closed two sided ideal of M. We let I_0 denote the norm closed two sided ideal of I generated by the abelian projections in I. (A projection e of M is abelian, or Type I_1, if the algebra eMe is abelian.)

We will need an extension of the solution of the Mackey-Gleason Problem given by the authors in [**1**], [**2**] and summarised in [**3**]. By way of preparation we note that if Ω is a maximal abelian C^*-subalgebra of I, then $\Omega = I \cap \overline{\Omega}$ and so is an ideal of $\overline{\Omega}$, the (abelian) weak closure of Ω in M. By spectral theory Ω is (as therefore is I) the norm closed linear span of its projections. Given $e, f \in P(M)$ with $e \leq f$ and $f \in P(I)$ we have $e \in P(I)$. For arbitrary $e, f \in P(I)$ we have $e \vee f - e \sim f - e \wedge f$.

A projection e of M is said to be a Type I_n projection, where $n < \infty$, if the von Neumann algebra eMe is of Type I_n.

Lemma 1. *Let $e_1, \dots, e_n$ be Type I_3 projections in M and let p be a (possibly zero) abelian projection in M. Then $(e \vee p)M(e \vee p)$ (and hence eMe) has no Type I_2 direct summand, where $e = e_1 \vee \dots \vee e_n \vee p$.*

Proof. If $(e \vee p)M(e \vee p)$ has a Type I_2 direct summand, then $z(e \vee p) = (ze_1) \vee \dots \vee (ze_n) \vee (zp)$ is a Type I_2 projection for some central projection z of M. Since for $i = 1, \dots, n$ e_iMe_i has no Type I_2 direct summand this means that $ze_i = 0$ so that $z(e \vee p) = zp$ is abelian, which is a contradiction. □

Lemma 2.

(i) *$P(I)$ is an increasing approximate unit of I.*

(ii) *If M has no Type I_1 direct summand (as well as no Type I_2 direct summand), then*

$$\{e_1 \vee \dots \vee e_n : e_i \text{ is a Type } I_3 \text{ projection in } I, i = 1, \dots n, n \in \mathbb{N}\}$$

is an increasing approximate unit of I_0.

Proof.

(i) Note that $P(I)$ is an increasing net in I. Let x be in I, where $x = x^*$, and let $\epsilon > 0$. By the preamble there exist real numbers $\lambda_1, \dots, \lambda_n$ and mutually orthogonal projections $e_1, \dots, e_n$ in I such that $\|x - \sum_1^n \lambda_i e_i\| < \epsilon$. So, $\|x(1-p)\| < \epsilon$ where $p = \sum_1^n e_i$. If $q \in P(I)$ with $p \leq q$, then

$$\|x(1-q)\| = \|x(1-q)x\|^{\frac{1}{2}} \leq \|x(1-p)x\|^{\frac{1}{2}} = \|x(1-p)\| < \epsilon.$$

(ii) Suppose that M has no Type I_1 direct summand and let p be an abelian projection of M contained in I and so in I_0. Because M has no Type I_2 direct summand either, there exist projections q and h in M such that p, q, h are mutually orthogonal and equivalent. The projection $e = p + q + h$ is then a Type I_3 projection in I dominating p. We note that e is in I_0.

For any projection p in I_0, pMp is a postliminal C^*-algebra because I_0 is postliminal as follows from [**5**, §6.1] for example. So pMp is a direct sum of finitely many Type I_k von Neumann algebras for certain $k < \infty$ and hence $p = p_1 + \dots + p_r$ for certain abelian projections p_i in I_0. So $p \leq e_1 \vee \dots \vee e_r$ from some Type I_3 projections $e_1, \dots, e_r$ of I_0 by the above.

Since, by (i), $P(I_0)$ is an increasing approximate unit for I_0 it now follows that the increasing net in $P(I_0)$ described in the statement is also an approximate unit of I_0. □

In the case when $A = M$, the following proposition is [**2**, Theorem A]. See also [**3**].

Proposition 2. *Let the C^*-algebra A be equal to a quotient of I, where I is a norm closed ideal of M, and let X be a Banach space. Let $\rho : P(A) \to X$ be a bounded function such that*

$$\rho(e+f) = \rho(e) + \rho(f), \qquad \text{whenever } ef = 0.$$

Then ρ extends uniquely to a continuous linear map $\overline{\rho} : A \to X$.

Proof. (a) Let $A = I$. Suppose first that $X = \mathbb{C}$.

Because the maximal abelian subalgebras of I are generated by projections we have, precisely as in [**1**, §1], that ρ extends uniquely to a function $\overline{\rho} : I \to \mathbb{C}$ bounded on the unit ball of I satisfying $\overline{\rho}(a+ib) = \overline{\rho}(a) + i\overline{\rho}(b)$ wherever $a = a^*$ and $b = b^*$, and which is linear on every abelian C^*-subalgebra of I. We make the harmless normalising assumption that $\overline{\rho}$ sends the unit ball of I into the unit disc. We show that $\overline{\rho}$ is linear.

If M has Type I_1 direct summand zM, where z is central projection of M, then $I = zI \oplus (1-z)I$ and $\overline{\rho}$ is already linear on zI. Passing to $(1-z)I$, we may therefore suppose that M has no Type I_1 direct summand (as well as having no Type I_2 direct summand).

We show first that ρ is linear on I_0. Invoking Lemmas 1 and 2 we can an increasing approximate unit (e_λ) of I_0 contained in $\rho(I_0)$ for which the algebras $e_\lambda M e_\lambda$ and $(e_\lambda \vee p)M(e_\lambda \vee p)$ have no Type I_2 direct summand, for all λ and for all abelian projections p in I_0. Consequently, $\overline{\rho}$ is linear on all of these algebras by [**2**, Theorem A].

Let x be a self-adjoint element of I_0 and let $\epsilon > 0$. There is a λ_0 such that $\|e_\lambda x e_\lambda - e_\mu x e_\mu\| < \epsilon$, for all $\lambda, \mu \geq \lambda_0$. Linearity of $\overline{\rho}$ on $(e_\lambda \vee e_\mu)M(e_\lambda \vee e_\mu)$ implies that $|\overline{\rho}(e_\lambda x e_\lambda) - \overline{\rho}(e_\mu x e_\mu)| < \epsilon$, for all $\lambda, \mu > \lambda_0$. Consequently, $\rho(e_\lambda x e_\lambda)$ converges for all x in I_0 and we see that the map $\tau : A \to \mathbb{C}$ defined by $\tau(x) = \lim \overline{\rho}(e_\lambda x e_\lambda)$ is linear.

When p is an abelian projection in I_0, linearity of $\overline{\rho}$ on all $(e_\lambda \vee p)M(e_\lambda \vee p)$ and the fact that $\|e_\lambda p e_\lambda - p\| < \epsilon$ for all large enough λ, gives $\rho(p) = \lim \overline{\rho}(e_\lambda p e_\lambda)$ by similar reasoning. Hence $\rho(e) = \tau(e)$ for all $e \in P(I_0)$, because every projection of I_0 is a finite orthogonal sum of abelian projections in I_0. It follows that $\overline{\rho}$ agrees with τ on all of I_0 and so is linear there.

Now let e be any projection in I. We have $eMe = M_1 \oplus M_2$, where M_1 is Type I_2 or zero and M_2 has no Type I_2 direct summand. But then $M_1 \subset I_0$ and so $\overline{\rho}$ is linear on M_1 by the above. By [**2**, Theorem A], ρ is linear on M_2. Hence $\overline{\rho}$ is linear on eMe, for all $e \in P(I)$. Now, repeating appropriately the arguments of the previous paragraf, this time for the approximate unit $P(I)$ of I, we have that $\overline{\rho}$ is linear on I.

The general case, for an arbitrary Banach space X, now follows as in Lemma 1.1. of [**2**] or [**3**].

(b) Let $A = I/J$, where J is a norm closed two-sided ideal of I and let $\pi : I \to I/J$ be the canonical homomorphism. By the above, $\rho\pi : P(I) \to X$ then extends to a continuous linear map $\varphi : I \to X$ vanishing on J. The induced linear map $\overline{\varphi} : I/J \to X$ extends ρ (uniquely) as desired. This completes the proof. □

We can now prove our main result.

Theorem . *Let M be a von Neumann algebra without Type I_2 part and let A be a norm closed ideal of M or be a quotient of a norm closed ideal of M. Let $d : P(A) \to A$ be a bounded function such that*

$$d(ef) = d(e)f + e(df), \qquad \textit{whenever } ef = fe.$$

Then d extends uniquely to a derivation of A.

Proof. We note that $d(p) = d(p)p + pd(p)$, for all projections p of A. Let e and f be orthogonal projections of A. Thus, as $ef = fe = 0$, we have

$$(*) \qquad d(e)f + ed(f) = d(f)e + fd(e) = 0.$$

In addition, since $e = e(e+f) = (e+f)e$, we have

$$d(e) = d(e)(e+f) + ed(e+f) = (d(e+f))e + (e+f)d(e),$$

so that

$$2d(e) = ed(e+f) + (d(e+f))e + d(e)(e+f) + (e+f)d(e),$$

and hence, using (*)

$$d(e) = ed(e+f) + (d(e+f))e + d(e)f + fd(e).$$

Similarly,

$$d(f) = fd(e+f) + (d(e+f))f + d(f)e + ed(f).$$

The final two equations, together with (*), imply that

$$d(e) + d(f) = (e+f)d(e+f) + (d(e+f))(e+f) = d(e+f).$$

Therefore, by Proposition 2, d extends uniquely to a continuous linear map $\overline{d} : A \to A$. It follows that the assignment

$$h(x,y) = \overline{d}(xy) - \overline{d}(x)y - x\overline{d}(y)$$

is continuous and bilinear on $A \times A$. Let Ω be an abelian C^*-subalgebra of A. By assumption $h(e,f) = 0$ for all projections e, f in Ω. But Ω is the norm closed linear span of its projections. Hence, by continuous bilinearity, h is identically zero on $\Omega \times \Omega$. Therefore,

$$\overline{d}(xy) = \overline{d}(x)y + x\overline{d}(y), \quad \text{for all } x, y \text{ in } A_{sa} \text{ with } xy = yx.$$

Hence, by Proposition 1, $\overline{d}$ is a derivation. □

Corollary . *Let A be as in the Theorem and let $d : A_{sa} \to A_{sa}$ be a velocity map bounded on the ball of A_{sa}. Then d extends to a derivation of A.*

The Theorem and its Corollary apply to all von Neumann algebras without Type I_2 direct summand and to their norm closed two sided ideals and quotients. So, for instance, they apply to the Calkin algebra and also to every dual C^*-algebra without Type I_2 representations. But the exclusion of Type I_2-algebras is necessary. The following example explains why.

Example. Let $M = M_2(\mathbb{C})$.

Let $p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $u_n = \begin{bmatrix} \cos(\theta_n) & \sin(\theta_n) \\ -\sin(\theta_n) & \cos(\theta_n) \end{bmatrix}$ and $p_n = u_n p u_n^*$ where $\theta = \pi/5^n$, for all $n \in \mathbb{N}$. Notice that the p_n are mutually non-orthogonal and that

$$\overline{p}_n = p_n\overline{p}_n + \overline{p}_n p_n, \quad \text{where } \overline{p}_n = u_n \begin{bmatrix} 0 & (-1)^n \\ (-1)^n & 0 \end{bmatrix} u_n^*,$$

for all $n \in \mathbb{N}$. Define a bounded function, $d : P(M) \to M_{sa}$, by

$$d(p_n) = \overline{p}_n, \text{ for all } n \in \mathbb{N}$$
$$d(1 - p_n) = -\overline{p}_n, \text{ for all } n \in \mathbb{N}$$
$$d(q) = 0, \text{ whenever } q \text{ is not in } \{p_n : n \in \mathbb{N}\} \cup \{1 - p_n : n \in \mathbb{N}\}.$$

Since commuting non-zero projections are either equal or are orthogonal with sum 1 it is easy to see that d satisfies

$$d(ef) = ed(f) + d(e)f, \text{ whenever } ef = fe$$
$$d(e + f) = d(e) + d(f), \text{ whenever } ef = 0.$$

Also, by direct calculation, (or a simple version of arguments deployed in the proof of the Theorem) we observe that d has a unique extension to a velocity map $\overline{d} : M_{sa} \to M_{sa}$ bounded on the unit ball. But $\overline{d}$ is not linear because d is discontinuous. For example, $p_n \to e$ whereas $d(p_n)$ fails to converge. So d (and $\overline{d}$) cannot extend to a derivation of M.

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