PACIFIC JOURNAL OF MATHEMATICS
Vol. 172, No. 1, 1996

ENTROPY OF A SKEW PRODUCT WITH A Z^2-ACTION

KYEWON KOH PARK

We consider the entropy of a dynamical system of a skew product $\widehat{T}$ on $X_1 x X_2$ where there is a Z^2-action on the fiber X_2. If the Z^2-action comes from a Cellular Automaton map, then the contribution of the fiber to the entropy of the skew product is the directional entropy in the direction of the integral of a skewing function φ from X_1 to Z^2.

1. Introduction.

J. Milnor has defined the notion of directional entropy in the study of dynamics of Cellular Automata [**Mi1**], [**Mi2**]. When the notion is applied to a Z^n action it is considered to be a generalization of the entropy of non co-compact subgroups of Z^n.

In the case of a Z^2-action, we denote the generators of the groups by $\{U, V\}$. Let P be a generating partition under the Z^2-action. We write $P_{i,j} = U^i V^j P$. If a subgroup is generated by $U^p V^q$, then there is a natural way to compute the entropy of $U^p V^q$ as a Z-action on the space. Milnor extended this idea to define the entropy of a vector by embedding Z^2 to the ambient vector space R^2 as follows.

$$h(\vec{v}) = \sup_{B:\text{bounded set}} \overline{\lim}_{t\to\infty} \frac{1}{t} H\left(\bigvee_{(i,j)\in B+[o,t)\vec{v}} P_{i,j}\right).$$

Given a vector $\vec{v}$, we let θ_o be the angle between two vectors $\vec{v}$ and $(1,0)$. Let $w = \frac{1}{\tan\theta_o}$ so that $(w,1)$ is a scalar multiple of the vector $\vec{v}$. It is easy to see that

$$h(\vec{v}) = \lim_{m\to\infty}\lim_{t\to\infty} \frac{1}{t} H\left(\bigvee_{j=0}^{[ty]} \bigvee_{-m+jw<i<m+jw} P_{i,j}\right),$$

where $[a]$ denote the greatest integer $\leq a$.

We note that if $\vec{v} = (p,q)$, then $h(\vec{v}) = h(U^p V^q)$. And it is easy to see that directional entropy is a homogeneous function, that is $h(c\vec{v}) = ch(\vec{v})$ for any $c \in R$.

Directional entropy in the case of a Z^2-action generated by a Cellular Automaton map has been investigated in [**Pa1, Pa3**] and [**Si**]. D. Lind defined a cone entropy, denoted by $h^c(\vec{v})$, of a vector $\vec{v}$. Given a vector $\vec{v}=(x,y)$ and a small angle θ, we consider the vectors $\vec{v}_\theta=(x_\theta,y)$ and $\vec{v}_{-\theta}=(x_{-\theta},y)$ where x_θ and $x_{-\theta}$ satisfy $\frac{y}{x_\theta}=\tan(\theta_o+\theta)$ and $\frac{y}{x_{-\theta}}=\tan(\theta_o-\theta)$ respectively. Cone entropy is defined as follows.

$$h^c(\vec{v})=\lim_{\theta\to 0}\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\bigvee_{jx_{-\theta}\leq i\leq jx_\theta}P_{i,j}\right).$$

From the definition, it is clear that we have $h^c(\vec{v})\geq h(\vec{v})$.

We say that a Z^2-action is generated by a Cellular Automaton if one of the generators of the Z^2-action, say V, is a block map (a finite code) of U. That is, $(V(x))_i$ depends only on the coordinates $x_{-r},x_{-r+1},\dots,x_r$ [**He**]. We call r the size of the block map V. We will show that in the case of a Z^2-action generated by a Cellular Automaton map, the directional entropy and the cone entropy are the same (Theorem 1).

Let (X_1,ζ_1,μ_1,G) and (X_2,ζ_2,μ_2,H) be two ergodic measure preserving dynamical systems with finite entropy, where G and H denote the respective group. Given an integrable skewing function $\varphi: X_1\to H$, we define a skew product G-action $\widehat{T}$ on $(X_1\times X_2,\zeta_1\times\zeta_2,\ \mu_1\times\mu_2)$ such that $\widehat{T}^g(x,y)=(T^gx,F^{\varphi(x)}y)$ where T denotes the G-action of X_1 and F denotes the H-action on X_2. When we have $G=H=Z$, then the entropy of $\widehat{T}$ has been extensively studied by many people (e.g. [**Ab**], [**Ad**], [**Ma**, **Ne**]). It is well known in this case that $h(\widehat{T})=h(T)+|\int\varphi\,d\mu|h(F)$. The above formula says that, as we expect, the fiber contribution to the entropy is $|\int\varphi\,d\mu|h(F)$.

We investigate the entropy of $\widehat{T}$ when $G=Z$ and $H=Z^2$. Note that the above formula cannot hold when the acting group on the fiber is a more general group, say Z^2. First of all, $\int\varphi\,d\mu$ is in general a vector. Secondly, if the skewing function takes a constant value, say $(1,1)$, then the fiber contribution should come from the entropy of UV, not necessarily from the whole Z^2-action. We prove that if the fiber Z^2-action is generated by a Cellular Automaton map, then we have the analogous theorem (Theorem 2) to the case when $H=Z$.

We may mention that directional entropy can be also defined in a topological setting. D. Lind constructed an example whose topological entropy does not satisfy the analogue of our Theorem 3 [**Li**]. His example involves a Z^2-action which is not generated by a Cellular Automaton map. It is not clear that Theorem 3 holds for topological entropy when we have a Z^2-action on the fiber generated by a Cellular Automaton map. Lind's example is not interesting in the measure theoretic sense because it has the trivial invariant measure.

We have constructed a counterexample which does not satisfy Theorem 3

[**Pa2**]. For the example we explicitly construct the base transformation and use the Z^2-action due to Thouvenot [**Th**] on the fiber. Both of them are constructed by cutting and staking method. It would be interesting to find out how generally Theorem 3 holds. For example, it is unknown if Theorem 3 is true when we have a topological Markov shift which does not satisfy the condition of Corollary 4. We are more interested in the case when the topological Markov shift has 0-entropy as a Z^2-action.

Although Theorem 2 and 4 are more general than Theorem 1 and 3, we will prove Theorem 1 and 3 because their proofs are easier and more geometric. It is also easy to see the proofs of Theorem 2 and 4 from those of Theorem 1 and 3.

We would like to thank Professor D. Ornstein for helpful discussions and the Referee for many valuable comments.

2. Cone entropy.

Throughout the section we assume that our Z^2-action is generated by a Cellular Automaton map. We denote by $H^m(\vec{v})$

$$\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1}\bigvee_{i=-m+jw}^{m+jw}P_{i,j}\right).$$

Note that $H^m(\vec{v})$ is independent of the size of the vector $\vec{v}$. Let τ denote $H(P_{0,0})$.

Lemma 1. $H^m(\vec{v}) = H^{m'}(\vec{v})$ *if* $m, m' > 2r + w$.

Proof. Case 1. $\vec{v}$ is not a scalar multiple of $(1,0)$.

Suppose $m' \geq m$. Clearly from the definition we have $H^{m'}(\vec{v}) \geq H^m(\vec{v})$. Hence it is enough to show $H^{m'}(\vec{v}) \leq H^m(\vec{v})$. Note that

$$\begin{aligned} H^m(\vec{v}) &= \lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{j=0}^{n-1}\bigvee_{-m+jw\leq i\leq m+jw}P_{i,j}\right)\\ &= \lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}H\left(\bigvee_{-m+jw\leq i\leq m+jw}P_{i,j}\,\middle|\,\bigvee_{0\leq k<j}\bigvee_{-m+kw\leq i\leq m+kw}P_{i,k}\right)\end{aligned}$$

$$= \lim_{n\to\infty} \frac{1}{n} \left\{ H\left(\bigvee_{-m\le i\le m} P_{i,0} \right) \right.$$
$$+ \sum_{j=1}^{n-1} H\left(\bigvee_{jw\le i\le (j-1)w+r} P_{i,j} \middle| \bigvee_{0\le k<j} \bigvee_{kw\le i\le 2m+kw} P_{i,k} \right)$$
$$+ \sum_{j=1}^{n-1} H\left(\bigvee_{(j-1)w-r\le i\le jw} P_{i,j} \middle| \bigvee_{1\le k<j} \bigvee_{-2m+kw\le i\le kw} P_{i,k} \right.$$
$$\left.\left. \bigvee \bigvee_{-2m+jw\le i\le -2m+(j-1)w+r} P_{i,j} \right) \right\}.$$

We make the following observations:

(1)

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{-m\le i\le m} P_{i,0} \right) = 0 = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{-m'\le i\le m'} P_{i,0} \right).$$

(2)

$$H\left(\bigvee_{jw\le i<(j-1)w+r} P_{i,j} \middle| \bigvee_{0\le k<j} \bigvee_{kw\le i\le 2m+kw} P_{i,k} \right)$$
$$\ge H\left(\bigvee_{jw\le i<(j-1)w+r} P_{i,j} \middle| \bigvee_{0\le k<j} \bigvee_{kw<i\le 2m'+kw} P_{i,k} \right),$$

because we condition on more information.

(3) By the same reason, we have

$$H\left(\bigvee_{(j-1)w-r\le i\le jw} P_{i,j} \middle| \bigvee_{0\le k<j} \bigvee_{-2m+kw<i<kw} P_{i,k} \right.$$
$$\left. \bigvee \bigvee_{-2m+jw\le i\le -2m+(j-1)w+r} P_{i,j} \right)$$
$$\ge H\left(\bigvee_{(j-1)w-r\le i\le jw} P_{i,j} \middle| \bigvee_{0\le k<j} \bigvee_{-2m'+kw\le i<kw} P_{i,k} \right.$$
$$\left. \bigvee \bigvee_{-2m'+jw\le i\le -2m'+(j-1)w+r} P_{i,j} \right).$$

These observations together with the formula for $H^m(\vec{v})$ above shows $H^{m'}(\vec{v}) \leq H^m(\vec{v})$.

Case 2. $\vec{v} = \eta(1,0)$ for some real η.

We analogously denote by $H^m(\vec{v})$

$$\lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^{m} P_{i,j}\right).$$

We note that

$$\begin{aligned}
H^{m'}(\vec{v}) &= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{[n\eta]}; \bigvee_{j=-m}^{m+2(m'-m)} P_{i,j}\right) \\
&= \lim_{n\to\infty} \frac{1}{n}\left(H\left(\bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^{m} P_{i,j}\right)\right. \\
&\quad \left. + H\left(\bigvee_{i=1}^{[n\eta]} \bigvee_{j=m+1}^{m+2(m'-m)} P_{i,j} \,\middle|\, \bigvee_{i=0}^{[n\eta]} \bigvee_{j=-m}^{m} P_{i,j}\right)\right) \\
&\leq H^m(\vec{v}) + \lim_{n\to\infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} H\left(\bigvee_{i=0}^{[n\eta]} P_{i,j} \,\middle|\, \bigvee_{i=0}^{[n\eta]} P_{i,j-1}\right) \\
&\leq H^m(\vec{v}) + \lim_{n\to\infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} H\left(\bigvee_{i=0}^{r} P_{i,j} \vee \bigvee_{i=[n\eta]-r}^{[n\eta]} P_{i,j}\right) \\
&\leq H^m(\vec{v}) + \lim_{n\to\infty} \frac{1}{n} \sum_{j=m+1}^{m+2(m'-m)} 2r\tau \\
&= H^m(\vec{v}) + \lim_{n\to\infty} \frac{4r\tau(m'-m)}{n} \\
&= H^m(\vec{v}).
\end{aligned}$$

Since we have $H^{m'}(\vec{v}) \geq H^m(\vec{v})$ by definition, the proof is complete. □

Corollary 1. *If $\vec{v}$ is not a scalar multiple of* $(1,0)$, *then we have*

$$\begin{aligned}
&\left|\frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \bigvee_{-m+jw\leq i\leq m+jw} P_{i,j}\right) - \frac{1}{n} H\left(\bigvee_{j=0}^{n-1} \bigvee_{-m'+jw\leq i\leq m'+jw} P_{i,j}\right)\right| \\
&\leq \frac{1}{n} H\left(\bigvee_{m<|i|\leq m'} P_{i,0}\right) \\
&\leq \tau \frac{2(m'-m)}{n}.
\end{aligned}$$

Theorem 1. $h^c(\vec{v}) = h(\vec{v})$.

Proof. It is enough to show that $h^c(\vec{v}) - h(\vec{v})$ is small. If $\vec{v} = (x, y)$ where $y \neq 0$, then by rescaling, we may assume that $\vec{v} = (x, 1)$. Given any $\varepsilon > 0$, there exists θ such that if $\kappa \leq \theta$, then

(i)

$$\lim_{n\to\infty} \frac{1}{n} H \left(\bigvee_{0\leq j<n} \bigvee_{jx_\kappa \leq \imath \leq jx_{-\kappa}} P_{\imath,j} \right) < h^c(\vec{v}) + \varepsilon$$

(ii) $|x_{-\theta} - x_\theta| < \gamma$ where γ satisfies that $\gamma\tau < \varepsilon$. There exists m_0 such that if $m \geq m_0$, then

$$\lim \frac{1}{n} H \left(\bigvee_{\jmath=0}^{n-1} \bigvee_{-m+\jmath x \leq i \leq m+jx} P_{i,j} \right) = h(\vec{v}).$$

We choose n_o such that if $n \geq n_o$, then we have

(iii)

$$h(\vec{v}) - \varepsilon < \frac{1}{n} H \left(\bigvee_{0\leq j \leq n-1} \bigvee_{-m_o+jx \leq i \leq m_o+jx} P_{\imath,j} \right) \leq h(\vec{v}) + \varepsilon,$$

(iv)

$$h^c(\vec{v}) - 2\varepsilon < \frac{1}{n} H \left(\bigvee_{o\leq j\leq n-1} \bigvee_{jx_\theta \leq i \leq jx_{-\theta}} P_{i,j} \right) \leq h^c(\vec{v}) + 2\varepsilon,$$

(v)

$$\frac{1}{n} H \left(\bigvee_{o\leq j<K} \bigvee_{-m_o+jx<i<m_o+jx} P_{i,j} \right) < \varepsilon, \text{ where}$$

$$K = \max\{j : j|x_\theta - x| < m_o \text{ and } j|x_{-\theta} - x| < m_o\},$$

and

(vi)

$$\frac{1}{n} H \left(\bigvee_{o\leq j<n} \bigvee_{jx_\theta \leq i \leq jx_{-\theta}} P_{i,j} \right) \geq \frac{1}{n} H \left(\bigvee_{0\leq \jmath<n} \bigvee_{-m_o+jx \leq i \leq m_o+jx} P_{i,j} \right).$$

We compute

$$
\begin{aligned}
&|h^c(\vec{v}) - h(\vec{v})| \\
&\leq \left| \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{jx_\theta \leq i \leq jx_{-\theta}} P_{i,j} \right) \right. \\
&\qquad \left. - \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{-m_o - jx \leq i < m_o + jx} P_{i,j} \right) \right| + 3\varepsilon \\
&\leq \left| \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{n(x_\theta - x) + jx \leq i \leq n(x_{-\theta} - x) + jx} P_{i,j} \right) \right. \\
&\qquad \left. - \frac{1}{n} H \left(\bigvee_{o \leq j < n} \bigvee_{-m_o + jx \leq i < m_o + jx} P_{i,j} \right) \right| + 3\varepsilon \\
&\leq \frac{1}{n} H \left(\bigvee_{n(x_\theta - x) \leq i \leq n(x_{-\theta} - x)} P_{i,o} \right) + 3\varepsilon \\
&\leq \frac{1}{n} \gamma n \tau + 3\varepsilon.
\end{aligned}
$$

Hence we have

$$|h(\vec{v}) - h^c(\vec{v})| < 4\varepsilon.$$

In the case of $\vec{v} = (x, o)$, it is not hard to see that the idea of the second part of the proof of Lemma 1 combined with the idea of the proof above will give the desired result. □

Theorem 2. *If* $\sum_{m=0}^{\infty} H\left(P_{0,1} \,\middle|\, \bigvee_{-m \leq i \leq m} P_{i,0}\right)$ *is finite, then we have* $h^c(\vec{v}) = h(\vec{v})$.

Proof. We note that if we choose M so that

$$\sum_{m=M}^{\infty} H\left(P_{0,1} \,\middle|\, \bigvee_{-m \leq i \leq m} P_{i,0}\right) < \varepsilon,$$

then we get

$$\sum_{k=-m+M}^{m-M} H\left(P_{k,1} \,\middle|\, \bigvee_{-m \leq i \leq m} P_{i,0}\right) < 2\varepsilon,$$

for all $m > M$. Using this, it is easy to see that if $m_2 \geq m_1 \geq M$, we have that for any n,

$$\frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\bigvee_{i=m_2+jw}^{-m_2+jw}P_{i,j}\right) < \frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\bigvee_{i=-m_1+jw}^{m_1+jw}P_{i,j}\right) + 2\varepsilon + \frac{m_2-m_1}{n}\tau$$

where $\frac{m_2-m_1}{n}\tau$ comes from the difference between $\frac{1}{n}H\left(\bigvee_{i=-m_1}^{m_1}P_{i,0}\right)$ and $\frac{1}{n}H\left(\bigvee_{i=-m_2}^{m_2}P_{i,0}\right)$.

Hence for a given $\varepsilon > 0$, there exist m_o as in Theorem 1 such that for a sufficiently large n,

$$\begin{aligned}
&|h^c(\vec{v}) - h(\vec{v})| \\
&\leq \left|\frac{1}{n}H\left(\bigvee_{o\leq j<n}\ \bigvee_{n(x_\theta-x)+jx\leq i\leq n(x_{-\theta}-x)+jx}P_{i,j}\right)\right. \\
&\quad\left.-\frac{1}{n}H\left(\bigvee_{o\leq j<n}\ \bigvee_{-m_o+jx\leq i<m_o+jx}P_{i,j}\right)\right| + 3\varepsilon \\
&\leq \frac{1}{n}\gamma n\tau + 2\varepsilon + 3\varepsilon.
\end{aligned}$$

□

Corollary 2. *If V is a finitary code with finite expected code length, then $h^c(\vec{v}) = h(\vec{v})$.*

Proof. It is easy to see that a finitary code with finite expected code length satisfies the condition of Theorem 2. See [**Pa3**]. □

3. Main Theorem.

Let $\lambda = \mu_1 \times \mu_2$. We denote $\sum_{i=0}^{n-1}\varphi_k(T^i z)$ by $\varphi_k^n(z)$ for $k = 1$ or 2 and $z \in X_1$. Given two partitions, β_1 and β_2, we write $\beta_1 \leq \beta_2$ if β_2 is a finer partition than β_1.

Theorem 3. *$h(\widehat{T}) = h(T) + h(\vec{v})$ where $\vec{v} = \int \varphi\, d\mu = (\int \varphi_1\, d\mu, \int \varphi_2\, d\mu)$.*

Proof. Since $\int \varphi\, d\mu$ is finite, as in the case of a Z-valued skewing function, there exists φ' which is bounded and cohomologous to φ. Hence we may assume that φ is bounded. Let $|\varphi_1(z)| \leq L$ and $|\varphi_2(z)| \leq L$. Suppose $\vec{v} = \int \varphi\, d\mu = (x, y)$ where $y \neq 0$. We let α denote the generating partition of the base. Let β denote a partition of X_2. Both of the partitions α and β can be considered in a natural way to be a partition of $X_1 \times X_2$. For a given $z \in X_1$, we denote the set $\{(z,u) : u \in X_2\}$ by I_z.

Since

$$\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i(\alpha \vee \beta)\right) = \frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\alpha\right) + \frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta\middle|\bigvee_{i=0}^{n-1}\widehat{T}^i\alpha\right)$$

and

$$\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta\middle|\bigvee_{i=0}^{n-1}\widehat{T}^i\alpha\right) = \int \frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta|I_z\right)d\mu,$$

we have

$$\begin{aligned}
\sup_{\beta} h\left(\widehat{T}, \alpha^{\vee}\beta\right) &= \sup_{\beta}\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i(\alpha^{\vee}\beta)\right)\\
&= h\left(\widehat{T},\alpha\right) + \sup_{\beta_m}\lim_{n\to\infty}\int\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right)d\mu\\
&= h\left(\widehat{T},\alpha\right) + \lim_{m\to\infty}\lim_{n\to\infty}\int\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right)d\mu,
\end{aligned}$$

where β_m denote the partition $\bigvee_{i=-m}^{m}\bigvee_{j=0}^{L-1}P_{i,j}$.

We denote $\lim_{n\to\infty}\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right)$ by $h_z\left(\widehat{T},\beta_m\right)$.

As in Lemma 1, it is not hard to see that for sufficiently large m and m', we have

$$h_z\left(\widehat{T},\beta_m\right) = h_z\left(\widehat{T},\beta_{m'}\right).$$

We will show that for sufficiently large m,

$$\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right) \to h(\vec{v}) \text{ as } n\to\infty, \text{ for a.e. } z \in X_1.$$

We denote by x_ℓ the x-intercept of a line in $\mathbb{R}^2$ passing through $\varphi^\ell(z)$ with the same slope as $\vec{v}$. Let

$$\begin{aligned}
s_n &= \max\{x_1,\dots,x_n\} \text{ and}\\
t_n &= \min\{x_1,\dots,x_n\}.
\end{aligned}$$

Given $\varepsilon > 0$, let k_o be the integer such that if $k \geq k_o$, then we have

(i)

$$\left| h(\vec{v}) - \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{0}^{[ny]} \bigvee_{i=-k+jw}^{k+jw} P_{i,j} \right) \right| < \varepsilon.$$

Given any $\delta > 0$ and $\varepsilon > 0$, there exists n_o such that if $n \geq n_o$, then we have

(ii)

$$\mu E_1 = \mu\left\{ z : \left| \int \varphi \, d\mu - \frac{1}{n}\varphi^n(z) \right| < \delta \right\} > 1 - \varepsilon,$$

(iii)

$$\left| h(\vec{v}) - \frac{1}{n} H\left(\bigvee_{j=0}^{[ny]} \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j} \right) \right| < \varepsilon,$$

(iv)

$$\mu E_2 = \mu\left\{ z : \left| h_z\left(\widehat{T}, \beta_{k_o}\right) - \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_{k_o} | I_z \right) \right| < \varepsilon \right\} > 1 - \varepsilon,$$

(v) $k_o < \frac{\varepsilon}{2} n_o$,

and

(vi) $|s_n - t_n| < 2n\delta$.

We choose $\delta < \varepsilon^2$ and choose n_o satisfying (ii)-(vi) above. We fix m_o such that $k_o < (\varepsilon/2)n_o < m_o < \varepsilon n_o$. For notational convenience, we write m and n instead of m_o and n_o respectively. We note that

$$\begin{aligned}
&\bigvee_{j=0}^{n-1} \widehat{T}^j \beta_m \text{ on } I_Z \\
&\leq \beta_m \bigvee F^{\varphi(z)}(\beta_m) \bigvee F^{\varphi^2(z)}(\beta_m) \bigvee \cdots \bigvee F^{\varphi^{n-1}(z)}(\beta_m) \text{ on } I_z \\
&\leq \bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1} \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j} \text{ on } I_Z.
\end{aligned}$$

Since t_n and s_n satisfy that

$$|(t_n + m) - (s_n - m)| = |2m + t_n - s_n| > |2m - 2n\delta| > k_o$$

and

$$|(s_n + m) - (t_n - m)| < \varepsilon n,$$

if $z \in E_1$, then by our Corollary and (ii), we have

$$\begin{aligned}
&\left|\frac{1}{n}H\left(\bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1}\ \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j}\right) - \frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\ \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j}\right)\right| \\
&< \frac{1}{n}H\left(\bigvee_{i=t_n-m}^{s_n+m} P_{i,0}\right) + \frac{1}{n}H\left(\bigvee_{j=q_1}^{q_2}\ \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j}\right) \\
&< \frac{1}{n}\tau\varepsilon n + \frac{1}{n}(q_2 - q_1)\tau(w+2r) \\
&< \tau(\varepsilon + \delta(w+2r)),
\end{aligned}$$

where $q_1 = \min\{[ny],\ \varphi_2^{n-1}(z)+L-1\}$ and $q_2 = \max\{[ny],\ \varphi_2^{n-1}(z)+L-1\}$.
Hence we have

$$\begin{aligned}
&\left|\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right) - h(\vec{v})\right| \\
&\le \left|\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right) - \frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\ \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j}\right)\right| + \varepsilon \\
&\le \left|\frac{1}{n}H\left(\bigvee_{j=0}^{\varphi_2^{n-1}(z)+L-1}\ \bigvee_{i=t_n-m+jw}^{s_n+m+jw} P_{i,j}\right) - \frac{1}{n}H\left(\bigvee_{j=0}^{[ny]}\ \bigvee_{i=-k_o+jw}^{k_o+jw} P_{i,j}\right)\right| + \varepsilon \\
&\le \tau(\varepsilon + \delta(w+2r)) + \varepsilon.
\end{aligned}$$

Let $E = E_1 \cap E_2$. If $z \in E$, then by our choice of m and Corollary 1, we have

$$\begin{aligned}
&\left|\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right) - h_z\left(\widehat{T},\ \beta_m\right)\right| \\
&\le \left|\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_m|I_z\right) - \frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_k|I_z\right)\right| \\
&\quad + \left|\frac{1}{n}H\left(\bigvee_{i=0}^{n-1}\widehat{T}^i\beta_k|I_z\right) - h_z\left(\widehat{T},\ \beta_k\right)\right| + \left|h_z\left(\widehat{T},\ \beta_k\right) - h_z\left(\widehat{T},\ \beta_m\right)\right| \\
&\le \varepsilon + \varepsilon + \frac{1}{n}m\tau < \varepsilon(2+\tau).
\end{aligned}$$

Since φ_1 and φ_2 are bounded, it is easy to see that there exists ω such that $\left|h_z\left(\widehat{T},\ \beta\right)\right| < \omega$ for all β and all z. We may also assume that $h(\vec{v})$ is bounded above by ω. Now we compute

$$
\begin{aligned}
&\left|\sup_\beta \int h_z\left(\widehat{T},\, \beta\right)\, d\mu - h(\vec{v})\right| \\
&\leq \int_E \left|h_z\left(\widehat{T},\, \beta_m\right) - h(\vec{v})\right|\, d\mu + \sup_\beta \int_{E^c} \left|h_z\left(\widehat{T},\, \beta\right) - h(\vec{v})\right|\, d\mu + \varepsilon \\
&\leq \int_E \left|h_z\left(\widehat{T},\, \beta_m\right) - \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right)\right|\, d\mu \\
&\quad + \int_E \left|\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right) - h(\vec{v})\right|\, d\mu \\
&\quad + \sup_\beta \int_{E^c} \left|h_z\left(\widehat{T},\, \beta\right) - h(\vec{v})\right|\, d\mu + \varepsilon \\
&\leq \varepsilon(2+\tau) + \tau(\varepsilon + \delta(w+2r)) + \varepsilon + 4\omega\varepsilon + \varepsilon \\
&\leq \varepsilon(4 + 2\tau + \tau(w+2r) + 4\omega).
\end{aligned}
$$

In the case when $\vec{v} = \int \varphi = \eta(1,0)$ for some real number η, we need to argue differently. We may assume $\eta > 0$. We construct φ' which is cohomologous to φ as follows. Let $\varphi' = (\varphi_1', \varphi_2')$.

(i) φ_1' takes the values $[\eta]-1$, $[\eta]$ and $[\eta]+1$
φ_2' takes the values $-1, 0, 1$.

(ii) In an orbit of a point, φ_2' value, 1 or -1, follows its value 0.

(iii) We use the ergodic theorem to construct φ_2' so that it takes the value 0 for all z's except a set of small measure.

Hence we may assume that φ satisfies these properties.

We let $\beta_m = \bigvee_{i=0}^{[\eta]} \bigvee_{j=-m}^{m} P_{i,j}$. Recall that r denote the size of the block map.

As in the previous case,we choose m_o so that if $m \geq m_o$, then

(i) $m_o \geq 10r$,

(ii) $|h(\vec{v}) - H^m(\vec{v})| < \varepsilon$,

(iii) $\mu\left\{z : \left|\sup_\beta \int h_z\left(\widehat{T},\, \beta\right) - h_z\left(\widehat{T},\, \beta_m\right)\right| < \varepsilon\right\} > 1 - \varepsilon$.
We fix $m \geq m_o$. We choose n_o so that if $n \geq n_o$, then

(iv) $\mu\left\{z : \left|\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right) - h_z\left(\widehat{T}, \beta_m\right)\right| < \varepsilon\right\} > 1 - \varepsilon$,

(v) $\mu\left\{z : \left|\frac{1}{n} \sum_{i=0}^{n-1} \varphi\left(T^i(z)\right) - \int \varphi\, d\mu\right| < \varepsilon\right\} > 1 - \varepsilon$,
and

(vi) $\mu\left\{z : \left|\frac{1}{n} \sum_{i=0}^{k} \varphi_2\left(T^i(z)\right)\right| < \varepsilon \text{ for all } 0 \leq k < n\right\} > 1 - \varepsilon$.

Let E denote the set satisfying the above conditions, (iii), (iv), (v) and (vi). We have $\mu E > 1 - 4\varepsilon$. Let $z \in E$.

Let

$$u = \max\left\{\sum_{i=0}^{k} \varphi_2\left(T^i(z)\right) : k = 0, 1, \dots, n-1\right\}$$

and

$$v = \min\left\{\sum_{i=0}^{k} \varphi_2\left(T^i(z)\right) : k = 0, 1, \dots, n-1\right\}.$$

Since $\eta > 0$, there exists $i_o = \max\{k : \varphi_1^k(z) \le i\}$ for a.e. $z \in X_1$. We denote by $\Psi_2^i(z)$

$$\max\left\{\sum_{\zeta=0}^{k} \varphi_2\left(T^\zeta(z)\right) : \ 0 \le k \le i_o, \ i - [\eta] \le \varphi_1^k(z) \le i\right\}.$$

Now we compute

$$\begin{aligned}
&\frac{1}{n} H\left(\bigvee_{j=-m}^{m} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right)\\
&= \frac{1}{n} H\left(\bigvee_{j=-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right)\\
&\le \frac{1}{n} H\left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\Psi_2^i(z)}^{u+m} P_{i,j}\right)\\
&\le \frac{1}{n}\left(H\left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\Psi_2^i(z)}^{m+\Psi_2^i(z)} P_{i,j}\right) + 2 \cdot 2(\varepsilon n) \cdot \tau \cdot r\right)\\
&\le \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right) + 4\varepsilon r \tau.
\end{aligned}$$

The second to the last inequality is clear because by the condition (i) on m_o, we have

$$\begin{aligned}
&H\left(\bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\psi_2^i(z)}^{u+m} P_{i,j} \,\middle|\, \bigvee_{i=0}^{\varphi_1^n(z)} \bigvee_{j=-m+\psi_2^i(z)}^{m+\psi_2^i(z)} P_{i,j}\right)\\
&\le H\left(\bigvee_{j=m}^{u+m} \bigvee_{i=0}^{r-1} P_{i,j} \vee \bigvee_{j=v+m}^{u+m} \bigvee_{i=\varphi_1^n(z)-r+1}^{\varphi_1^n(z)} P_{i,j}\right)
\end{aligned}$$

$$\leq u \cdot r \cdot \tau + (u-v) \cdot r \cdot \tau.$$

Since the following inequality is also true

$$\begin{aligned}
&\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right) \\
&\leq \frac{1}{n} H\left(\bigvee_{j=-m+v}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right) \\
&= \frac{1}{n} H\left(\bigvee_{j=-m}^{m+u-v} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right) \\
&\leq \frac{1}{n} H\left(\bigvee_{j=-m}^{m} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right) + \frac{2}{n}(u-v) r \cdot \tau \\
&= \frac{1}{n} H\left(\bigvee_{j=-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right) + 4\varepsilon r \tau,
\end{aligned}$$

we have

$$\left|\frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \widehat{T}^i \beta_m | I_z\right) - \frac{1}{n} H\left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right)\right| < 4\varepsilon r \tau.$$

We note that

$$\frac{1}{n} H\left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right) = \frac{\varphi_1^n(z)}{n} \frac{1}{\varphi_1^n(z)} H\left(\bigvee_{-m+u}^{m+u} \bigvee_{i=0}^{\varphi_1^n(z)} P_{i,j}\right)$$

converges to $h(\vec{v})$.

As in the case of $\vec{v} = \int \varphi \, d\mu = (x, y)$ where $y \neq 0$, it is now clear that

$$\left|\sup_{\beta} \int h_z\left(\widehat{T}, \beta\right) d\mu - h(\vec{v})\right|$$

can be made arbitraily small. □

Similarly we can prove the following theorem.

Theorem 4. *If* $\sum_{m=0}^{\infty} H\left(P_{0,1} \middle| \bigvee_{-m \leq i \leq m} P_{i,0}\right)$ *is finite, then we have* $h(\widehat{T}) = h(T) + h(\vec{v})$ *where* $\vec{v} = \int \varphi \, d\mu = \left(\int \varphi_1 \, d\mu, \int \varphi_2 \, d\mu\right)$.

The following Corollaries are also almost immediate from the proof of Theorem 3.

Corollary 3. *If* $\sum_{m=0}^{\infty} H\left(P_{0,1} \middle| \bigvee_{-k\le j\le k} \bigvee_{-m\le i\le m} P_{i,j}\right)$ *is finite for some* k, *then we have* $h(\widehat{T}) = h(T) + h(\vec{v})$ *where* $\vec{v}$ *is given as above.*

Corollary 4. *If a fiber* Z^2*-action,* F*, satisfies the condition of Corollary 3 after a linear transformation by a matrix* A *in* $SL(2,Z)$*, that is,* $A \circ F$ *satisfies the condition, then we have the above formula in Corollary 3 for the entropy.*

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Received February 10, 1993 and revised June 24, 1993. This research has been supported in part by NSF DMS 8902080 and GARC-KOSEF.

AJOU UNIVERSITY
SUWON, 441-749
KOREA