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PARTITIONING PRODUCTS OF $\mathcal{P}(\omega)/\operatorname{fin}$

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We generalize the cardinal invariant $\mathfrak{a}$ to products of $\mathcal{P}(\omega)/\operatorname{fin}$ and then sharpen the well-known inequality $\mathfrak{b} \leq \mathfrak{a}$ by proving $\mathfrak{b} \leq \mathfrak{a}(\lambda)$ for every $\lambda \leq \omega$. Here $\mathfrak{a}(n)$, for $n < \omega$, is the least size of an infinite partition of $(\mathcal{P}(\omega)/\operatorname{fin})^n$, $\mathfrak{a}(\omega)$ is the least size of an uncountable partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$, and $\mathfrak{b}$ is the least size of an unbounded family of functions from ω to ω ordered by eventual dominance. We also prove the consistency of $\mathfrak{b} < \mathfrak{a}(n)$ for every $n < \omega$.

0. Introduction and notation.

In this paper we generalize the cardinal invariant $\mathfrak{a}$ to products of $\mathcal{P}(\omega)/\operatorname{fin}$, where $\mathcal{P}(\omega)$ is the power set of ω and fin the ideal of finite subsets of ω, and show that Solomon's inequality $\mathfrak{b} \leq \mathfrak{a}$ remains true for these numbers. The definitions of $\mathfrak{a}$ and $\mathfrak{b}$ will be given below.

Throughout the paper by $\mathcal{P}(\omega)/\operatorname{fin}$ we really mean $\mathcal{P}(\omega)/\operatorname{fin}\setminus\{0\}$, and will confuse members of $\mathcal{P}(\omega)/\operatorname{fin}$ with their representatives in $[\omega]^\omega$, the set of infinite subsets of ω. For λ an ordinal, by $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ we denote the set of all $C : \lambda \to \mathcal{P}(\omega)/\operatorname{fin}\setminus\{0\}$, ordered coordinatewise. Members of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ will be called *λ-dimensional cubes*, or just *cubes* if λ is clear from the context. As the terminology suggests we will sometimes confuse C with $\prod_{\alpha<\lambda} C(\alpha)$. Cubes C, D are called *compatible*, and we write $C||D$, if there exists a cube contained in C and D. Otherwise C, D are *incompatible* and we write $C \perp D$. A subset of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ is called an *antichain* if any two members are incompatible. For $\mathcal{A} \subseteq (\mathcal{P}(\omega)/\operatorname{fin})^\lambda$, by $\mathcal{A}\restriction C$ we denote the set of cubes in $\mathcal{A}$ which are compatible with C. For $F \subseteq \lambda$, let $\operatorname{pr}_F\mathcal{A} = \{D\restriction F : D \in \mathcal{A}\}$. It is well-known that $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ can be densely embedded into a complete Boolean algebra which is denoted $\text{r.o.}(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$. Even though $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ is not itself a Boolean algebra we use the terminology of Boolean algebras and call a maximal antichain of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ a *partition.* Note that the meet in $\text{r.o.}(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ of a subset of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ is the coordinatewise intersection of its members. As usual, we will use the symbol $\bigwedge$ to denote the meet operation.

The cardinal invariant $\mathfrak{a}$ is defined as the least size of an infinite partition of $\mathcal{P}(\omega)/\operatorname{fin}$. It is well-known that $\mathfrak{a}$ is uncountable and that the axioms of

ZFC do not determine its value (see [**vD**] or [**K**, 2.15 p. 57 and 2.3 p. 256]). Example 2.1. below will show that there exist partitions of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ of size γ for every cardinal γ with $0 < \gamma \leq |\lambda|$. This motivates the following definition.

Definition. For $0 < n < \omega$ let $\mathfrak{a}(n)$ be the least size of an infinite partition of $(\mathcal{P}(\omega)/\operatorname{fin})^n$. If λ is an infinite cardinal we define $\mathfrak{a}(\lambda)$ as the least size $> \lambda$ of a partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$.

In this paper we investigate $\mathfrak{a}(\lambda)$ for $0 < \lambda \leq \omega$.

It is easy to see that for any $0 < \lambda < \gamma \leq \omega$ we have $\mathfrak{a}(\gamma) \leq \mathfrak{a}(\lambda)$. However for arbitrary cardinals $0 < \lambda < \gamma$ this may be false; in a model where CH fails and $\mathfrak{a} = \omega_1$, e.g. in the Cohen model, $\mathfrak{a} < \mathfrak{a}(\omega_1)$.

Most other familiar cardinal invariants such as $\mathfrak{p}$, $\mathfrak{t}$, $\mathfrak{h}$, $\mathfrak{s}$ can be generalized to $(\mathcal{P}(\omega)/\operatorname{fin})^\lambda$ in an obvious way. Straightforward generalizations of the well-known inequalities $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{h} \leq \mathfrak{s}$ (see [**vD**] and [**BS**] for the definitions and proofs) give $\mathfrak{p}(\lambda) \leq \mathfrak{t}(\lambda) \leq \mathfrak{h}(\lambda) \leq \mathfrak{s}(\lambda)$. Moreover it is not difficult to see that $\mathfrak{p} = \mathfrak{p}(\lambda)$, $\mathfrak{t} = \mathfrak{t}(\lambda)$ and $\mathfrak{s} = \mathfrak{s}(\lambda)$ hold for every λ. However for the $\mathfrak{h}(\lambda)$ this is not true. In [**ShSp1**] and [**ShSp2**] the consistency of $\mathfrak{h}(n+1) < \mathfrak{h}(n)$ has been proved for every $n < \omega$, thus solving an open problem from [**BPS**]. See also [**GRShSp**] for another natural situation where the $\mathfrak{h}(\lambda)$ occur.

It is an open problem how to construct a model for $\mathfrak{a}(n+1) < \mathfrak{a}(n)$.

In this paper we are concerned with the following problem. From the equalities and inequalities stated above it follows that $\mathfrak{t}$ is a lower bound for all the $\mathfrak{h}(\lambda)$. Hence clearly Martin's axiom implies $\mathfrak{h}(\lambda) = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum.

For $\mathfrak{a}(\lambda)$ obvious lower bounds are missing, and it is not trivial to show that Martin's axiom implies $\mathfrak{a}(\omega) = \mathfrak{c}$; moreover we do not know whether MA implies $\mathfrak{a}(\lambda) = \mathfrak{c}$ for $\omega_1 \leq \lambda < \mathfrak{c}$[1].

One way to show that Martin's axiom implies $\mathfrak{a} = \mathfrak{c}$ is by using the inequality $\mathfrak{b} \leq \mathfrak{a}$ which is due to Solomon and can be found in [**vD**, 3.1(a)]. Here the bounding number $\mathfrak{b}$ is defined as the least size of an unbounded family in $({}^\omega\omega, \leq^*)$, where ${}^\omega\omega$ is the set of all functions from ω to ω and $\leq^*$ is eventual dominance.

In Sections 1 and 2 we sharpen this inequality by proving $\mathfrak{b} \leq \mathfrak{a}(\lambda)$ for every $0 < \lambda \leq \omega$.

In Section 3 we sketch the proof of the consistency of $\mathfrak{b} < \mathfrak{a}(n)$ for every $n < \omega$. We only sketch it since it is a variation of Shelah's consistency proof of $\mathfrak{b} < \mathfrak{a}$ in [**Sh1**]. We do not know how to construct a model for $\mathfrak{b} < \mathfrak{a}(\omega)$.

Our notation is the standard set-theoretic one. A function is identified with its graph, i.e. it is a set of pairs. The concatenation of two functions

[1]Stefan Grieder has shown that the answer is yes for $\lambda = \omega_1$.

f, g is denoted by $f^\frown g$. By ${}^X Y$ we denote the set of all functions from X to Y. Given a set X and a cardinal κ, then $[X]^\kappa$ ($[X]^{\leq\kappa}$) denotes the set of all subsets of X of cardinality κ (at most κ).

Acknowledgment. In a preliminary version of this paper we had proved only $\mathfrak{a}(\omega) \geq \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$, where $\mathcal{M}$ is the ideal of meagre subsets of the real line and $\mathrm{cov}(\mathcal{M})$ is the least size of a subset of $\mathcal{M}$ whose union is $\mathbb{R}$. The only occurrence of $\mathrm{cov}(\mathcal{M})$ was in the proof of Proposition 2.9. The referee observed that $\mathrm{cov}(\mathcal{M})$ can be eliminated by proving Lemma 2.10.

1. Infinite partitions of finite products of $\mathcal{P}(\omega)/\mathrm{fin}$.

Lemma 1.1. *Suppose $n^* < \omega$ and $\mathcal{A}$ is an infinite partition of $(\mathcal{P}(\omega)/\mathrm{fin})^{n^*}$. Let $F \subseteq n^*$ be maximal (with respect to $\subseteq$) such that there exists $\mathcal{F} \in [\mathcal{A}]^\omega$ with the property that $\{p \restriction F : p \in \mathcal{F}\}$ has the finite intersection property (i.e. finite subsets have a lower bound). Then $|F| = n^* - 1$.*

Proof. Suppose $|n^* \setminus F| \geq 2$. Let C be a lower bound of $\{p \restriction F : p \in \mathcal{F}\}$. Fix $i^* \in n^* \setminus F$, and let $F^* = F \cup \{i^*\}$.

By assumption we may choose finite $\mathcal{F}_0 \subseteq \mathcal{A}$ such that $\mathrm{pr}_{F^*}(\mathcal{F}_0)$ has the finite intersection property, $\bigwedge\{p \restriction F : p \in \mathcal{F}_0\} \wedge C \neq 0$ and $\mathrm{pr}_{n^* \setminus F^*} \mathcal{F}_0$ is a partition of $(\mathcal{P}(\omega)/\mathrm{fin})^{n^* \setminus F^*}$. Let $C_0 = \bigwedge\{p \restriction F : p \in \mathcal{F}_0\} \wedge C$, and let $U_0 = \bigcap\{p(i^*) : p \in \mathcal{F}_0\}$.

Suppose $\langle \mathcal{F}_i : i < n\rangle$, $\langle C_i : i < n\rangle$ and $\langle U_i : i < n\rangle$ have been constructed. Suppose $\bigcup\{p(i^*) : p \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{n-1}\} \neq^* \omega$. By assumption we may choose finite $\mathcal{F}_n \subseteq \mathcal{A}$ such that $\mathrm{pr}_F(\{C_{n-1}\} \cup \mathcal{F}_n)$ has the finite intersection property, $\bigcap\{p(i^*) : p \in \mathcal{F}_n\} \setminus \bigcup\{p(i^*) : p \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{n-1}\}$ is infinite and $\mathrm{pr}_{n^* \setminus F^*} \mathcal{F}_n$ is a partition of $(\mathcal{P}(\omega)/\mathrm{fin})^{n^* \setminus F^*}$. Let $C_n = \bigwedge\{p \restriction F : p \in \mathcal{F}_n\} \wedge C_{n-1}$ and $U_n = \bigcap\{p(i^*) : p \in \mathcal{F}_n\} \setminus \bigcup\{p(i^*) : p \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_{n-1}\}$.

Suppose that in this way we can define $\mathcal{F}_n, C_n, U_n$ for every $n < \omega$. Then $\mathcal{F}_n \cap \mathcal{F}_m = \emptyset$ for distinct m, n, and $\bigcup\{p \restriction F : p \in \bigcup\{\mathcal{F}_n : n < \omega\}\}$ has the finite intersection property. Let $\mathcal{U}$ be an ultrafilter on the Boolean algebra $\mathrm{r.o.}(\mathcal{P}(\omega)/\mathrm{fin})^{n^* \setminus F^*}$. Since $\{p \restriction n^* \setminus F^* : p \in \mathcal{F}_n\}$ is a finite partition we can choose unique $p_n \in \mathcal{F}_n$ such that $p_n \restriction n^* \setminus F^* \in \mathcal{U}$. But then $\{p_n \restriction n^* \setminus \{i^*\} : n < \omega\}$ has the finite intersection property, and hence we obtain a contradiction to the maximal choice of F.

Consequently, the construction above stops at some stage n, because $\bigcup\{p(i^*) : p \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_n\} =^* \omega$. The family $\mathcal{F}$ witnesses that $\mathcal{A} \restriction {}^{n^* \setminus F}\{\omega\}^\frown C_n$ is infinite. Hence there exist $n_0 \leq n$ and $p_0 \in \mathcal{F}_{n_0}$ such that $\mathcal{A} \restriction \{\langle i^*, p_0(i^*)\rangle\} \cup {}^{n^* \setminus F^*}\{\omega\}^\frown C_n$ is infinite. Hence there exists $j \in n^* \setminus F^*$ such that $\mathcal{A} \restriction \{\langle i^*, p_0(i^*)\rangle, \langle j, \omega \setminus p_0(j)\rangle\} \cup {}^{n^* \setminus (F^* \setminus \{j\})}\{\omega\}^\frown C_n$ is infinite. Define cube $K_0 = \{\langle i^*, p_0(i^*)\rangle, \langle j, \omega \setminus p_0(j)\rangle\} \cup {}^{n^* \setminus (F^* \setminus \{j\})}\{\omega\}^\frown C_n$. So $K_0 \perp p_0$ and

$K_0 \restriction F^* \leq p_0 \restriction F^*$.

Now we repeat the above construction inside K_0 for $\mathcal{A} \restriction K_0$. By the same argument as above it must stop after finitely many steps, and hence we obtain $p_1 \in \mathcal{A}$ and cube $K_1 \leq K_0$ such that $p_1 || K_0$, $p_1 \perp K_1$, $K_1 \restriction F^* \leq p_1 \restriction F^*$ and $\mathcal{A} \restriction K_1$ is infinite. Then clearly $p_0 \neq p_1$. Proceeding similarly we construct a descending chain of cubes $\langle K_n : n < \omega \rangle$ and family $\langle p_n : n < \omega \rangle$ in $\mathcal{A}$ such that $p_{n+1} || K_n$, $p_n \perp K_n$, $K_n \restriction F^* \leq p_n \restriction F^*$ and $\mathcal{A} \restriction K_n$ is infinite for every $n < \omega$. Then clearly $p_n \neq p_m$ for distinct n, m, and $\{p_n \restriction F^* : n \in \omega\}$ has the finite intersection property. This contradicts the maximality of F. □

Theorem 1.2. $\mathfrak{b} \leq \mathfrak{a}(n)$ *holds for every* $n < \omega$.

Proof. Let n^* be fixed and let $\mathcal{A}$ be an infinite partition of $(\mathcal{P}(\omega)/\operatorname{fin})^{n^*}$. Let $F \subseteq n^*$ be of maximal size such that there exists $\mathcal{F} \in [\mathcal{A}]^\omega$ with the property that $\{p \restriction F : p \in \mathcal{F}\}$ has the finite intersection property. By Lemma 1.1, F has size $n^* - 1$. Let $n^* \setminus F = \{i\}$ and choose cube C such that $C \leq p \restriction F$ for every $p \in \mathcal{F}$. Certainly $\{p(i) : p \in \mathcal{F}\}$ is almost disjoint. For $p \in \mathcal{F}$ choose $A_p \in [p(i)]^\omega$ such that $A_p \cap A_q = \emptyset$ for distinct $p, q \in \mathcal{F}$. We claim that the set

$$\mathcal{A}' = \{p \in \mathcal{A} : \forall q \in \mathcal{F}(|p(i) \cap A_q| < \omega)\}$$

has size $\geq \mathfrak{b}$. Otherwise, for $p \in \mathcal{A}'$ define $g_p \in {}^{\mathcal{F}}\omega$ by $g_p(q) = \max(p(i) \cap A_q)$, and choose $g \in {}^{\mathcal{F}}\omega$ such that for all $p \in \mathcal{A}'$, $g(q) \geq g_p(q)$ for almost all $q \in \mathcal{F}$. Let $A \in [\omega]^\omega$ be such that $|A \cap A_p| = |A \cap (A_p \setminus g(p))| = 1$ for all $p \in \mathcal{F}$. By the maximality of $\mathcal{A}$, there exists $p \in \mathcal{A}$ which is compatible with the cube $\{\langle i, A \rangle\} \cup C$. Certainly $p \notin \mathcal{F}$ and $p \restriction F || q \restriction F$ for every $q \in \mathcal{F}$, and hence $p \in \mathcal{A}'$. But this is impossible. □

A simple application of Ramsey's Theorem shows that given an infinite partition of $(\mathcal{P}(\omega)/\operatorname{fin})^{n^*}$, there exists $i < n^*$ such that $\operatorname{pr}_{\{i\}}\mathcal{A}$ contains an infinite almost disjoint family. As a corollary of the proof of Lemma 1.1 we obtain the stronger result that for some $i < n^*$, $\operatorname{pr}_{\{i\}}\mathcal{A}$ even contains an almost disjoint family of size $\mathfrak{t}$.

Corollary 1.3. *Suppose* $\mathcal{A}$ *is an infinite partition of* $(\mathcal{P}(\omega)/\operatorname{fin})^{n^*}$. *There exists* $i < n^*$ *such that* $\operatorname{pr}_{\{i\}}(\mathcal{A})$ *contains an almost disjoint family of size* $\mathfrak{t}$.

Proof. By Lemma 1.1 there exist $\mathcal{F}_0 \in [\mathcal{A}]^\omega$ and $i < n^*$ such that $\operatorname{pr}_{n^* \setminus \{i\}}\mathcal{F}_0$ has a lower bound, say cube C_0. Hence $\operatorname{pr}_{\{i\}}\mathcal{F}_0$ is countable and almost disjoint.

Suppose that for $\gamma < \mathfrak{t}$, $\langle \mathcal{F}_\alpha : \alpha < \gamma \rangle$ and $\langle C_\alpha : \alpha < \gamma \rangle$ have been constructed such that $\langle \mathcal{F}_\alpha : \alpha < \gamma \rangle$ is a $\subseteq$ $-$increasing chain in $[\mathcal{A}]^{<\mathfrak{t}}$, $\langle C_\alpha : \alpha < \gamma \rangle$ is a decreasing chain of cubes in $(\mathcal{P}(\omega)/\operatorname{fin})^{n^* \setminus \{i\}}$ and $C_\alpha \leq p \restriction n^* \setminus \{i\}$ for every $p \in \mathcal{F}_\alpha$.

Let γ be a successor ordinal. Note that $\{p(i) : p \in \mathcal{F}_{\gamma-1}\}$ is an almost disjoint family. As $\mathfrak{t} \leq \mathfrak{a}$, it is not maximal. Choose $A \in [\omega]^\omega$ such that $|A \cap p(i)| < \omega$ for every $p \in \mathcal{F}_{\gamma-1}$. Find $p \in \mathcal{A}$ which is compatible with $\{\langle i, A\rangle\} \cup C_{\gamma-1}$. Let $\mathcal{F}_\gamma = \mathcal{F}_{\gamma-1} \cup \{p\}$ and $C_\gamma = C_{\gamma-1} \wedge p{\restriction}n^* \setminus \{i\}$.

If γ is a limit let $\mathcal{F}_\gamma = \bigcup\{\mathcal{F}_\alpha : \alpha < \gamma\}$ and choose C_γ such that $C_\gamma \leq C_\alpha$ for every $\alpha < \gamma$. Such C_γ can be found as $\gamma < \mathfrak{t}$. Then the inductive assumption is easily verified.

Finally let $\mathcal{F}_\mathfrak{t} = \bigcup\{\mathcal{F}_\alpha : \alpha < \mathfrak{t}\}$. Given distinct $p, q \in \mathcal{F}_\mathfrak{t}$, by construction $p{\restriction}n^* \setminus \{i\}, q{\restriction}n^* \setminus \{i\}$ are compatible, and hence $p(i), q(i)$ are almost disjoint. □

2. Uncountable partitions of countable products of $\mathcal{P}(\omega)/\text{fin}$.

Partitions of $(\mathcal{P}(\omega)/\text{fin})^\lambda$, for $\lambda \geq \omega$, are considerably more difficult to understand than those of finite products. First note that there always exist partitions of every size $\leq \lambda$, as is shown by the following example.

In order to have a simple notation for defining cubes, if A, B are sets and B has one element, in the sequel we will identify the (one-element) set ${}^AB$ with its member.

Example 2.1. Let $\gamma \leq \lambda$. Choose $A \in [\omega]^\omega$ such that $-A \in [\omega]^\omega$. For $\alpha < \gamma$ set

$$p_\alpha = -A^\alpha \times A \times \omega^{\lambda \setminus (\alpha+1)},$$

and let $\mathcal{F} = \{p_\alpha : \alpha < \gamma\}$. Moreover, set $q = -A^\gamma \times \omega^{\lambda\setminus\gamma}$ and $\mathcal{A} = \mathcal{F} \cup \{q\}$.

Given cube p which is incompatible with q, there exists minimal $\alpha < \gamma$ such that $p(\alpha) \subseteq^* A$. Then clearly $p||p_\alpha$. It is easily seen that $\mathcal{A}$ is pairwise incompatible. Hence $\mathcal{A}$ is a partition of $(\mathcal{P}(\omega)/\text{fin})^\lambda$.

The main result of this section is the following.

Theorem 2.2. $\mathfrak{a}(\omega) \stackrel{\cdot}{\geq} \mathfrak{b}$.

A simple observation is that the analogue of Corollary 1.3 badly fails in general for infinite products.

Example 2.3. Let $A \in [\omega]^\omega$ with $-A \in [\omega]^\omega$. Then $\mathcal{A} = {}^\lambda\{A, -A\}$ is a partition of $(\mathcal{P}(\omega)/\text{fin})^\lambda$.

However there exist partitions of $(\mathcal{P}(\omega)/\text{fin})^\omega$ which resemble ones of finite products, and to these we will often reduce more difficult situations during the proof of Theorem 2.2 later. Then we will apply the following generalizations of Lemma 1.1 and Theorem 1.2.

Lemma 2.4. *Suppose $\lambda \geq \omega$ and $\mathcal{A}$ is a partition of $(\mathcal{P}(\omega)/\text{fin})^\lambda$ with the property that there exist $F \subseteq \lambda$ and $\mathcal{F} \in [\mathcal{A}]^\omega$ such that $\lambda \setminus F$ is finite and*

$\{p{\restriction}F : p \in \mathcal{F}\}$ has the finite intersection property. If F is maximal (with respect to $\subseteq$) such that there is $\mathcal{F}$ as stated, then $\lambda \setminus F$ has one element.

Proof. The proof is completely analogous to the one of Lemma 1.1. Instead of a descending chain of finite-dimensional cubes C_n in $(\mathcal{P}(\omega)/\operatorname{fin})^F$ we construct one of infinite-dimensional cubes. The family $\mathcal{F}$ ensures that $\mathcal{A}{\restriction}^{\lambda\setminus F}\{\omega\}^\frown C_n$ is infinite always, and hence that the construction yields infinitely many cubes K_n and $p_n \in \mathcal{A}$. □

By a similar modification of the proof of Theorem 1.2 we obtain the following.

Proposition 2.5. *Suppose that $\mathcal{A}$ is as in Lemma 2.4. Then $|\mathcal{A}| \geq \mathfrak{b}$.*

Consider another example, which will motivate the definition to follow.

Example 2.6. Let $\sigma \in {}^n2$, and $i < 2$. Define cube $p^{\sigma,i}$ as follows. For $j < n$ let

$$p^{\sigma,i}{\restriction}\{2j, 2j+1\} = \begin{cases} A \times -A & \text{if } \sigma(j) = 0 \\ -A \times A & \text{if } \sigma(j) = 1. \end{cases}$$

Moreover

$$p^{\sigma,i}{\restriction}\{2n, 2n+1\} = \begin{cases} A \times A & \text{if } i = 0 \\ -A \times -A & \text{if } i = 1. \end{cases}$$

For $j \geq 2n+2$ let $p^{\sigma,i}(j) = \omega$. Set $\mathcal{F} = \{p^{\sigma,i} : \sigma \in {}^{<\omega}2,\ i < 2\}$. Clearly $\mathcal{F}$ is pairwise incompatible. For $x \in {}^\omega 2$ define cube q_x by

$$q_x{\restriction}\{2j, 2j+1\} = \begin{cases} A \times -A & \text{if } x(j) = 0 \\ -A \times A & \text{if } x(j) = 1 \end{cases}$$

for all $j < \omega$.

Now it is easy to see that $\mathcal{F} \cup \{q_x : x \in {}^\omega 2\}$ is a partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$. Moreover, if $\mathcal{A}$ is any partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$ with $\mathcal{F} \subseteq \mathcal{A}$, then for every $p \in \mathcal{A} \setminus \mathcal{F}$ there exists $x \in {}^\omega 2$ such that $p \leq q_x$. Hence $\mathcal{A}$ has size $\mathfrak{c}$.

Definition 2.7. Given cubes $C_0, C_1, C_2 \in (\mathcal{P}(\omega)/\operatorname{fin})^\omega$ we say that C_0 *separates* C_1, C_2, if $C_0 \perp C_1$ and $C_0 \perp C_2$ and every cube which is compatible with both C_1 and C_2 is compatible with C_0. If $\mathcal{A}$ is a partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$ we say that C_0 *$\mathcal{A}$−separates* C_1, C_2 if C_0 separates C_1, C_2 and $\mathcal{A}{\restriction}C_0$ is countable, but both $\mathcal{A}{\restriction}C_1$ and $\mathcal{A}{\restriction}C_2$ are uncountable. Given cube

C, we say "*$\mathcal{A}$ has no separation (below C)*" if there exist no cubes C_0, C_1, C_2 (below C) such that C_0 $\mathcal{A}$–separates C_1, C_2.

Proposition 2.8. *Let $\mathcal{A}$ be an uncountable partition of $(\mathcal{P}(\omega)/\mathrm{fin})^\omega$, and suppose that for every cube C such that $\mathcal{A}\restriction C$ is uncountable there exist cubes $C_0, C_1, C_2 \leq C$ such that C_0 $\mathcal{A}$–separates C_1, C_2. Then $\mathcal{A}$ has size $\mathfrak{c}$.*

Proof. By hypothesis it is trivial to construct two families of cubes, $\langle C_s : s \in {}^{<\omega}2\rangle$ and $\langle D_s : s \in {}^{<\omega}2\rangle$, such that for all $s \in {}^{<\omega}2$ the following properties hold:

(1) $C_\emptyset = C$;

(2) $s \subseteq t \Rightarrow D_s \leq C_s$ and $C_t \leq C_s$;

(3) D_s $\mathcal{A}$–separates $C_{s^\frown\langle 0\rangle}, C_{s^\frown\langle 1\rangle}$.

Let $\mathcal{C}_s = \mathcal{A}\restriction D_s$ and $\mathcal{C} = \bigcup\{\mathcal{C}_s : s \in {}^{<\omega}2\}$. So $\mathcal{C}$ is countable. For $x \in {}^\omega 2$ let C_x be a cube with $C_x \leq C_{x\restriction n}$ for every $n < \omega$. Suppose $x, y \in {}^\omega 2$ are distinct, and let $s = x \cap y$. Choose $p_x, p_y \in \mathcal{A}$ such that $p_x || C_x$ and $p_y || C_y$. Since D_s separates C_x, C_y, if $p_x = p_y$ then $p_x \in \mathcal{C}_s$. Hence, in order to finish the proof it suffices to modify the construction of $\langle C_s : s \in {}^{<\omega}2\rangle$, $\langle D_s : s \in {}^{<\omega}2\rangle$ to make sure that $C_x \perp p$ for every $x \in {}^\omega 2$ and $p \in \mathcal{C}$. For this we need the following simple claim.

Claim . *Suppose $\mathcal{A}\restriction C$ is uncountable and $p \in \mathcal{A}$. There exists cube $C_0 \leq C$ such that $C_0 \perp p$ and $\mathcal{A}\restriction C_0$ is uncountable.*

Proof. It is obvious that every $q \in \mathcal{A}\restriction C \setminus \{p\}$ is compatible with $\{\langle n, C(n) \setminus p(n)\rangle\} \cup C\restriction\omega\setminus\{n\}$, for some n. Hence there exists n^* such that $\mathcal{A}\restriction\{\langle n^*, C(n^*)\setminus p(n^*)\rangle\} \cup C\restriction\omega\setminus\{n^*\}$ is uncountable. So let $C_0 = \{\langle n^*, C(n^*)\setminus p(n^*)\rangle\} \cup C\restriction\omega\setminus\{n^*\}$. □

Using the Claim and a suitable bookkeeping it is easy to construct $\langle C_s : s \in {}^{<\omega}2\rangle$, $\langle D_s : s \in {}^{<\omega}2\rangle$ with (1) – (3) such that for every $s \in {}^{<\omega}2$ and $p \in \mathcal{C}_s$ there exists $n > \ell h(s)$ so that for every $t \in {}^n 2$ we have $C_t \perp p$. By the observation above, this is enough to ensure $p_x \neq p_y$ for distinct $x, y \in {}^\omega 2$, and hence to conclude $|\mathcal{A}| = \mathfrak{c}$. □

By Proposition 2.8 we may assume that there exists cube C such that $\mathcal{A}\restriction C$ is uncountable and $\mathcal{A}$ has no separation below C. By replacing $\mathcal{A}$ with $\{p \wedge C : p \in \mathcal{A}\restriction C\}$, without loss of generality we may assume $C(n) = \omega$ for all $n < \omega$ and so that $\mathcal{A}$ has no separation.

Examples 2.1 and 2.6 show that given a partition $\mathcal{A}$, the set $\mathcal{F}$ of $p \in \mathcal{A}$ with $p(n) =^* \omega$ for almost all n essentially affects the size and shape of $\mathcal{A}$. If $\mathcal{F}$ is uncountable, then $\mathcal{A}$ has size $\geq \mathfrak{b}$ by Proposition 2.5. If $\mathcal{F}$ is empty, then $|\mathcal{A}| \geq \mathfrak{b}$ by the following proposition. For its proof we do not make use of our assumption that $\mathcal{A}$ has no separation. This will only be used later in the case that $\mathcal{F}$ is countable and not empty.

Proposition 2.9. *Let $\mathcal{A}$ be an uncountable partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$, and suppose that there exists cube C_0 such that for every $p \in \mathcal{A}$, either $p \perp C_0$ or there exist infinitely many $n < \omega$ with the property that $C_0(n) \setminus p(n)$ is infinite. Then $|\mathcal{A}| \geq \mathfrak{b}$.*

Proof. Choose $p_0 \in \mathcal{A} \restriction C_0$ such that $|p_0(n) \cap C_0(n)| = \omega$ for almost all n. Such p_0 exists by maximality of $\mathcal{A}$. Suppose that $\langle p_\alpha : \alpha < \gamma\rangle, \langle C_\alpha : \alpha < \gamma\rangle$ have been constructed. If γ is a successor let $C_\gamma \leq C_{\gamma-1}$ be defined by setting $C_\gamma(n) = C_{\gamma-1}(n) \cap p_{\gamma-1}(n)$ if this intersection is infinite, and $C_\gamma(n) = C_{\gamma-1}(n)$ otherwise. Moreover, choose $p_\gamma \in \mathcal{A} \restriction C_0 \setminus \{p_\alpha : \alpha < \gamma\}$, if possible, such that $|p_\gamma(n) \cap C_\gamma(n)| = \omega$ for almost all n. If γ is a limit choose cube C_γ, if possible, such that $C_\gamma \leq C_\alpha$ for all $\alpha < \gamma$, and then choose $p_\gamma \in \mathcal{A} \restriction C_0 \setminus \{p_\alpha : \alpha < \gamma\}$, if possible, as in the successor case.

Suppose that this construction does not stop at any stage $\gamma < \omega_1$, so we construct $\langle p_\alpha : \alpha < \omega_1\rangle$, $\langle C_\alpha : \alpha < \omega_1\rangle$. Then by construction, for every $\alpha < \omega_1$ there exists minimal $n_\alpha \in \omega$ such that $C_\beta(n) \subseteq^* p_\alpha(n)$, for all $\beta > \alpha$ and $n \geq n_\alpha$. Find $\{\alpha_k : k < \omega\} \in [\omega_1]^\omega$ such that $\langle n_{\alpha_k} : k < \omega\rangle$ is constant, and let $\alpha^* = \sup\{\alpha_k + 1 : k < \omega\}$. But then C_{α^*}, $\{p_{\alpha_k} : k < \omega\}$ are as in Proposition 2.5, and hence we conclude $|\mathcal{A}| \geq \mathfrak{b}$.

Otherwise the construction stops at some $\gamma < \omega_1$. So we get $\langle C_\alpha : \alpha < \gamma\rangle$, $\langle p_\alpha : \alpha < \gamma\rangle$ and by σ-closedness of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$ also C_γ, but we cannot find p_γ as desired, i.e.

$$\forall p \in \mathcal{A} \restriction C_0 \setminus \{p_\alpha : \alpha < \gamma\} \exists^\infty n < \omega \, (p(n) \cap C_\gamma(n) =^* \emptyset)\,. \tag{1}$$

Define cube D by $D(n) = C_0(n) \setminus C_\gamma(n)$ if this set is infinite, and $D(n) = C_0(n)$ otherwise. By hypothesis and construction we know

$$\forall \alpha < \gamma \exists^\infty n < \omega \, (|D(n) \setminus p_\alpha(n)| = \omega)\,. \tag{2}$$

Enumerate $\{p_\alpha : \alpha < \gamma\}$ by $\langle q_n : n < \omega\rangle$. Define the following sets

$$X_m = \{n < \omega : |D(n) \setminus q_m(n)| = \omega\} \tag{3}$$

$$Y_p = \{n < \omega : p(n) \cap C_\gamma(n) =^* \emptyset\}\,. \tag{4}$$

By (1) and (2) we conclude that each X_m and Y_p for $p \in \mathcal{A} \restriction C_0 \setminus \{p_\alpha : \alpha < \gamma\}$ is infinite.

Lemma 2.10. *For any $\{X_n : n < \omega\}, \{Y_\alpha : \alpha < \kappa\}$ sets of infinite subsets of ω where $\kappa < \mathfrak{b}$ there exists an increasing sequence $\langle k_n : n < \omega\rangle$ such that*

$k_n \in X_n$ for every n, and for each $\alpha < \kappa$, for almost all n, $Y_\alpha \cap (k_n, k_{n+1})$ is nonempty.

Proof. For $\alpha < \kappa$ let $f_\alpha \in {}^\omega\omega$ be defined by

$$f_\alpha(n) = \min\{m \in Y_\alpha : m > n\}.$$

Since $\kappa < \mathfrak{b}$ we may find a strictly increasing $f \in {}^\omega\omega$ which eventually dominates every f_α. Let f' be the iterate of f, defined by $f'(0) = f(0), f'(n+1) = f(f'(n))$. Then f' has the property that for every $\alpha < \kappa$, for almost all n, $(f'(n), f'(n+1)) \cap Y_\alpha \neq \emptyset$. In fact, if $f(n) > f_\alpha(n)$ for all $n > k$ and if $f'(n) > k$ then $f'(n+1) = f(f'(n)) > f_\alpha(f'(n)) > f'(n)$.

Now define $\langle k_n : n < \omega\rangle$ as follows: $k_0 = \min(X_0)$, k_{n+1} is the minimum element of X_{n+1} such that for some i, $k_n < f'(i) < f'(i+1) < k$. Then $\langle k_n : n < \omega\rangle$ is as desired. □

If $|\mathcal{A}| < \mathfrak{b}$, find $\langle k_n : n < \omega\rangle$ as in Lemma 2.10 for $\{X_m : m < \omega\}, \{Y_p : p \in \mathcal{A}\restriction C_0 \setminus \{p_\alpha : \alpha < \gamma\}\}$ as defined in (3), (4). Define cube C as follows: $C(k_n) = D(k_n) \setminus q_n(k_n)$, $C(n) = C_\gamma(n)$ if $n \notin \{k_n : n < \omega\}$. Then using (1) and (2) we easily see that C is incompatible with every member of $\mathcal{A}$, a contradiction. Hence $|\mathcal{A}| \geq \mathfrak{b}$. □

By Proposition 2.9 we may assume that for every cube C_0 there exists $p \in \mathcal{A}\restriction C_0$ such that $C_0(n) \subseteq^* p(n)$ for almost all $n < \omega$. Note that $\mathcal{A}$ in Example 2.6 has this property. The following lemma will be crucial in this case.

Lemma 2.11. *Let $\mathcal{A}$ be a partition of $(\mathcal{P}(\omega)/\text{fin})^\omega$. Suppose that $\mathcal{A}$ has no separation, cube C is such that $\mathcal{A}\restriction C$ is uncountable, and $p \in \mathcal{A}$ is such that $p||C$ and $C(n) \subseteq^* p(n)$ for almost all n. Then there exists $n^* < \omega$ such that $\mathcal{A}\restriction\{\langle n^*, C(n^*) \cap p(n^*)\rangle\} \cup C\restriction\omega \setminus \{n^*\}$ is countable.*

Proof. Let $A = \{n < \omega : |C(n) \setminus p(n)| = \omega\}$. By assumption $A \neq \emptyset$ and A is finite. For $B \subseteq A$ let C_B be the cube defined by $C_B(n) = C(n) \setminus p(n)$ if $n \in B$, and $C_B(n) = p(n) \cap C(n)$ otherwise. Moreover let $\mathcal{F} = \{C_B : B \subseteq A\}$, and for $k \leq |A|$ let $\mathcal{F}_k = \{C_B : |B| = k\}$. Clearly $\mathcal{F}$ is a finite partition of C and hence there exists a minimal $k^* \leq |A|$ such that for some $q \in \mathcal{F}_{k^*}$, $\mathcal{A}\restriction q$ is uncountable. Clearly $k^* > 0$.

If $k^* = 1$ choose $n^* \in A$ such that $\mathcal{A}\restriction C_{\{n^*\}}$ is uncountable. But p separates $C_{\{n^*\}}$, q for every $q \in \mathcal{F} \setminus \{p\}$ with $q(n^*) = p(n^*) \cap C(n^*)$. Hence $\mathcal{A}\restriction q$ is countable for all these q, and hence $\mathcal{A}\restriction\{\langle n^*, C(n^*) \cap p(n^*)\rangle\} \cup C\restriction\omega \setminus \{n^*\}$ is countable.

If $k^* > 1$, choose $B \subseteq A$ of size k^* such that $\mathcal{A}\restriction C_B$ is uncountable. Let $n^* \in B$ be arbitrary. We claim that $\mathcal{A}\restriction\{\langle n^*, C(n^*) \cap p(n^*)\rangle\} \cup C\restriction\omega \setminus \{n^*\}$ is countable. Otherwise there exists $B' \subsetneq A$ such that $\mathcal{A}{\restriction}C_{B'}$ is uncountable and $n^* \notin B'$. Then $|B'| \geq k^*$ and hence $B' \setminus B \neq \emptyset$. Consequently $C_{B\cap B'}$ separates C_B, $C_{B'}$. But $|B \cap B'| < k^*$, and so $\mathcal{A}{\restriction}C_{B\cap B'}$ is countable. This contradicts the hypothesis. $\square$

Note that if in Lemma 2.11 we let $C_0 = \{\langle n^*, C(n^*) \setminus p(n^*)\rangle\} \cup C{\restriction}\omega \setminus \{n^*\}$ and $\mathcal{C} = \mathcal{A}{\restriction}\{\langle n, C(n^*) \cap p(n^*)\rangle\} \cup C{\restriction}\omega \setminus \{n^*\}$, then clearly $C_0 \perp p$, $\mathcal{C}$ is countable, and for every $q \in \mathcal{A}{\restriction}C \setminus \mathcal{C}$, we have $q \wedge C \leq C_0$.

Proposition 2.12. *Let $\mathcal{A}$ be an uncountable partition of $(\mathcal{P}(\omega)/\operatorname{fin})^\omega$ which has no separation. Then $\mathcal{A}$ has size at least $\mathfrak{b}$.*

Proof. Let $\mathcal{F}_0 = \{p \in \mathcal{A} : \forall^\infty n\, (\omega \subseteq^* p(n))\}$. If $\mathcal{F}_0$ is empty, by Proposition 2.9 we conclude $|\mathcal{A}| \geq \mathfrak{b}$. If $\mathcal{F}_0$ is uncountable, then by an easy application of Proposition 2.5 we conclude $|\mathcal{A}| \geq \mathfrak{b}$. Hence we may assume that $\mathcal{F}_0$ is nonempty and countable. Then by applying Lemma 2.11 ω times we can construct a descending chain $\langle C_i : i < \omega\rangle$ of cubes and a sequence $\langle \mathcal{C}_i : i < \omega\rangle$ of countable subsets of $\mathcal{A}$ such that for every $p \in \mathcal{F}_0$ there exists $i < \omega$ such that $p \perp C_i$, and moreover for every $q \in \mathcal{A}$, either $q \leq C_i$ for all $i < \omega$, or else $q \in \mathcal{C}_i$ where i is minimal with $q \not\leq C_i$.

Now suppose that for some limit ordinal $\gamma < \omega_1$, $\langle C_\alpha : \alpha < \gamma\rangle$, $\langle \mathcal{C}_\alpha : \alpha < \gamma\rangle$ and $\langle \mathcal{F}_\alpha : \alpha \in \lim(\gamma)\rangle$ have been constructed such that the following properties are satisfied:

(1) $\langle C_\alpha : \alpha < \gamma\rangle$ is a descending chain of cubes;

(2) $\mathcal{C}_\alpha \in [\mathcal{A}]^{\leq\omega}$;

(3) $\forall p \in \mathcal{A}\, (p \in \bigcup\{\mathcal{C}_\alpha : \alpha < \gamma\} \vee \forall \alpha < \gamma\, (p \leq C_\alpha))$;

(4) $\mathcal{F}_\alpha = \{p \in \mathcal{A}{\restriction}C_\alpha : \forall^\infty n\, (C_\alpha(n) \subseteq^* p(n))\}$, $\mathcal{F}_\alpha$ is nonempty and countable, and $\forall p \in \mathcal{F}_\alpha \exists i < \omega\, (p \perp C_{\alpha+i})$.

Set $\mathcal{C}_\gamma = \bigcup\{\mathcal{C}_\alpha : \alpha < \gamma\}$. Clearly $\mathcal{C}_\gamma$ is countable. If $|\mathcal{A}| < \mathfrak{b}$, we define cube C_γ as follows. In case $\langle C_\alpha(n) : \alpha < \gamma\rangle$ is eventually constant, let $C_\gamma(n)$ be this constant value. Otherwise, $\langle C_\alpha(n) : \alpha < \gamma\rangle$ contains a cofinal $\subset^*$-decreasing subsequence of length ω. Moreover, by (3), $q(n) \subseteq^* C_\alpha(n)$ holds for all $q \in \mathcal{A} \setminus \mathcal{C}_\gamma$ and $\alpha < \gamma$. By Rothberger's characterization of $\mathfrak{b}$ (see [**vD**, Theorem 3.3, p. 117]), there exists $X \in [\omega]^\omega$ such that $q(n) \subseteq^* X \subseteq^* C_\alpha(n)$ holds for all $q \in \mathcal{A} \setminus \mathcal{C}_\gamma$ and $\alpha < \gamma$. Set $C_\gamma(n) = X$.

Consequently, we have constructed cube C_γ such that $q \leq C_\gamma$ for all $q \in \mathcal{A} \setminus \mathcal{C}_\gamma$, and moreover $C_\gamma \leq C_\alpha$ for every $\alpha < \gamma$ and, by (4), $C_\gamma \perp p$ for all $p \in \mathcal{F}_\alpha$ where $\alpha \in \lim(\gamma)$. Especially $\mathcal{A}{\restriction}C_\omega$ is uncountable.

If now no $p \in \mathcal{A}{\restriction}C_\gamma$ satisfies $C_\gamma(n) \subseteq^* p(n)$ for almost all $n < \omega$, by Proposition 2.9 we conclude $|\mathcal{A}| \geq \mathfrak{b}$. Otherwise let

$$\mathcal{F}_\gamma = \{p \in \mathcal{A}{\restriction}C_\gamma : \forall^\infty n\, (C_\gamma(n) \subseteq^* p(n))\}.$$

So $\mathcal{F}_\gamma \neq \emptyset$. By Proposition 2.5 we may assume that $\mathcal{F}_\gamma$ is countable. Note that $\mathcal{F}_\gamma$ and $\mathcal{F}_\alpha$ are disjoint for any $\alpha < \gamma$. Since by assumption $\mathcal{A}$ has no separation below C_γ, using Lemma 2.11 we can repeat the construction above and obtain families $\langle C_\alpha : \gamma < \alpha < \gamma + \omega \rangle$ and $\langle \mathcal{C}_\alpha : \gamma < \alpha < \gamma + \omega \rangle$ such that (1) – (4) hold for $\gamma + \omega$ instead of γ.

Since $\mathcal{A}$ has no separation, only the fact $|\mathcal{A}| \geq \mathfrak{b}$ can cause this construction to stop at some stage $\gamma < \omega_1$. Hence we may assume that it does not stop below ω_1, and so we obtain $\langle C_\alpha : \alpha < \omega_1 \rangle$, $\langle \mathcal{C}_\alpha : \alpha < \omega_1 \rangle$ and $\langle \mathcal{F}_\alpha : \alpha \in \lim(\omega_1) \rangle$ with (1) – (4).

Pick $p_\alpha \in \mathcal{F}_\alpha$ for every $\alpha \in \lim(\omega_1)$. So by (4) there exists $n_\alpha < \omega$ such that $C_\alpha(n) \subseteq^* p_\alpha(n)$ for all $n \geq n_\alpha$. Find $A \in [\omega_1]^{\omega_1}$ and n^* such that $n_\alpha = n^*$ for all $\alpha \in A$. Let $\{\alpha_k : k < \omega\}$ be the first ω members of A and let $\alpha^* = \sup\{\alpha_k : k < \omega\}$. Then by (1) and (4) we conclude

$$\forall k < \omega \forall n \geq n^* \left(C_{\alpha^*}(n) \subseteq^* p_{\alpha_k}(n) \right).$$

But then by Proposition 2.5 we conclude that $\mathcal{A}$ has size at least $\mathfrak{b}$. □

3. The consistency of $\mathfrak{b} < \mathfrak{a}(n)$.

Following [**Sh1**, §§1, 2] closely we will sketch the proof of the following.

Theorem 3.1. *If ZFC is consistent then so is* $ZFC + \forall n < \omega(\mathfrak{b} < \mathfrak{a}(n))$.

A revised version of [**Sh1**] will appear in [**Sh2**]. Since there are several gaps in the proof of the consistency of $\mathfrak{b} < \mathfrak{a}$ in [**Sh1**] the reader should also consult [**Sh2**].

Definition 3.2. Given cubes $C, D \in (\mathcal{P}(\omega)/\text{fin})^\lambda$ we say that C *splits* D if $C||D$ and there exists cube $D' \leq D$ such that $D' \perp C$. We call $\mathcal{F} \subseteq (\mathcal{P}(\omega)/\text{fin})^\lambda$ a *splitting family* if every member of $(\mathcal{P}(\omega)/\text{fin})^\lambda$ is split by some member of $\mathcal{F}$. Let $\mathfrak{s}(\lambda)$ be the least size of a splitting family in $(\mathcal{P}(\omega)/\text{fin})^\lambda$.

Fix $n^* < \omega$. First we define a forcing Q (really $Q(n^*)$) which is almost ${}^\omega\omega$–bounding in the sense of [**Sh1**, 1.4] and adds a n^*–dimensional cube which is not split by any cube from the ground model.

Definition 3.3. For $n < \omega$ let K_n be the set of pairs (s, h), s a finite set, h a partial function of $\mathcal{P}(s)$ to $n+1$ such that $h(s) = n$, and for every $t \subseteq s$, if $h(t) = l + 1$ and $t = t_1 \cup t_2$ then $h(t_1) \geq l$ or $h(t_2) \geq l$.

Let $K_{\geq n} = \bigcup\{K_i : i \geq n\}$, and let $K = \bigcup\{K_n : n < \omega\}$.

Then K is partially ordered as defined in [**Sh1**, 2.2.].

Definition 3.4. For $n < \omega$ let L_n be the set of pairs (S, H) such that

(1) S is a finite tree with a root such that $\text{in}(S)$, the set of nonmaximal nodes of S, is contained in ω, and $\text{int}(S)$, the set of maximal nodes of S, is contained in ${}^{n^*}\omega$;

(2) H is a function with domain $\mathrm{in}(S)$ and value H_x for $x \in \mathrm{in}(S)$ such that $(\mathrm{succ}_S(x), H_x) \in K_{\geq n}$, where $\mathrm{succ}_S(x)$ is the set of immediate successors of x in S.

Then L_n is partially ordered as in [**Sh1**, 2.4(2)]. Let $\mathrm{lev}(S, H) = \max\{n : (S, H) \in L_n\}$; for $t \in L_n$ we write $t = (S^t, H^t)$.

The following fact is proved by induction on the height of the tree S.

Fact 3.5. *If $(S, H) \in L_{n+1}$, $\mathrm{int}(S) = A_0 \cup A_1$, then there exists $(S^1, H^1) \in L_n$ such that $(S^1, H^1) \leq (S, H)$ and either $\mathrm{int}(S^1) \subseteq A_0$ or $\mathrm{int}(S^1) \subseteq A_1$.*

Definition 3.6. We define a forcing notion Q as follows. Members of Q are pairs (w, T) such that $w \subseteq {}^{n^*}\omega$ is finite, and T is a countable infinite subset of $\bigcup\{L_n : n < \omega\}$ such that $T \setminus L_n$ is finite for every $n < \omega$ and moreover, for every $n < \omega$, $\mathrm{field}(\mathrm{int}(S^t)) \cap n$ is empty for almost all $t \in T$.

The ordering of Q is analogously defined as in [**Sh1**, 2.8(2,3)].

Lemma 3.7.

(1) *Q is proper;*

(2) *Suppose G is Q-generic over V. Define n^*-dimensional cube C_G by $C_G(i) = \{\sigma(i) : \sigma \in \mathrm{pr}_0 G\}$, for every $i < n^*$. Then C_G is not split by any cube in V.*

Proof. (1) is proved analogously as [**Sh**, 2.11(2)].

For (2), let $C \in (\mathcal{P}(\omega)/\mathrm{fin})^{n^*} \cap V$ and $(w, T) \in Q$. To every $\sigma \in \bigcup\{\mathrm{int}(t) : t \in T\}$ assign a colour $c \in {}^{n^*}2$ in such a way that for every $i < n^*$, $\sigma(i) \in C(i)$ if and only if $c(i) = 1$. So for every $t \in T$, $\mathrm{int}(t)$ gets coloured by (at most) 2^{n^*} colours. As $T \setminus L_n$ is finite for every $n < \omega$, by applying Fact 3.5 repeatedly we may find $(w, T') \in Q$ extending (w, T) and colour $c \in {}^{n^*}2$ such that for every $t \in T'$, every member of $\mathrm{int}(t)$ has colour c. By genericity we may assume $(w, T') \in G$.

Suppose first that c is constant with value 1. Then clearly $C_G \leq C$. Otherwise $c(i) = 0$ for some $i < n^*$, hence $C_G(i) \cap C(i)$ is finite, and so $C_G \perp C$. Hence in either case, C does not split C_G. □

Similarly as in [**Sh1**, 2.12, 2.13] one proves that forcing Q is almost ${}^{\omega}\omega$-bounding in the sense of [**Sh1**, 1.4]. Moreover in [**Sh1**, §1] it is proved that the property of being almost ${}^{\omega}\omega$-bounding is preserved by countable support iterations.

Fix an infinite partition $\mathcal{A}$ of $(\mathcal{P}(\omega)/\mathrm{fin})^{n^*}$.

Definition 3.8. Define a subforcing $Q[\mathcal{A}] \subseteq Q$ by letting $(w, T) \in Q[\mathcal{A}]$ if and only if for every finite subset $F \subseteq \mathcal{A}$ there exist infinitely many $t \in T$ such that $\mathrm{int}(S^t) \cap C = \emptyset$ for all $C \in F$. (Note that here we identify C with $\prod_{i<n^*} C(i)$.)

Similarly as in [**Sh1**, 2.16A] one proves that if forcing with Q preserves $\mathcal{A}$, then $Q[\mathcal{A}]$ has a nice dense subset.

Lemma 3.9. *Suppose that after forcing with Q, $\mathcal{A}$ is still a partition of $(\mathcal{P}(\omega)/\text{fin})^{n^*}$. Then the set of all $(w,T) \in Q[\mathcal{A}]$ with the property that there exists $\mathcal{F} \in [\mathcal{A}]^\omega$ such that for every $C \in \mathcal{F}$ the set $\{t \in T : \text{int}(S^t) \subseteq C\}$ is infinite and $\bigcup\{\text{int}(S^t) : t \in T\} \subseteq \bigcup \mathcal{F}$ is dense in $Q[\mathcal{A}]$.*

Proof. Let $(w,T) \in Q[\mathcal{A}]$. Since forcing with Q preserves $\mathcal{A}$ and $(w,T) \in Q$ there exist $(w_0,T_0) \in Q$, $(w_0,T_0) \le (w,T)$, and $C_0 \in \mathcal{A}$ such that $(w_0,T_0) \Vdash_Q C_G \| C_0$. By the argument from the proof of Lemma 3.6(2) we may assume that $\text{int}(S^t) \subseteq C_0$ for all $t \in T_0$.

Suppose that $T_0, \dots, T_n$ and $C_0, \dots, C_n$ have been constructed. Note that the condition

$$\left(w, \left\{t \in T : \text{int}(S^t) \cap \bigcup_{i \le n} C_i = \emptyset\right\}\right)$$

belongs to $Q[\mathcal{A}]$. Hence, as in the first step, there exist an extension (w_{n+1}, T_{n+1}) of it in Q and $C_{n+1} \in \mathcal{A}$ such that $\text{int}(S^t) \subseteq C_{n+1}$ for all $t \in T_{n+1}$.

Finally let $T' = \bigcup\{T_n : n \in \omega\}$. Then $(w,T') \in Q[\mathcal{A}]$, $(w,T') \le (w,T)$, and (w,T') is as desired. □

Corollary 3.10. *Suppose that after forcing with Q, $\mathcal{A}$ is still a partition of $(\mathcal{P}(\omega)/\text{fin})^{n^*}$. Then forcing with $Q[\mathcal{A}]$ destroys $\mathcal{A}$.*

Proof. Let $C \in \mathcal{A}$ and (w,T) belong to the dense set from Lemma 3.9, witnessed by $\{C_n : n < \omega\} \in [\mathcal{A}]^\omega$. Let $T_n = \{t \in T : \text{int}(S^t) \subseteq C_n\}$.

Note that except for at most one n (for which $C_n = C$), for almost all $t \in T_n$, $\text{int}(S^t) \cap C = \emptyset$. Hence $(w, \{t \in T : \text{int}(S^t) \cap C = \emptyset\})$ belongs to $Q[\mathcal{A}]$, extends (w,T) and forces $C_G \perp C$. □

Now using 3.9 and 3.10, similarly as in [**Sh2**] one shows that if Q preserves $\mathcal{A}$, then first adding $\aleph_1$ Cohen reals and then forcing with $Q[\mathcal{A}]$ is almost ${}^\omega\omega$–bounding and destroys $\mathcal{A}$. Now it is clear how to construct a model for Theorem 3.1. □

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