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THE WEIL REPRESENTATION AND GAUSS SUMS

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We use the Weil representation to evaluate certain Gauss sums over a local field, up to ± 1. Also we construct a cocycle on $\mathrm{Sp}(2m, \mathbb{R})$ with a simple formula on the maximal compact torus and we show how to lift homomorphisms $j : \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{Sp}(2m, \mathbb{R})$ to the double covers of these groups.

1. Introduction.

Let F be a self-dual local field of char $\neq 2$, for instance $F = \mathbb{R}$, $\mathbb{C}$, or a finite extension of $\mathbb{Q}_p$. For most of the paper we will assume $F \neq \mathbb{C}$. Let χ be a nontrivial additive character of F. Then all additive characters of F have the form $\lambda\chi$ for some $\lambda \in F$, where $\lambda\chi(t) = \chi(\lambda t)$. We consider the following unitary operators on $L^2(F^m)$:

$$(\mathbf{a}(A)\Phi)(X) = |\det A|_F^{1/2}\, \Phi(XA) \qquad \text{for } A \in \mathrm{GL}_m(F), \tag{1.1}$$

$$\mathbf{n}(B)\Phi(X) = \chi(XBX^T/2)\, \Phi(X) \qquad \text{for } B = B^T \in \mathrm{M}_m(F), \tag{1.2}$$

(1.3)
$$(\mathcal{F}_j\Phi)(X) = \int_{F^j} \Phi(Y_1, \dots, Y_j, X_{j+1}, \dots, X_m)\, \chi(X_1Y_1 + \cdots + X_jY_j)\, dY,$$

$$(\iota(t)\Phi)(X) = t\Phi(X), \qquad t \in \mathbf{T} = \{z \in \mathbb{C} \mid z\bar{z} = 1\}. \tag{1.4}$$

Here Φ is a nice function in $L^2(F^m)$ (to be precise, Φ belongs to the Schwartz space $\mathcal{S}(F^m)$), dY is an additive Haar measure on F^j normalized so that $\mathcal{F}_j^2 = \mathbf{a}(\mathrm{diag}\{-I_j, I_{m-j}\})$ for $0 \leq j \leq m$, and $|a|_F$ for $a \in F$ is the modulus function, determined by $d(ya) = |a|_F\, dy$ for a Haar measure dy on $(F, +)$. All our vectors are row vectors. We will usually suppress the symbol ι; that is, identify t with $\iota(t)$ for $t \in \mathbf{T}$. Let $\mathrm{Mp} = \mathrm{Mp}(F^m)$ be the topological group generated by all the above operators. We call this the metaplectic group. This group is independent of χ since $\lambda\chi(XBX^T/2) = \chi(X(\lambda B)X^T/2)$ and $\mathbf{a}(\left(\begin{smallmatrix}\lambda I_j & \\ & I_{m-j}\end{smallmatrix}\right))\mathcal{F}_{j,\chi} = \mathcal{F}_{j,\lambda\chi}$, where we have added a subscript to the Fourier transform $\mathcal{F}_j$ for clarification. Let $\mathrm{Sp} = \mathrm{Sp}(2m, F)$ denote the symplectic group consisting of all $2m \times 2m$ matrices τ such that $\tau^T w_m \tau = w_m$, where $w_m = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, $I = I_m$. This group is generated by the matrices $\{\underline{a}(A) \mid A \in \mathrm{GL}_m(F)\}$, $\{\underline{n}(B) \mid B = B^T \in \mathrm{M}_m(F)\}$, and w_m, where

$$\underline{a}(A) = \begin{pmatrix} A & \\ & A^{-T} \end{pmatrix} \quad \underline{n}(B) = \begin{pmatrix} I & B \\ & I \end{pmatrix} \quad (A \in \mathrm{GL}_m(F),\ B = B^T \in \mathrm{M}_m(F)).$$

Theorem (Segal, Shale, Weil). *There is a homomorphism $\pi = \pi_\chi : \mathrm{Mp} \to \mathrm{Sp}$ such that*

$$\pi(\mathbf{a}(A)) = \underline{a}(A), \qquad \pi(\mathbf{n}(B)) = \underline{n}(B), \qquad \pi(\iota(t)) = I_{2m}, \qquad \pi(\mathcal{F}_j) = w_j,$$

where

$$(1.5) \qquad w_j = \begin{pmatrix} 0_j & & I_j & \\ & I_{m-j} & & 0_{m-j} \\ -I_j & & 0_j & \\ & 0_{m-j} & & I_{m-j} \end{pmatrix}.$$

Moreover, there is an exact sequence of topological groups

$$(1.6) \qquad 1 \to \mathbf{T} \xrightarrow{\iota} \mathrm{Mp} \xrightarrow{\pi} \mathrm{Sp} \to 1.$$

If $F = \mathbb{C}$, this sequence splits. If $F \neq \mathbb{C}$ then Mp contains a subgroup $\widetilde{\mathrm{Sp}} = \widetilde{\mathrm{Sp}}(2m, F)$ such that $\pi|_{\widetilde{\mathrm{Sp}}} : \widetilde{\mathrm{Sp}} \to \mathrm{Sp}$ is a nontrivial two-fold cover of Sp.

From now on assume $F \neq \mathbb{C}$. The realization $\widetilde{\mathrm{Sp}} \subset U(L^2(F^m))$ is known as the Weil, oscillator, or Segal-Weil-Shale representation. The group Mp and the projection π have much better definitions in terms of a certain centralizing property these operators have with respect to a unitary representation of the Heisenberg group; see [**R1**, §3.2]. Note that

$$(1.7) \qquad \pi_{\lambda\chi}(\sigma) = \begin{pmatrix} I & \\ & \lambda I \end{pmatrix} \pi_\chi(\sigma) \begin{pmatrix} I & \\ & \lambda^{-1} I \end{pmatrix} \qquad \text{for } \sigma \in \mathrm{Mp}.$$

The main result of this paper is that a certain Gauss sum is computed up to ± 1. The idea of the proof is to compare two sections of the homomorphism π. If $F = \mathbb{R}$ or if F is a nonarchimedean field whose residue characteristic is $\neq 2$ then there is a splitting homomorphism $\mathbf{k} : K \to \mathrm{Mp}$, where K is a certain maximal compact subgroup of Sp. This section can be compared to the standard section $r_0 : \mathrm{Sp} \to \mathrm{Mp}$ (see (4.1)) which was defined by Rao. Define $x_0 : K \to \mathbf{T}$ by the formula $r_0(k) = x_0(k)\,\mathbf{k}(k)$. We will find an expression for $x_0(k)$ as a Gauss sum. The operators in $\widetilde{\mathrm{Sp}}$ are known explicitly, thus $y_0(g)$ and $y(k)$ may be found (Prop. 2 and Lemma 7) such that $y_0(g)\, r_0(g) \in \widetilde{\mathrm{Sp}}$ and $y(k)\, \mathbf{k}(k) \in \widetilde{\mathrm{Sp}}$ for each $g \in \mathrm{Sp}$, $k \in K$. Then $x_0(k)\, y_0(k)\, y(k)^{-1} \in \widetilde{\mathrm{Sp}} \cap \mathbf{T} = \{\pm 1\}$. In this way the exact value of the Gauss sum $x_0(k)$ may be computed, up to ± 1. A technical difficulty is that one must find an explicit formula for the Bruhat decomposition on K in order to compare the sections r_0 and $\mathbf{k}$; this is done in Lemma 9.

Now we state our result explicitly. Let $F = \mathbb{R}$ or $F =$ a self-dual nonarchimedean local field such that 2 is a unit in the ring $\mathfrak{o}_F$ of algebraic integers of F. Let $\chi(t) = e^{2\pi i t}$ if $F = \mathbb{R}$, and let χ be any additive character such that $\mathfrak{a}_\chi = \mathfrak{o}_F$ if F is nonarchimedean, where

$$\mathfrak{a}_\chi = \{ x \in F \mid \chi(xy) = 1 \quad \text{for all } y \in \mathfrak{o}_F \}. \tag{1.8}$$

For example, if $F = \mathbb{Q}_p$, one could take $\chi = \chi_p$ to be the unique additive character such that $\chi_p(a/p^n) = e^{-2\pi i a/p^n}$ when $a, n \in \mathbb{Z}$, and if F is a finite extension of $\mathbb{Q}_p$ one could take $\chi = \lambda \chi_p \circ \mathrm{tr}_{F/\mathbb{Q}_p}$, where λ is a generator for the inverse different of F. Let $K = \mathrm{Sp}(2m, \mathfrak{o}_F)$ if F is nonarchimedean and

$$K = \left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \mid (\alpha + i\beta)(\alpha - i\beta)^T = I_m \right\} \qquad \text{if } F = \mathbb{R}.$$

Define $\Phi_0 \in L^2(F^m)$ to be the characteristic function of $\mathfrak{o}_F^m$ if F is nonarchimedean and $\Phi_0(X) = e^{-\pi X \cdot X}$ if $F = \mathbb{R}$. Let $k = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in K$ and $j = \mathrm{rank}(C)$. It is not hard to see (Lemma 8) that there exist $\pi_1, \pi_2 \in \mathrm{SO}(m, F)$ such that all entries of π_i are $0, 1$, or -1 and such that the top left $j \times j$ minor of $\pi_1 C \pi_2$ is invertible. Let

$$\pi_1 C \pi_2 = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}, \qquad \pi_1 D \pi_2 = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}, \tag{1.9}$$

where C_1, D_1 are $j \times j$ matrices and C_4, D_4 are $(m-j) \times (m-j)$ matrices. For $a, b \in F^\times$ let $(a,b)_F \in \{\pm 1\}$ denote the Hilbert symbol: $(a,b)_F = 1$ or -1 according as a is or is not a norm in $F(\sqrt{b})$. In particular, if $F = \mathbb{R}$ then $(a,b)_\mathbb{R} = -1$ iff a and b are both negative.

Main Theorem. *Let $k \in K - P$ and $\pi_1, \pi_2, C_1, C_2, \ldots, D_4$ be as above. Put $\eta = \det \left(\begin{smallmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{smallmatrix}\right)$. Then $\eta \neq 0$, and the quantity*

(1.10)
$$x_0(k) = |\det C_1|\, |\eta|^{-1/2} \int_{F^j} \chi(y(D_1 {C_1}^T + D_2 {C_2}^T) y^T/2)\, \Phi_0(yC_1, yC_2)\, dy$$

is independent of the choice of π_1 and π_2. Furthermore, $x_0(k)^2 = (\eta, -1)_F$ if F is nonarchimedean and $x_0(k)^2 = \mathrm{sign}(\eta)\, (-i)^j \det(D - iC)$ if $F = \mathbb{R}$.

In §2 we discuss the Weil index, which is fundamental to the study of the Weil representation, and we give a new method to compute the operators belonging to $\widetilde{\mathrm{Sp}}$. In §3 we discuss splittings of the maximal compact subgroup of Sp into Mp. The main theorem is proved in §4, and as an application we construct in Proposition 10 a section $r_+ : \mathrm{Sp} \to \widetilde{\mathrm{Sp}}$ such that the associated cocycle c_+ has the following nice properties: a) $c_+(pg_1, g_2) = c_+(g_1, g_2) = c_+(g_1, g_2p)$ for all $p = \left(\begin{smallmatrix} A & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{Sp}$ with $\det A > 0$ and all $g_1, g_2 \in \mathrm{Sp}$; b) c_+ has a simple formula on the standard maximal compact torus of Sp; and c) c_+ coincides with the Kubota cocycle when $m = 1$. In §5 we show how to lift homomorphisms $j : \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{Sp}(2m, \mathbb{R})$ to the double covers of these groups.

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2. The Weil index.

We first show that $\widetilde{\mathrm{Sp}}$ is equal to the commutator subgroup of Mp (this fact is known). By computing some commutators explicitly, we are led to consider some products of Gauss sums which Rao calls the Weil index. We will give a new proof that the Weil index defines a homomorphism of the Witt group and a new way to compute the operators in $\widetilde{\mathrm{Sp}}$. If H is any group, let (H, H) denote its commutator.

Lemma 1. *$\widetilde{\mathrm{Sp}}$ is equal to the commutator subgroup of Mp. If G_1 is a subgroup of Mp and $\pi(G_1) = \mathrm{Sp}$ then $G_1 = H\widetilde{\mathrm{Sp}} = H(\mathrm{Mp}, \mathrm{Mp})$, where $H = G_1 \cap \mathbf{T}$. Also, $\widetilde{\mathrm{Sp}} = (\widetilde{\mathrm{Sp}}, \widetilde{\mathrm{Sp}})$.*

Proof. Let $G_0 = (\mathrm{Mp}, \mathrm{Mp})$. Then $\pi|G_0$ is surjective, because $\mathrm{Sp} = (\mathrm{Sp}, \mathrm{Sp})$. Now let G_1 be any subgroup of Mp such that the restriction of π to G_1 is surjective and let $H = G_1 \cap \mathbf{T}$. Then (G_1, G_1) contains G_0 because given any A, B in Mp there are constants t_1 and t_2 in $\mathbf{T}$ such that t_1A and t_2B belong to G_1; thus $ABA^{-1}B^{-1} = (t_1A)(t_2B)(t_1A)^{-1}(t_2B)^{-1} \in (G_1, G_1)$. In particular, $G_0 \subset (\widetilde{\mathrm{Sp}}, \widetilde{\mathrm{Sp}}) \subset \widetilde{\mathrm{Sp}}$. If the inclusion of G_0 in $\widetilde{\mathrm{Sp}}$ were proper, then the exact sequence (1.6) would be split. So $\widetilde{\mathrm{Sp}} = (\widetilde{\mathrm{Sp}}, \widetilde{\mathrm{Sp}}) = G_0 \subset G_1 \subset \mathrm{Mp}$. Given any $g \in G_1$ there exists $a \in \mathbf{T}$ such that $ag \in \widetilde{\mathrm{Sp}}$. Since $\widetilde{\mathrm{Sp}} \subset G_1$, $a = (ag)g^{-1} \in G_1 \cap \mathbf{T} = H$. Since $g = a^{-1}(ag)$, we see $G_1 \subset H\widetilde{\mathrm{Sp}} \subset G_1$, □

Let P be the subgroup of matrices $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}$ such that $C = 0$. Then P is generated by $\underline{a}(A)\,\underline{n}(B)$ such that $A \in \mathrm{GL}_m(F)$ and $B = B^T \in \mathrm{M}_m(F)$, and the section $r_P : P \to \mathrm{Mp}$ given by

$$r_P(\underline{a}(A)\,\underline{n}(B)) = \mathbf{a}(A)\,\mathbf{n}(B) \tag{2.1}$$

is easily seen to be a homomorphism. Since P and $\{w_j \mid j = 0, \dots, m\}$ generate Sp ([**R1**, Lemma 2.14]), their lifts $\{r_P(p) \mid p \in P\}$ and $\mathcal{F}_j$, together with $\mathbf{T}$, generate Mp. We will use this information to compute $\widetilde{\mathrm{Sp}} = (\mathrm{Mp}, \mathrm{Mp})$ and see how Gauss sums arise in the process. In the following let $\mathcal{F} = \mathcal{F}_m$.

Since $(\mathrm{GL}(m, F), \mathrm{GL}(m, F)) = \mathrm{SL}(m, F)$ for any field F (Milnor, J., *Introduction to Algebraic K-Theory*, Ann. of Math. Studies No. 72, p. 25 and 28), it is not hard to see that (P, P) is generated by $\underline{a}(A)$ such that $A \in \mathrm{SL}_m(F)$ and $\underline{n}(B)$ such that $B = B^T \in \mathrm{M}_m(F)$, where $I = I_m$. Since $\mathbf{a}$ and $\mathbf{n}$ are homomorphisms,

(2.2)
$$\mathbf{a}(A) \in \widetilde{\mathrm{Sp}} \quad \text{and} \quad \mathbf{n}(B) \in \widetilde{\mathrm{Sp}} \quad \text{for all } A \in \mathrm{SL}_m(F) \quad \text{and } B = B^T \in \mathrm{M}_m(F).$$

Since $\widetilde{\mathrm{Sp}} = (\mathrm{Mp}, \mathrm{Mp})$ is normal in Mp and $\left(\begin{smallmatrix} I & \\ B & I \end{smallmatrix}\right) = w_m \, \underline{n}(-B) \, w_m^{-1}$,

$$\mathcal{F}\mathbf{n}(-B)\mathcal{F}^{-1} \in \widetilde{\mathrm{Sp}} \cap \pi^{-1} \begin{pmatrix} I & \\ B & I \end{pmatrix}. \tag{2.3}$$

Now we compute the fiber in $\widetilde{\mathrm{Sp}}$ over $\underline{a}(B^{-1})w_m$ when B is symmetric and invertible. The Bruhat decomposition on the big cell ($\det C \neq 0$) has the form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \underline{n}(AC^{-1}) \, \underline{a}(-C^{-T}) \, w_m \, \underline{n}(C^{-1}D). \tag{2.4}$$

Note that AC^{-1} and $C^{-1}D$ are symmetric. Taking $\left(\begin{smallmatrix} I & \\ -B & I \end{smallmatrix}\right)$ in place of $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$ when B is invertible gives

$$\underline{n}(B^{-1}) \begin{pmatrix} I & \\ -B & I \end{pmatrix} \underline{n}(B^{-1}) = \underline{a}(B^{-1})w_m.$$

An element of $\widetilde{\mathrm{Sp}}$ which lies in the fiber over the left side of the equation is

$$\mathbf{n}(B^{-1}) \, \mathcal{F} \, \mathbf{n}(B) \, \mathcal{F}^{-1} \, \mathbf{n}(B^{-1}) \in \widetilde{\mathrm{Sp}}.$$

On the other hand, the fiber in Mp over the right hand side contains $\mathbf{a}(B^{-1}) \, \mathcal{F}$. Hence there is a constant $\gamma(B) \in \mathbf{T}$ (which depends on χ) such that

(2.5)
$$\gamma(B) \, \mathbf{a}(B^{-1}) \, \mathcal{F} = \mathbf{n}(B^{-1}) \, \mathcal{F} \, \mathbf{n}(B) \, \mathcal{F}^{-1} \mathbf{n}(B^{-1}) \in \widetilde{\mathrm{Sp}} \cap \pi^{-1} \left(\begin{pmatrix} & B^{-1} \\ -B & \end{pmatrix} \right).$$

This constant is the same as $\gamma(f)$ of [**W**, No. 14] or the "Weil index" $\gamma_F(f)$ of [**R1**, Appendix], where f is the character of second degree on F^m given by $f(X) = \chi(XBX^T/2)$.

Weil has shown that γ is a homomorphism of the Witt group ([**W**, No. 25]); that is,

$$\gamma(R^TBR) = \gamma(B), \qquad \gamma\begin{pmatrix} B_1 & \\ & B_2 \end{pmatrix} = \gamma(B_1)\gamma(B_2) \tag{2.6}$$

for all $R \in \mathrm{GL}_m(F)$ and all square matrices B_1, B_2. Here is a different proof of these facts. Take any $R \in \mathrm{GL}_m(F)$ such that $R^TBR = \mathrm{diag}\{b_1, \dots, b_m\}$ with $b_i \in F^\times$. It will suffice to show $\gamma(B) = \prod \gamma(b_i)$. Let φ be any nonzero element of $L^2(F)$, and define $\varphi^{\otimes m} \in L^2(F^m)$ by $\varphi^{\otimes m}(X) = \prod \varphi(X_i)$. Let $\Phi(X) = \varphi^{\otimes m}(XR)$. The right side of (2.5) evaluated at Φ is the function

$$\begin{aligned}
&\Phi_1(X) \\
&= \chi(XB^{-1}X^T/2) \int_{F^m} \chi(YBY^T/2)\,(\mathcal{F}^{-1}\,\mathbf{n}(B^{-1})\,\Phi)(Y)\,\chi(XY^T)\,dY \\
&= \chi(XB^{-1}X^T/2) \int_{F^m} \chi(YBY^T/2) \int_{F^m} \chi(ZB^{-1}Z^T/2)\Phi(Z)\chi((X-Z)Y^T)\,dZ\,dY.
\end{aligned}$$

Let $X' = XR = (X'_1, \dots, X'_m)$. Now change variables $Y \mapsto YR^{-T}$, $Z \mapsto ZR$. Then

$$\begin{aligned}
&\Phi_1(X) \\
&= \prod_{i=1}^m \chi(b_i^{-1}X_i'^2/2) \int_F \chi(b_iy^2/2) \int_F \chi(b_i^{-1}z^2/2)\,\varphi(z)\,\chi((X'_i - z)y)\,dz\,dy.
\end{aligned}$$

On the other hand, the left side of (2.5) evaluated at Φ is

$$\begin{aligned}
\Phi_1(X) &= \gamma(B)|\det B|_F^{-1/2} \int_{F^m} \Phi(Y)\,\chi(XB^{-1}Y^T)\,dY \\
&= \gamma(B) \prod_{i=1}^m |b_i|_F^{-1/2} \int_F \varphi(y)\,\chi(b_i^{-1}X'_iy)\,dy \\
&= \gamma(B) \prod |b_i|_F^{-1/2}\hat{\varphi}(b_i^{-1}X'_i),
\end{aligned}$$

where $\hat{\varphi}$ is the Fourier transform of φ. Comparing the two expressions for $\Phi_1(X)$, we find $\gamma(B) = \prod \gamma(b_i)$, where $\gamma(b)$ for $b \in F^\times$ is the constant in $\mathbf{T}$ such that

$$\begin{aligned}
&\gamma(b)|b|^{-1/2}\hat{\varphi}(b^{-1}x) \\
&\qquad = \chi(b^{-1}x^2/2) \int_F \chi(by^2/2) \int_F \chi(b^{-1}z^2/2)\varphi(z)\chi((x-z)y)\,dz\,dy.
\end{aligned} \tag{2.7}$$

Note that our notation is consistent: if $m = 1$ and $B = b$ then $\gamma(B) = \prod_{i=1}^{1} \gamma(b) = \gamma(b)$. This proves (2.6).

If $F = \mathbb{R}$ and $\chi(t) = e^{2\pi i \lambda t}$, $\gamma(b)$ may be evaluated by taking $\phi(x) = e^{-\pi x^2}$ in formula (2.7) and applying [**Ig**, Ch. I, §2, Lemma 1]. The result is

$$\gamma(b) = \begin{cases} e^{2\pi i/8} & \text{if } \lambda b > 0 \\ e^{-2\pi i/8} & \text{if } \lambda b < 0. \end{cases} \tag{2.8}$$

If F is nonarchimedean with a discrete valuation v, then by [**W**, No. 27],

$$\gamma(b) = \tilde{\gamma}(b)/|\tilde{\gamma}(b)|, \qquad \tilde{\gamma}(b) = \sum_{\nu=-\infty}^{+\infty} \int_{v(y)=\nu} \chi(by^2/2)\, dy, \tag{2.9}$$

where dy is an additive Haar measure. This is recognizable as a Gauss sum.

In ([**W**, No. 28]) it is shown that for all $a, b \in F^\times$,

$$\gamma(-b) = \bar{\gamma}(b), \qquad \gamma(1)\,\gamma(-a)\,\gamma(-b)\,\gamma(ab) = (a,b)_F, \tag{2.10}$$

where $(a,b)_F = 1$ or -1 according as a is or is not a norm in $F(b^{1/2})$. From this it is easy to deduce that $\gamma(b)^8 = 1$, and even $\gamma(a)^4\,\gamma(b)^4 = 1$, for all $a, b \in F^\times$. Other formulas are gathered in the appendix of [**R1**].

Proposition 2. *For all $A \in \mathrm{GL}_m(F)$ and $B = B^T \in \mathrm{M}_m(F)$,*

$$(\det A, -1)_F^{1/2}\mathbf{a}(A) \in \widetilde{\mathrm{Sp}}, \qquad \mathbf{n}(B) \in \widetilde{\mathrm{Sp}}, \qquad \gamma(1)^j \mathcal{F}_j \in \widetilde{\mathrm{Sp}}.$$

Proof. These formulas can be deduced from [**R1**, Def. 5.2 and Cor. A.5] or from [**R2**, Th. 4.1]; here we give a different proof. The first assertion when $\det A = 1$, the second assertion in general, and the third assertion when $j = m$ have already been shown (equations (2.2) and (2.5)). For arbitrary $A \in \mathrm{GL}_m(F)$, write $A = BA_1$ with $B = \operatorname{diag}\{\det A, I_{m-1}\}$ and $A_1 \in \mathrm{SL}_m(F)$. Then

$$\frac{\gamma(\det A)}{\gamma(1)}\mathbf{a}(A) = \{\gamma(B)\,\mathbf{a}(B)\mathcal{F}\}\,\{\gamma(I)\mathcal{F}\}^{-1}\,\{\mathbf{a}(A_1)\}.$$

Here we have used that $\gamma(B)/\gamma(I) = \gamma(\det A)/\gamma(1)$ by (2.6). Each term in brackets belongs to $\widetilde{\mathrm{Sp}}$ by (2.5) and (2.2). Now

$$\{\gamma(1)/\gamma(a)\}^2 = \gamma(1)^2\,\gamma(-a)^2 = (a,-1)_F$$

by (2.10), so $\gamma(\det A)/\gamma(1) = \pm(\det A, -1)_F^{1/2}$. This proves the first assertion. One can deduce the last assertion by embedding $\mathrm{Sp}(2j, F)$ into $\mathrm{Sp}(2m, F)$ ($\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A = \left(\begin{smallmatrix} a & \\ & I_{m-j} \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} b & \\ & 0_{m-j} \end{smallmatrix}\right)$, $C = \left(\begin{smallmatrix} c & \\ & 0_{m-j} \end{smallmatrix}\right)$, $D = \left(\begin{smallmatrix} d & \\ & I_{m-j} \end{smallmatrix}\right)$) and applying the argument of (2.4-5) to the image of $\mathrm{Sp}(2j, F)$. $\square$

3. Splitting of K.

In this section we recall how to obtain the splitting of a maximal compact subgroup K of Sp. First let us discuss the case where $F = \mathbb{R}$. Let $U_m = \{ Z \in \mathrm{M}_m(\mathbb{C}) \mid Z\bar{Z}^T = I_m \}$, and define $u : U_m \to \mathrm{Sp}$ by the formula

$$u(A+iB) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}. \tag{3.1}$$

It is well known that Sp acts on the Siegel upper half space, and by an easy calculation $u(U_m)$ is the stabilizer of iI_m, hence it is a maximal compact subgroup of Sp. The nontrivial additive character χ which appears in (1.2) and (1.3) has the form $\chi(t) = e^{2\pi i\lambda t}$, where $\lambda \in \mathbb{R}^\times$. Define $\Phi_0 \in L^2(\mathbb{R}^m)$ by

$$\Phi_0(X_1, \dots, X_m) = e^{-\pi|\lambda|\sum X_j^2}.$$

Lemma 3. *Let $F = \mathbb{R}$ and $K = u(U_m)$. There is a splitting homomorphism $\mathbf{k} : K \to \mathrm{Mp}$ such that $\mathbf{k}(k)(\Phi_0) = \Phi_0$ for all $k \in K$.*

Proof. It is well known that $\pi^{-1}(K)$ stabilizes $\mathbb{C}^\times \Phi_0$; see [**Ig**, Ch. I, §**9**] or [**B**, Prop. 3.2(a)]. Let $\tilde{\tau} \in \mathrm{Mp}$ and $\pi(\tilde{\tau}) = k \in K$. Then $\tilde{\tau}\Phi_0 = c\Phi_0$, and $c \in \mathbf{T}$ since $\tilde{\tau}$ is a unitary operator. Thus one takes $\mathbf{k}(k) = c^{-1}\tilde{\tau}$. □

There is a similar result in the nonarchimedean case ([**W**, No. 19] or [**Kz**, Lemma 2]). Let F be a nonarchimedean field, $\mathfrak{o}_F$ its ring of integers, Λ a lattice in F^m, and Λ' the dual lattice:

$$\Lambda' = \{ x \in F^m \mid \chi(x \cdot y) = 1 \qquad \text{for all } y \in \Lambda \}.$$

Let $K \subset \mathrm{Sp}$ be the stabilizer of $\Lambda \oplus \Lambda'$:

$$K = \{ \sigma \in \mathrm{Sp} \mid (x, y)\sigma \in \Lambda \oplus \Lambda' \qquad \text{for all } x \in \Lambda,\ y \in \Lambda' \}.$$

Then K is a maximal compact subgroup of Sp. Since $\Lambda = \mathfrak{o}_F^m \alpha$ for some $\alpha \in \mathrm{GL}_m(F)$, the various K for different choices of Λ are conjugate to one another. If $\Lambda = \mathfrak{o}_F^m$ and $\mathfrak{a}_\chi = \mathfrak{o}_F$ (see (1.8)) then $\Lambda' = \mathfrak{o}_F^m$ and $K = \mathrm{Sp}(2m, \mathfrak{o}_F)$.

Lemma 4. *Let F be a nonarchimedean field such that $2 \in \mathfrak{o}_F^\times$. Let Φ_0 be the characteristic function of Λ. There is a splitting homomorphism $\mathbf{k} : K \to \widetilde{\mathrm{Sp}}$ such that $\mathbf{k}(k)\Phi_0 = \Phi_0$ for all $k \in K$.*

Proof. This is proved in [**Kz**, Lemma 2]. (N.B. Kazhdan's article contains an error in the group law for the Heisenberg group, but the reasoning is correct in the case where $2 \in \mathfrak{o}_F^\times$.) □

Lemma 5. *Let $F = \mathbb{R}$ or F as in the previous lemma. If $p \in P \cap K$ then $r_P(p) = \mathbf{k}(p)$. Let $\widehat{P} = w_m^{-1} P w_m = \{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp} \mid B = 0 \}$ and define $r_{\widehat{P}} : \widehat{P} \to \mathrm{Mp}$ by $r_{\widehat{P}}(w_m^{-1} p w_m) = \mathcal{F}^{-1} r_P(p) \mathcal{F}$. If $\hat{p} \in \widehat{P} \cap K$ then $r_{\widehat{P}}(\hat{p}) = \mathbf{k}(\hat{p})$.*

Proof. Suppose $p \in P \cap K$. We can write $p = \underline{n}(B)\underline{a}(A)$. From the definition of K it is clear that $\det A$ is a unit. Write $r_P(p) = c\mathbf{k}(k)$, with $c \in \mathbf{T}$. Since $\Phi_0(0) = 1$ and $\mathbf{k}(k)\Phi_0 = \Phi_0$, we know $c = r_P(p)\,\Phi_0(0) = \mathbf{n}(B)\mathbf{a}(A)\Phi_0(0) = |\det A|^{1/2} = 1$. Next suppose $\hat{p} = w_m^{-1} p w_m \in \widehat{P} \cap K$. Let $\Phi_0' = \mathcal{F}\Phi_0$ and $K' = w_m K w_m^{-1}$. If $F = \mathbb{R}$ then $\Phi_0' = \Phi_0$ and $K' = K$. If F is nonarchimedean then Φ_0' is a constant multiple of the characteristic function of Λ' and K' is the stabilizer of $\Lambda' \oplus \Lambda$; that is, $\Phi_0'(0)^{-1}\,\Phi_0', K'$ are defined like Φ_0, K with the roles of Λ and Λ' reversed. Moreover $p \in P \cap K'$, hence $r_P(p)\,\Phi_0' = \Phi_0'$. Thus $r_{\widehat{P}}(\hat{p})\,\Phi_0 = \mathcal{F}^{-1} r_P(p)\,\Phi_0' = \mathcal{F}^{-1}\Phi_0' = \Phi_0$, so that $r_{\widehat{P}}(\hat{p}) = \mathbf{k}(\hat{p})$. □

Lemma 6. *If F is nonarchimedean and $2 \in \mathfrak{o}_F^\times$ then $(a,b)_F = 1$ for all $a, b \in \mathfrak{o}_F^\times$.*

Proof. This proof may be found in [**W**, Theorem 5]. Choose χ such that $\mathfrak{a}_\chi = \mathfrak{o}_F$ (see (1.8)). Let $\Lambda = \mathfrak{o}_F^m$; thus $K = \mathrm{Sp}(2m, \mathfrak{o}_F)$. Note that $\mathbf{k}(w_m) = \mathcal{F}$, since $\mathcal{F}\,\Phi_0 = \Phi_0$. Consider equation (2.5) with $m = 1$ and $B = b$, where $b \in \mathfrak{o}_F^\times$. By Lemma 5, (2.5) says

$$\gamma(b)\,\mathbf{k}(\underline{a}(b^{-1}))\,\mathbf{k}(w_m) = \mathbf{k}(\underline{n}(b^{-1}))\,\mathbf{k}(w_m)\mathbf{k}(\underline{n}(b))\,\mathbf{k}(w_m)^{-1}\mathbf{k}(\underline{n}(b^{-1})).$$

Since $\mathbf{k}$ is a homomorphism, $\gamma(b) = 1$. Then $(a,b)_F = 1$ by (2.10). □

Lemma 7. *If F is nonarchimedean and $2 \in \mathfrak{o}_F^\times$ then $\mathbf{k}(k) \in \widetilde{\mathrm{Sp}}$ for all $k \in K$. If $F = \mathbb{R}$ then $\pm(\det Z)^{1/2}\,\mathbf{k}(u(Z_\lambda)) \in \widetilde{\mathrm{Sp}}$ for all $k \in K$, where $Z_\lambda = Z$ or $\bar{Z}$ according as $\lambda > 0$ or $\lambda < 0$.*

Proof. First assume F is nonarchimedean. It is well known that K is generated by $K \cap P$ and $K \cap \widehat{P}$, where $\widehat{P} = w_m^{-1} P w_m$. Suppose $p = \underline{n}(B)\,\underline{a}(A) \in P \cap K$. Then $\mathbf{k}(p) = r_P(p)$ by Lemma 5. Now $\det A \in \mathfrak{o}_F^\times$, so $r_P(p) \in \widetilde{\mathrm{Sp}}$ by Lemma 6 and Prop. 2. Next suppose $\hat{p} = w_m^{-1} p w_m \in \widehat{P} \cap K$. By Lemma 5, $\mathbf{k}(\hat{p}) = r_{\widehat{P}}(\hat{p})$. Now $r_P(p) \in \widetilde{\mathrm{Sp}}$ by the same argument as above. (Note that $p \in P \cap K'$ in the proof of Lemma 5.) Since $\widetilde{\mathrm{Sp}}$ is normal in Mp, $r_{\widehat{P}}(\hat{p}) \in \widetilde{\mathrm{Sp}}$ also.

Next assume $F = \mathbb{R}$. In [**R2**, Th. 4.1] a homomorphism $\tilde{\sigma} \mapsto r(\tilde{\sigma})$ from the universal cover of Sp into Mp is constructed. From the construction it is easy to see that for each $\sigma \in \mathrm{Sp}$, the set $\{ r(\tilde{\sigma}) \mid \pi(\tilde{\sigma}) = \sigma \}$ has cardinality two. By Lemma 1 it follows that the image of r is $\widetilde{\mathrm{Sp}}$. When $\lambda = 1$, the result now follows from [**R2**, Prop. 4.2]. Let us add a subscript χ to remind ourselves that the definition of $\mathbf{k}$ depends on χ. From (1.7) and the definition of Φ_0 it is not hard to deduce that $\mathbf{k}_\chi(u(Z)) = \mathbf{k}_{\lambda\chi}(u(Z_\lambda))$ for all $\lambda \in \mathbb{R}^\times$, $Z \in U_m$. The result follows. $\square$

4. Proof of the Main Theorem.

In this section we will prove the Main Theorem, which was stated in the introduction, and give an application. Let $F = \mathbb{R}$ or $F =$ a self-dual nonarchimedean field such that 2 is a unit in $\mathfrak{o}_F$. Let $\chi(t) = e^{2\pi i t}$ if $F = \mathbb{R}$, and let $\chi(t)$ be an additive character of F such that $\mathfrak{a}_\chi = \mathfrak{o}_F$ (see (1.8)) if F is nonarchimedean. Let $K = u(U_m)$ if $F = \mathbb{R}$ (see (3.1)) and $K = \mathrm{Sp}(2m, \mathfrak{o}_F)$ if F is nonarchimedean. For each $k \in K$ we will define a number $x_0(k) \in \mathbf{T}$, which will turn out to be the same as the Gauss sum $x_0(k)$ that appears in the statement of the Main Theorem. Moreover $x_0(k)^2$ can easily be evaluated using results of §§2-3.

Let us explain the definition of $x_0(k)$. By the Bruhat decomposition ([**R1**, Lemma 2.14]), every element $g \in \mathrm{Sp}$ can be written in the form $g = p_1 w_j p_2$ for some $p_1, p_2 \in P$ and $0 \leq j \leq m$. Consider the *standard section* $r_0 : \mathrm{Sp} \to \mathrm{Mp}$ given by

$$r_0(p_1 w_j p_2) = r_p(p_1)\, \mathcal{F}_j\, r_P(p_2). \tag{4.1}$$

This is well-defined by virtue of [**R1**, Th. 3.5(3)]. Define $x_0(k) \in \mathbf{T}$ for $k \in K$ by the formula

$$r_0(k) = x_0(k)\, \mathbf{k}(k). \tag{4.2}$$

Suppose $y(k)$, $y_0(g)$ are elements of $\mathbf{T}$ such that $y(k)\,\mathbf{k}(k) \in \widetilde{\mathrm{Sp}}$ and $y_0(g)\, r_0(g) \in \widetilde{\mathrm{Sp}}$. Then $x_0(k)\, y_0(k)\, y(k)^{-1} \in \widetilde{\mathrm{Sp}} \cap \mathbf{T} = \{\pm 1\}$, thus

$$x_0(k)^2 = y(k)^2\, y_0(k)^{-2}. \tag{4.3}$$

By virtue of Lemma 7, Prop. 2, and (2.8), we can take (for $g = \left(\begin{smallmatrix} A_1 & * \\ 0 & * \end{smallmatrix}\right) w_j \left(\begin{smallmatrix} A_2 & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{Sp}$ and $k \in K$)

(4.4)
$$y(k) = \det(u^{-1}(k))^{1/2} \quad \text{and} \quad y_0(g) = e^{2\pi i j/8}(\mathrm{sign}(\det A_1 A_2))^{1/2} \qquad \text{if} \quad F = \mathbb{R},$$

(4.5)
$$y(k) = 1 \quad \text{and} \quad y_0(g) = (\det A_1 A_2, -1)_F^{1/2} \qquad \text{if} \quad F \text{ is nonarchimedean.}$$

Now we begin the computation of $x_0(k)$. Suppose $k = p_1 w_j p_2$ is the Bruhat decomposition of k and $p_1 = \underline{n}(B_1)\,\underline{a}(A_1)$. Note that j is the rank of the bottom left $m \times m$ minor of k. Then

$$(4.6) \qquad x_0(k) = (r_0(k)\Phi_0)(0) = |\det A_1|^{1/2} \int_{F^j} (r_P(p_2)\,\Phi_0)(y,0)\,dy.$$

Thus we need to calculate A_1 and p_2. This is accomplished in the next two lemmas.

Lemma 8. *Given $C \in \mathrm{M}_m(F)$ of rank j, there exist $\pi_1, \pi_2 \in \mathrm{SO}(m,F)$ with all entries $0, 1$, or -1 such that the top left $j \times j$ minor of $\pi_1 C \pi_2$ is invertible.*

Proof. One can permute columns to make the first j columns of C linearly independent, then one can exchange rows to make the top left $j \times j$ minor of C invertible. The resulting matrix is $\pi_1' C \pi_2'$, where π_1' and π_2' are permutation matrices. One can always arrange for an even number of row exchanges (to make $\det \pi_i' = 1$) unless $m = 2$, $j = 1$. When $m = 2$, one could choose $\pi_i = I_2$ or $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. □

Lemma 9. *Suppose $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(2m,F)$, the rank of C is j, and the top left $j \times j$ minor of C is invertible. Write*

$$C = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix} \qquad D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$$

in block form, where C_1, D_1 are $j \times j$ matrices and C_4, D_4 are $(m-j)\times(m-j)$ matrices. Then $g = p_1 w_j p_2$, where $w_j = \pi(\mathcal{F}_j)$ is the matrix of (1.5) *and $p_1, p_2 \in P$ have the form*

$$p_1 = \begin{pmatrix} (\alpha'^T)^{-1} & * \\ & \alpha' \end{pmatrix}, \qquad \alpha' = \begin{pmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{pmatrix},$$

$$p_2 = \begin{pmatrix} I_m & B' \\ & I_m \end{pmatrix} \begin{pmatrix} \alpha & \\ & (\alpha^T)^{-1} \end{pmatrix}, \qquad \alpha = \begin{pmatrix} I_j & C_1^{-1}C_2 \\ & I_{m-j} \end{pmatrix},$$

$$B' = \begin{pmatrix} \beta & \\ & 0_{m-j} \end{pmatrix}, \qquad \beta = C_1^{-1}(D_1 + D_2 {C_2}^T {C_1}^{-T}) \in \mathrm{M}_j(F).$$

(In particular, we are asserting that α' is invertible and β is symmetric, so that p_1, p_2 really belong to P.)

Proof. First, DC^T is symmetric since $g \in \mathrm{Sp}$, thus $D_1 {C_1}^T + D_2 {C_2}^T$ is symmetric. This implies $\beta = \beta^T$, so $p_2 \in P$. Now $C\alpha^{-1} = \left(\begin{smallmatrix} C_1 & 0 \\ C_3 & * \end{smallmatrix}\right)$. Since

$\operatorname{rank}(C\alpha^{-1}) = \operatorname{rank}(C) = \operatorname{rank}(C_1)$, the * is really 0. Thus

$$gp_2^{-1} = \begin{pmatrix} A\alpha^{-1} & * \\ \left(\begin{smallmatrix} C_1 & 0 \\ C_3 & 0 \end{smallmatrix}\right) & D' \end{pmatrix}, \qquad D' = \begin{pmatrix} -C_1\beta & 0 \\ -C_3\beta & 0 \end{pmatrix} + D\alpha^T.$$

By direct calculation, $D' = \left(\begin{smallmatrix} 0 & D_2 \\ D_3' & D_4 \end{smallmatrix}\right)$ for some matrix D_3'. Since $gp_2^{-1} \in \mathrm{Sp}$, we know $\left(\begin{smallmatrix} C_1 & 0 \\ C_3 & 0 \end{smallmatrix}\right) D'^T = \left(\begin{smallmatrix} 0 & C_1 D_3'^T \\ 0 & * \end{smallmatrix}\right)$ is symmetric; since C_1 is invertible, this forces $D_3' = 0$. Thus the bottom m rows of gp_2^{-1} are $\left(\begin{smallmatrix} C_1 & 0 & 0 & D_2 \\ C_3 & 0 & 0 & D_4 \end{smallmatrix}\right)$. These rows must be linearly independent, so α' is invertible. Moreover, the bottom m rows of $\operatorname{diag}\{\alpha'^T, \alpha'^{-1}\} gp_2^{-1}$ are the same as the bottom m rows of w_j. Since this matrix is symplectic, it is easy to check from the relations $A^T C = C^T A$, $B^T D = D^T B$, $A^T D - C^T B = I$ for all $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}$ that the top m rows of $\operatorname{diag}\{\alpha'^T, \alpha'^{-1}\} gp_2^{-1}$ must have the form

$$\begin{pmatrix} A_1' & 0 & I_j & -A_3'^T \\ A_3' & I_{m-j} & 0 & B_4' \end{pmatrix}$$

with A_1' and B_4' symmetric. (Here the matrices in the top row have height j, the matrices in the first and third columns have width j.) Thus

$$\begin{pmatrix} I & E \\ 0 & I \end{pmatrix} \operatorname{diag}\{\alpha'^T, \alpha'^{-1}\} gp_2^{-1} = w_j, \quad \text{where} \quad E = \begin{pmatrix} A_1' & A_3'^T \\ A_3' & -B_4' \end{pmatrix}.$$

This completes the proof. □

Proof of Main Theorem. Define $x_0(k)$ as in (4.2). Let $k = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in K - P$ and choose π_1, π_2 as in Lemma 8. Let $\alpha_i = \underline{a}(\pi_i)$ for $i = 1, 2$ and $k' = \alpha_1 \, k \, \alpha_2$. Then $x_0(k) = x_0(k')$ because

(4.7) $r_0(k') = \mathbf{a}(\pi_1)\, r_0(k)\, \mathbf{a}(\pi_2) = \mathbf{k}(\alpha_1)\, x_0(k)\, \mathbf{k}(k)\, \mathbf{k}(\alpha_2) = x_0(k)\mathbf{k}(k')$.

Define $C_1, \ldots, C_4$ and $D_1, \ldots, D_4$ by (1.9). Now apply Lemma 9 to the matrix k'. By (4.6) and Lemma 9, the matrix $\alpha' = \left(\begin{smallmatrix} -C_1 & D_2 \\ -C_3 & D_4 \end{smallmatrix}\right)$ is invertible, and in the notation of Lemma 9,

$$\begin{aligned} x_0(k') &= |\det(\alpha')|^{-1/2} \int_{F^j} (\mathbf{n}(B')\, \mathbf{a}(\alpha)\Phi_0)(y, 0)\, dy \\ &= |\det(\alpha')|^{-1/2} \int_{F^j} \chi(y\beta y^T/2)\, \Phi_0(y, yC_1^{-1}C_2)\, dy. \end{aligned}$$

By changing variables $y \mapsto yC_1$ we obtain (1.10). Finally, $x_0(k)^2$ can be computed from (4.3), (4.4), and (4.5). □

Example. For $F = \mathbb{Q}_p$ ($p \geq 3$) and $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(2, \mathbb{Z}_p)$, we have $x_0(k) = 1$ if $c = 0$, and otherwise

$$x_0(k) = |c|^{1/2} \int_{c^{-1}\mathbb{Z}_p} \chi(dcy^2/2)\, dy = |c|^{-1/2} \int_{\mathbb{Z}_p} \chi(dc^{-1}y^2/2)\, dy.$$

If c is a unit, this integral is one. If c is not a unit, then $\int_{p^{-\nu}\mathbb{Z}_p} \chi(dc^{-1}y^2/2)\, dy = 0$ for all $\nu > 0$ by a standard argument in the theory of Gauss sums, so $x_0(k) = \gamma(d/c)$ by (2.9). In summary, for $k = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(2, \mathbb{Z}_p)$,

$$x_0(k) = \begin{cases} \gamma(c/d) & \text{if } c \neq 0 \text{ and } p|c \\ 1 & \text{if } c = 0 \ \text{ or } c \in \mathbb{Z}_p^\times. \end{cases}$$

Proposition 10. *Let $F = \mathbb{R}$, $\chi(t) = e^{2\pi i t}$. There is a section $r_+ : \mathrm{Sp}(2m, \mathbb{R}) \to \widetilde{\mathrm{Sp}}(2m, \mathbb{R})$ such that*
a) $r_+(p_1 g p_2) = r_P(p_1)\, r_+(g)\, r_P(p_2)$ whenever $p_1, p_2 \in P_+ = \{ \left(\begin{smallmatrix} A & * \\ 0 & * \end{smallmatrix}\right) \in P \mid \det(A) > 0 \}$; and
b) If $k = u(\mathrm{diag}\{ e^{i\theta_1}, \dots, e^{i\theta_m} \})$ $(-\pi < \theta_j \leq \pi)$ then

$$r_+(k) = e^{-\delta\pi i/4} (e^{\delta\pi i/2})^{1/2} e^{i\theta/2} \mathbf{k}(k), \tag{4.8}$$

where $\theta = \sum \theta_j$, $\delta = \sum \delta_j$, and $\delta_j = -1, 0, 1, 2$ according as $\theta_j \in (-\pi, 0)$, $\theta_j = 0$, $\theta_j \in (0, \pi)$, or $\theta_j = \pi$. Define a cocycle $c_+ : \mathrm{Sp} \times \mathrm{Sp} \to \{ \pm 1 \}$ by the formula $r_+(g_1)\, r_+(g_2) = c_+(g_1, g_2)\, r_+(g_1 g_2)$ for $g_1, g_2 \in \mathrm{Sp}$. If $m = 1$ then c_+ coincides with the Kubota cocycle.

Proof. Let $W_+ = \{ w_j, w_- w_j \mid j = 0, \dots, m \}$, where $w_- = \underline{a}\left(\begin{smallmatrix} -1 & \\ & I_{m-1} \end{smallmatrix}\right)$. It can easily be seen from the Bruhat decomposition that $\mathrm{Sp} = \sqcup_{w \in W_+} P_+\, w\, P_+$. Let $\varepsilon : W_+ \to \mathrm{Mp}$ be the restriction to W_+ of $k \mapsto \det(u^{-1}(k))^{1/2}\, \mathbf{k}(k)$, where we choose the branch of the square root function which has argument in $(-\pi/2, \pi/2]$, and define $r_+ : \mathrm{Sp} \to \widetilde{\mathrm{Sp}}$ by

$$r_+(p_1 w p_2) = r_P(p_1)\, \varepsilon(w)\, r_P(p_2) \qquad \text{for } w \in W_+ \text{ and } p_1, p_2 \in P_+.$$

This section is well-defined by [**R1**, Theorem 3.5(3)], and it takes values in $\widetilde{\mathrm{Sp}}$ by Prop. 2 and Lemma 7. Clearly (a) is satisfied. Now we prove (b). Define $\nu : \mathrm{Sp} \to \mathbf{T}$ by $r_+(g) = \nu(g)\, r_0(g)$, where $r_0(g)$ is the standard section (4.1). Note that $\nu(w) = (\det(u^{-1}(w)))^{1/2}$ for $w \in W_+$ because $r_0(w)\, \Phi_0 = \Phi_0$. By (4.2),

$$r_+(k) = \nu(k)\, x_0(k)\, \mathbf{k}(k).$$

Thus we need to show that for k as in part (b),

$$\nu(k)\, x_0(k) = e^{-\delta\pi i/4} (e^{\delta\pi i/2})^{1/2} e^{i\theta/2}. \tag{4.9}$$

If $A_1, A_2 \in \mathrm{SO}_m(\mathbb{R})$ and $k' = \underline{a}(A_1)\, k\underline{a}(A_2)$ then $\nu(k') = \nu(k)$ because r_0 and r_+ are constant on double cosets P_+wP_+, and $x_0(k') = x_0(k)$ by (4.7). Thus the left side of (4.9) does not change if the θ_j are permuted or if an even number of the θ_j are shifted by π. The right side is also invariant under such manipulations of the θ_j, because $\theta - \delta\pi/2$ and $e^{\delta\pi i/2}$ are unchanged. Thus we are reduced to the case where $\sin\theta_\ell \neq 0$ for $1 \leq \ell \leq j$, $\sin\theta_\ell = 0$ for $\ell > j$, and $\theta_\ell \in [0,\pi)$ for all $\ell \geq 2$.

If $j = 0$ then $k = I$ or $k = w_-$ according as $\theta_1 = 0$ or $\theta_1 = \pi$. If $k = I$, both sides of (4.9) are 1. If $k = w_-$ then $\delta = 2$, $\theta = \pi$, $x_0(k) = 1$, $\nu(k) = i$, and again both sides are equal. Now consider the case $j \geq 1$. Then $\sin\theta_1 \neq 0$, so $\delta_1 = \pm 1$, $\delta_2 = \cdots = \delta_j = 1$, and $\delta_\ell = 0$ for all $\ell > j$. Let $w = w_j$ or w_-w_j according as $\delta_1 = 1$ or $\delta_1 = -1$. Then $k \in P_+wP_+$, so $\nu(k) = (\delta_1 i^j)^{1/2} = (e^{\delta\pi i/2})^{1/2}$. It only remains to show $x_0(k) = e^{-\delta\pi i/4}e^{i\theta/2}$. By (1.10),

$$x_0(k) = \prod_{\ell=1}^{j} |\sin\theta_\ell|^{-1/2} \int_{\mathbb{R}} e^{-\pi i y^2 \sin\theta_\ell \cos\theta_\ell} e^{-\pi y^2 \sin^2\theta_\ell}\, dy.$$

By [**Ig**, Ch. I, Lemma 1], it can easily be verified that each term in the product has positive real part, and that up to a positive constant coming from the self-dual Haar measure, its square is $|\sin\theta_\ell|^{-1}(1 + i\cot\theta_\ell)^{-1} = e^{i\theta_\ell - (\pi i\delta_\ell/2)}$. Since $x_0(k) \in \mathbf{T}$ *a priori*, the positive constant is one. Note that $\theta_\ell - (\pi\delta_\ell/2) \in (-\pi/2, \pi/2)$ for all $\ell \leq j$, therefore $\mathrm{Re}(e^{i\theta_\ell/2}\, e^{-\delta_\ell\pi i/4}) > 0$. This completes the proof of (b).

Finally we want to show that if $m = 1$ then c_+ coincides with the Kubota cocycle c_K which is defined in [**K**]. We recall the definition of c_K: if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ let $x(g) = c$ or d according as $c \neq 0$ or $c = 0$. In terms of the Hilbert symbol which was defined after (1.9),

$$c_K(g_1, g_2) = (x(g_1), x(g_2))_{\mathbb{R}}\, (-x(g_1)\, x(g_2), x(g_1 g_2))_{\mathbb{R}}.$$

We can reduce the proof of (c) to the case where $g_i \in K$ by the following argument, which was suggested by the referee. It is clear that $x(pg)$, $x(gp)$, and $x(g)$ have the same sign if $p \in P_+$, thus $c_K(pg_1, g_2) = c_K(g_1, g_2 p) = c_K(g_1, g_2)$. Moreover $c_K(g, p) = 1 = c_K(p, g)$ because $(x(g), 1)_{\mathbb{R}} = (-x(g), x(g))_{\mathbb{R}} = 1$. Obviously c_+ satisfies the same relations because of the property (a) of r_+ and because $r_+(I_{2m}) = 1$. Also we know the cocycle relation

$$c(x,y)\, c(xy,z) = c(y,z)\, c(x,yz)$$

for $c = c_+$ or $c = c_K$, where $x, y, z \in \mathrm{SL}_2(\mathbb{R})$. For arbitrary g_1 and g_2 write $g_1 = p_1 k_1$, $g_2 = p_2 k_2$, $g_3 = g_1 g_2 = p_1 p_3 k_3$ with $p_i \in P_+$ and $k_i \in K$. Take

$x = k_1, y = p_2, z = k_2$ in the cocycle relation. Then $xy = k_1p_2 = p_3k_3k_2^{-1}$, hence

(4.10)
$$c(g_1, g_2) = c(k_1, p_2k_2) = c(p_3k_3k_2^{-1}, k_2)\, c(k_1, p_2)\, c(p_2, k_2)^{-1} = c(k_3k_2^{-1}, k_2).$$

Thus it suffices to show c_+ and c_K coincide on $K \times K$. Let $g_j = u(e^{i\theta_j})$ for $j = 1, 2, 3$, where $\theta_j \in (-\pi, \pi]$ and $e^{i(\theta_1+\theta_2)} = e^{i\theta_3}$, and let $x_j = 1$ or -1 according as $\theta_j \in (-\pi, 0]$ or $\theta_j \in (0, \pi]$. Then $c_+(k_1, k_2) = e^{i(\theta_1+\theta_2)/2}/e^{i\theta_3/2}$, and this is equal to -1 iff $x_1 = x_2 = -x_3$. Since $x_j \equiv x(g_j) \pmod{\mathbb{R}_+}$, it can easily be seen that the two cocycles agree. □

5. Embedding $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ into $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$.

In this section let $F = \mathbb{R}$, $\chi(t) = e^{2\pi i t}$. Let $P_+ = \{\, \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \in \mathrm{Sp} \mid \det A > 0 \,\}$. We will prove the following proposition.

Proposition 11. *Consider a continuous homomorphism $j : \mathrm{Sp}(2n, \mathbb{R}) \to \mathrm{Sp}(2m, \mathbb{R})$ such that $j(P_+^{(n)}) \subset P_+^{(m)}$ and $j(K^{(n)}) \subset K^{(m)}$, where for clarity we have superscripted our symbols with the degree of the underlying symplectic space. Let $\psi = \det \circ u^{-1} : K \to \mathbf{T}$. There is an integer N such that*

$$(5.1) \qquad (\psi^{(m)} \circ j)(k) = (\psi^{(n)}(k))^N \qquad \textit{for } k \in K^{(n)}.$$

Define $\tilde{j} : \mathrm{Mp}(\mathbb{R}^n) \to \mathrm{Mp}(\mathbb{R}^m)$ by

(5.2)
$$\tilde{j}(\zeta\, r_P(p)\, \mathbf{k}(k)) = \zeta^N\, r_P(j(p))\, \mathbf{k}(j(k)) \qquad \textit{for } \zeta \in \mathbf{T},\ p \in P_+^{(n)},\ k \in K^{(n)}.$$

Then $\tilde{j}$ is a continuous homomorphism, $\pi \circ \tilde{j} = j \circ \pi$, and $\tilde{j}|_{\widetilde{\mathrm{Sp}}(2n,\mathbb{R})}$ takes values in $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$. Moreover, $\tilde{j}|_{\widetilde{\mathrm{Sp}}}$ is the only continuous function from $\widetilde{\mathrm{Sp}}(2n, \mathbb{R})$ into $\widetilde{\mathrm{Sp}}(2m, \mathbb{R})$ such that $\pi \circ \tilde{j} = j \circ \pi$ and $\tilde{j}(1) = 1$.

If one takes $n = 1$ and $j\left(\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} \alpha I & \beta T \\ \gamma T^{-1} & \delta I \end{smallmatrix}\right)$, where T is a symmetric invertible $m \times m$ real matrix, and if one identifies $\widetilde{\mathrm{Sp}}(2, \mathbb{R})$ with $\mathrm{Sp} \times \{\pm 1\}$ via the bijection $\delta\, r_+(g) \mapsto (g, \delta)$ for r_+ as given in Prop. 10 and $\delta \in \{\pm 1\}$, then the formula (6.2) is a concise statement of [**RS**, Th. 1.2].

Proof. Let $\tilde{j}$ be as in (5.2), $\widetilde{K} = \widetilde{\mathrm{Sp}} \cap \pi^{-1}(K) = \{\pm\psi(k)^{1/2}\, \mathbf{k}(k) \mid k \in K\}$ (by Lemma 7). If $\zeta\, \mathbf{k}(k) \in \widetilde{K}^{(n)}$ then $\tilde{j}(\zeta\, \mathbf{k}(k)) = \zeta^N\, \mathbf{k}(j(k)) \in \widetilde{\mathrm{Sp}}(2m, F)$ by (5.1). (An integer N exists as in (5.1) because any element of $\mathrm{Hom}(U_n, \mathbf{T})$

is a power of determinant.) It is clear that $\tilde{j}$ is well-defined, continuous, and $\pi \circ \tilde{j} = j \circ \pi$, thus we have a commutative diagram

$$\begin{array}{ccc} \widetilde{\mathrm{Sp}}(2n,\mathbb{R}) & \xrightarrow{\tilde{j}} & \widetilde{\mathrm{Sp}}(2m,\mathbb{R}) \\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow \\ \mathrm{Sp}(2n,\mathbb{R}) & \xrightarrow{j} & \mathrm{Sp}(2m,\mathbb{R}). \end{array}$$

We still need to show $\tilde{j}$ is a homomorphism. Let

$$U = \{ pk \in \mathrm{Sp} \mid p \in P_+,\ \arg \psi(k) \in [-\pi/2, \pi/2] \}$$

and $\phi : U \to \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$ the homeomorphism: $\phi(pk) = \underline{\psi(k)^{1/2}\, r_P(p)\, \mathbf{k}(k)}$ (where $\arg \psi(k)^{1/2} \in [-\pi/4, \pi/4]$). Since $\phi(U)$ generates $\widetilde{\mathrm{Sp}}(2n,\mathbb{R})$, it will suffice to show

$$\tilde{j}(\tilde{u}\tilde{g}) = \tilde{j}(\tilde{u})\, \tilde{j}(\tilde{g})$$

for all $\tilde{u} = \phi(u) \in \phi(U)$, $\tilde{g} \in \widetilde{\mathrm{Sp}}(2n,\mathbb{R})$. Both sides lie in the fiber of $\widetilde{\mathrm{Sp}}(2m,\mathbb{R})$ over the point $j(u)\, j(g) \in \mathrm{Sp}(2m,\mathbb{R})$ $(g = \pi(\tilde{g}))$, so the equality holds up to ± 1. Let $F : U \to \{\pm 1\}$, $F(u) = \tilde{j}(\tilde{u}\tilde{g})\, \tilde{j}(\tilde{g})^{-1}\, \tilde{j}(\tilde{u})^{-1}$, where $\tilde{g} \in \widetilde{\mathrm{Sp}}$ is fixed. Then F is continuous and $F(I_{2m}) = 1$. Since U is connected, F is identically one. This proves $\tilde{j}$ is a homomorphism. The characterization of $\tilde{j}$ as the only continuous lift of j taking 1 to 1 follows from an elementary fact about covering projections ([**Sp**, Ch. 2, Sec. 2, Th. 2]).

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