

ON THE MINIMAL FREE RESOLUTION OF GENERAL EMBEDDINGS OF CURVES

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Here we study the minimal free resolution of general embeddings in \mathbf{P}^n of genus g curves with general moduli. We prove that if p is an integer with, roughly, $g \leq n^2/(2p+2)$, then the embedding has the property N_p , i.e., the first p pieces of the resolution are as simple as possible.

We work over an algebraically closed field. Let C be a smooth curve embedded in \mathbf{P}^n . We are interested in the minimal free resolution of C . Here we will consider the case in which the curve has general moduli and the embedding is general. Recall the following definition ([5], [6]).

Definition 0.1. Let $C \subset \mathbf{P}^n$ be a reduced curve; fix an integer $p \geq 1$; C satisfies the property N_p if C is arithmetically Cohen - Macaulay and for every integer i with $1 \leq i \leq p$ the i^{th} -sheaf appearing in the minimal free resolution of the homogeneous ideal of C is the direct sum of line bundles of degree $-i - 1$.

For instance if we say that N_0 means “ C is arithmetically Cohen-Macaulay”, then N_1 means that the curve C is N_0 and its homogeneous ideal is generated by quadrics. Furthermore, if $p > 0$, then N_p implies N_{p-1} .

In this paper, using degeneration techniques, we will prove the following results (Theorems 0.2 and 0.3).

Theorem 0.2. Fix an integer $p \geq 1$. For every integer u , set:

$$(1) \quad \alpha_p(u) := (u^2)/(2p+2) - (u/2).$$

Fix an integer $n \geq 3$ with $n \geq p+1$, and set:

$$(2) \quad G_p(n) := \alpha_p((p+1)[n/(p+1)])$$

where $[y]$ is the greatest integer $\leq y$. Then for every integer $g \leq G_p(n)$ the general linearly normal non special curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $\deg(C) = g + n$ satisfies the property N_p .

Note that $G_p(n)$ has order $(n^2)/(2p+2)$ and hence $d := g + n$ is usually much smaller than $2g + p$ if n is much larger than p .

In the case of special linearly normal embeddings we have the following “conditional” result.

Theorem 0.3. *Fix an integer $p \geq 1$ and an integer $s > 2p$. Assume that a general canonical curve of genus $s + 1$ in \mathbf{P}^s has the property N_p . Fix an integer $n > p + s$; write $n = s + a(p + 1) + b$ with a, b integers and $0 \leq b \leq p$. Set:*

$$(3) \quad S_{p,s}(n) := (a + 1)s + 1 + a(a - 1)(p + 1)/2.$$

Then for every integer g with $s + 1 \leq g \leq S_{p,s}(n)$ a general linearly normal curve $C \subset \mathbf{P}^n$ with $p_a(C) = g$ and $h^1(C, \mathcal{O}_C(1)) = 1$ (hence of degree $g + n - 1$) has the property N_p .

Quoting existing references on N_p for the canonical model of a curve, C , with general moduli (e.g. [7], [3]), one obtains corresponding statements for special embeddings of C . We stress that the proof of Theorems 0.2 and 0.3, being a kind of induction on n using as inductive tool Lemma 1.2, may be used to obtain many other cases not covered by the statements of 0.2 and 0.3; the proof of 0.2 should be helpful to the reader interested in other cases.

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1. In this section we will prove Theorems 0.2 and 0.3. A key quotation for the proofs here is the criterion for condition N_p given in [6, Prop. 1.3.3]; the base field for all [6] was the complex number field, but the statement of the quoted criterion 1.3.3 works in arbitrary characteristic because each of the steps of its proof either works verbatim in positive characteristic or it is known to hold in general; furthermore, [6, Prop. 1.3.3], although stated only for smooth curves, works with the same proof for all reduced curves. It is the use of this criterion for N_p which gives the condition of linear normality in the statements of 0.2, 0.3 and 1.3.

Fix an integer $n \geq 3$; following the notations of [6], M_n will denote the rank n vector bundle on \mathbf{P}^n with $M_n(-1)$ isomorphic to the cotangent bundle.

The following result is well known (see e.g. [1, Lemma 1.3]):

Lemma 1.1. *Let $D \subset \mathbf{P}^n$ be a rational normal curve. Then $M \mid D$ is the direct sum of n line bundles of degree -1 .*

Lemma 1.2. *Fix integers n, p, t, j and m with $n \geq t \geq j > p > 1$ and $m \geq 0$. Let $C \subset \mathbf{P}^{n-j}$ be a reduced curve satisfying condition N_p . See \mathbf{P}^{n-j}*

as a linear subspace V of \mathbf{P}^n and let D be a smooth rational curve of degree t in \mathbf{P}^n , D spanning a linear subspace W of dimension t , with $\dim(V \cap W) = t - j$ and $\text{card}(C \cap D) = t - j + 1$, D intersecting quasi transversally C , (hence $C \cup D$ spanning \mathbf{P}^n); assume that $H^1(C, (\Lambda^{(p+1)}M_{n-j}(1))(m) | C) = 0$; then $H^1(C \cup D, (\Lambda^{(p+1)}M_n(1))(m) | (C \cup D)) = 0$.

Proof. Consider the following Mayer-Vietoris exact sequence:

$$(4) \quad \begin{aligned} 0 &\rightarrow (\Lambda^{(p+1)}M_n(1))(m) | (C \cup D) \\ &\rightarrow (\Lambda^{(p+1)}M_n(1))(m) | C \oplus (\Lambda^{(p+1)}M_n(1))(m) | D \\ &\rightarrow (\Lambda^{(p+1)}M_n(1))(m) | (C \cap D) \rightarrow 0. \end{aligned}$$

Note that $M_n | C \cong (M_{n-j} | C) \oplus \mathbf{O}_C^j$. Hence $\Lambda^{(p+1)}M_n | C$ is a direct sum of trivial factors and factors isomorphic to $\Lambda^u M_{n-j} | C$ for some integer $u \leq p$. Since N_p implies N_u for every $u < p$ we have

$$H^1(C, (\Lambda^{(p+1)}M_n(1))(m) | C) = 0.$$

Note that $M_n | D$ is the direct sum of t line bundles of degree -1 and $n - t$ copies of \mathbf{O}_D . Hence $(\Lambda^{(p+1)}M_n(1))(m) | D$ is a direct sum of line bundles of degree at least $mt + t - p - 1$. To conclude it is sufficient to use (4) and to check (see below) the surjectivity of the restriction map $p : H^0((\Lambda^{(p+1)}M_n(1))(m) | D) \rightarrow H^0((\Lambda^{(p+1)}M_n(1))(m) | (C \cap D))$. For the surjectivity of p , note that, since $\text{deg}(\mathbf{O}_D(-(C \cap D))) = -\text{card}(C \cap D)$ and $j > p$, we have $H^1(D, E(-(C \cap D))) = 0$ for every line bundle E on D with $\text{deg}(E) \geq t - p - 1$. \square

Note that $p_a(C \cup D) = p_a(C) + t - j$ and that $C \cup D$ spans \mathbf{P}^n . An easy Mayer-Vietoris exact sequence gives $h^1(C \cup D, \mathbf{O}_{C \cup D}(1)) = h^1(C, \mathbf{O}_C(1))$ and $h^0(C \cup D, \mathbf{O}_{C \cup D}(1)) = n + 1$. Hence any smoothing of $C \cup D$ will give linearly normal smooth curves “near” $C \cup D$.

Proof of 0.2. We fix the integer p . First we will prove N_p for the genus $G_p(n)$ and every integer n . Set $n = a(p+1) + r$. As a starting point we assume the property N_p for the rational normal curve of \mathbf{P}^{p+1} . Of course, better results and other example can be obtained using other curves in \mathbf{P}^{p+1} with property N_p , e.g. the ones given by an important theorem of M. Green (proven in any characteristic in [5, Prop. 3.2]) saying that a linearly normal embedding of degree at least $2k + 1 + p$ of any smooth curve of genus k has the property N_p . The main property of the function G_p is the property $G_p(m + p + 1) = G_p(m) + (p + 1)[m/(p + 1)]$; its normalization $G_p(p + 1) = 0$ comes from the choice we made for the starting point of the induction. Then we apply $(a - 1)$ times Lemma 1.2, always with $j = p + 1$ and at each step with the maximal

possible t ; in the k^{th} -step we pass from a curve of genus $G_p((k+1)(p+1))$ in $\mathbf{P}^{(k+1)(p+1)}$ to a curve of arithmetic genus $G_p((k+2)(p+1))$ in $\mathbf{P}^{(k+2)(p+1)}$. Then we apply Lemma 1.2 for the integers n, t, j with $t = n$ and $j = r$, concluding the case $g = G_p(n)$. Now we will check N_p in \mathbf{P}^n for any non negative integer $g < G_p(n)$. There is an integer $x \leq n - p - 1$, x divisible by $p + 1$, with $G_p(x) \leq g < G_p(x + 1)$, say $x = m(p + 1)$. We take a curve, C , in \mathbf{P}^x with N_p and genus $G_p(x)$ and we apply Lemma 1.2 for the integers n, t, j , with $j = n - x$ and $t - j = g - G_p(x)$.

At each step of the induction the possibility of deforming the reducible curve to a smooth linearly normal curve (i.e., a smooth curve, T , with the correct $h^1(T, \mathbf{O}_T(1))$) is proven (in a much stronger form than needed here and for 0.3) independently in several papers: see for instance [BE, Lemma 1.2, (1)], or [4, Th.4.1], or [8, §5]; one can also see a discussion of the way the smoothing concerns the moduli spaces in [8] and [2, §1, §2, §3]. \square

Note that the bound (2) on the genus is just a byproduct of the inductive proof.

Proof of 0.3. The proof of 0.2 works with the following modifications. Instead of quoting [6, Prop. 1.3.3], use Remark (2) after the proof of [6, Prop. 1.3.3]. The starting point of the induction is a general canonical curve in \mathbf{P}^s which in the statement of 0.3 is assumed to have property N_p . To check the smoothability of reduced curves, use for instance [2, Lemma 1.2 (1)]. To check the condition " $h^1(T, \mathbf{O}_T(1)) = 1$ ", first (as remarked before the proof of 0.2) use a Mayer - Vietoris exact sequence to prove it for the reducible curves; then use semicontinuity to obtain that $h^1(T, \mathbf{O}_T(1)) \leq 1$ if T is general, while $h^1(T, \mathbf{O}_T(1)) \geq 1$ by Riemann - Roch. \square

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