PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 2, 1996

HEAT FLOW OF EQUIVARIANT HARMONIC MAPS FROM $\mathbb{B}^3$ INTO $\mathbb{CP}^2$

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We construct equivariant maps from $\mathbb{B}^3$ into $\mathbb{CP}^2$ and prove the global existence of heat flow of such equivariant harmonic maps for equivariant initial-boundary data which are not a priori required to have small range. We also show subconvergence of the solution. This supplies a regular harmonic extension of the given boundary condition.

1. Introduction.

The boundary value problem for harmonic maps has been studied by many mathematicians. For target manifolds with nonpositive sectional curvature, R. Hamilton [**H**] proved that such a boundary value problem is solvable by the heat flow method. In the case when the target manifolds have positive sectional curvature the situation becomes more complicated. If the boundary condition lies in a geodesic convex neighbourhood of the target manifold, S. Hildebrandt, H. Kaul and K.O. Widman [**H-K-W**] proved the existence of the boundary value problem by the direct method of the calculus of variations.

Although there exist examples to show the optimality of Hildebrandt-Kaul-Widman's theorem, one still expects the solvability for the boundary value problem with large image range when the boundary condition is "sufficiently nice". In [**J-K**] and [**E-L1**] the authors consider the rotationally symmetric harmonic maps from $\mathbb{B}^m$ into S^n whose boundary values lie just outside of a geodesic convex neighbourhood. Recently, many works have been written on maps from $\mathbb{B}^3$ into S^2 ([**Ha**], [**H-K-L1**], [**H-K-L2**], [**H-L-P**] and [**Z**]). Among them D. Zhang obtained a regular axially symmetric harmonic extension of $\mathbb{B}^3$ into S^2 for any regular axially symmetric boundary data which omit a neighbourhood of the south pole [**Z**]. As is well-known there are only two different kinds of isoparametric hypersurfaces in Euclidean space: umbilical ones and generalized cylinders. It is interesting to see that they correspond to the two kinds of reductions given by Jäger-Kaul and Zhang [**J-K**], [**Z**], respectively. By putting the problem in this framework with some essential technical improvement the result in [**Z**] has been improved in author's previous work [**X2**].

Zhang's result can also be proved by the heat flow method, as shown in [**G1**]. Coron-Ghidaglia first studied harmonic heat-flow into S^n ($n \geq 3$) for equivariant data [**C-G**]. In addition, there are several works on heat flow of equivariant harmonic maps; see [**C-D**], [**C-D-Y**], [**G2**] and [**G3**]. All of them treated the case when the target manifold is S^2 which can be viewed as a complex projective line $\mathbb{CP}^1$.

It is natural to study the similar problem when the target manifold is a higher dimensional complex projective space. This is the subject of the present paper. We concentrate on the case where the domain is the 3-dimensional unit ball, and the target manifold is $\mathbb{CP}^2$.

Let $\mathbb{B}^3$ be the 3-dimensional unit ball. Under an S^1 action the base region $D \in \mathbb{R}^2$ is given by

$$D = \{(r,z) \in \mathbb{R}^2; \quad r^2 + z^2 < 1, \quad r > 0\}.$$

Then $\tilde{r} = (r,z) : \mathbb{B}^3 \to D$ is an isoparametric map of rank 2. On the other hand the distance function from a fixed point in $\mathbb{CP}^2$ is an isoparametric function ϕ $(0 \leq \phi \leq \frac{\pi}{2})$. Let $f_1 : S^1 \to S^1, f_2 : S^1 \to \mathbb{CP}^1$ be harmonic maps of constant energy densities $\frac{\lambda_1}{2}$ and $\frac{\lambda_2}{2}$, respectively. In fact, f_1 and f_2 are harmonic polynomial maps and $\lambda_1 = k_1^2$, $\lambda_2 = k_2^2$, where k_1, $k_2 \in \mathbb{Z}$. We will explain in Section 2 how f_1 and f_2 can be used to define an equivariant map f from $\mathbb{B}^3$ into $\mathbb{CP}^2$. We obtain the following result.

Main Theorem 1 *For any equivariant initial-boundary condition with respect to the isoparametric map $\tilde{r}$ and the isoparametric function ϕ, whose restriction to $\bar{D}$ is a regular function ϕ_0 $(0 \leq \phi_0 \leq \frac{\pi}{2}, \quad \phi(0,z) = 0)$ on $\bar{D}$ and is of order $O\left(r^{\sqrt{\lambda_1+\lambda_2}}\right)$ as $r \to 0$, there exists a unique global solution to the evolution equation for the boundary value problem of harmonic maps from $\mathbb{B}^3$ into $\mathbb{CP}^2$. Furthermore, this solution subconverges to an equivariant harmonic map as $t \to \infty$.*

Remark. It is well-known that for complex projective space with the Fubini-Study metric, the sectional curvature lies between 1 and 4, the radius of the geodesic convex ball is $\frac{\pi}{4}$ and its diameter is $\frac{\pi}{2}$. The boundary condition in our theorem overpasses the convex ball and can reach any possible range.

The reminder of the paper is organized as follows:

2. The Geometry of $\mathbb{CP}^n$;
3. Construction of the Equivariant Maps into $\mathbb{CP}^2$;
4. Heat Flow;
    4.1. Short Time Existence,
    4.2. Barrier Functions,

4.3. Proof of the Main Theorem;
5. Final Remarks.

Acknowledgement. Parts of this work were carried out while the author was visiting to Tohôku University in Japan. He appreciates its hospitality. He is also grateful to Professor H. Urakawa for his kindness and helpful discussions on this work.

2. The Geometry of $\mathbb{CP}^n$.

Let $\pi : S^{2n+1} \to \mathbb{CP}^n$ be the usual Riemannian submersion with totally geodesic fibers S^1. For any $Z \in S^{2n+1}$ there exist $X \in S^{2n-1}$ and $Y \in S^1$ such that

$$Z = (X \sin\phi, Y \cos\phi), \quad 0 \le \phi \le \frac{\pi}{2}, \tag{2.1}$$

where ϕ is an isoparametric function on S^{2n+1} which is equivariant with respect to Riemannian submersion π. This induces an isoparametric function on $\mathbb{CP}^n$. We denote it by the same letter ϕ. The level hypersurfaces of ϕ are given by

$$M_\phi = S^{2n-1}(\sin\phi) \times S^1(\cos\phi)/S^1, \tag{2.2}$$

$$0 < \phi < \frac{\pi}{2}$$

with the focal point $A \in \mathbb{CP}^n$ and the focal variety $\mathbb{CP}^{n-1}$. One can easily see that M_ϕ is the geodesic sphere at the distance ϕ from A.

Every geodesic emanating from the point A lies in certain complex projective line passing through A. It follows that these projective lines are the integral manifolds of the distribution $\{n = \operatorname{grad}\phi, Jn\}$, where J is the complex structure of $\mathbb{CP}^n$. We know that $\mathbb{CP}^1 = S^2(\frac{1}{2})$ of constant sectional curvature 4, which is totally geodesic in $\mathbb{CP}^n$. The integral curves of $n = \operatorname{grad}\phi$ are geodesics in $\mathbb{CP}^n$. Thus $\mathbb{CP}^1$ has the metric form in polar coordinates

$$d\phi^2 + \left(\frac{1}{2}\sin 2\phi\right)^2 d\alpha^2, \tag{2.3}$$

where $0 \le \alpha \le 2\pi$. It follows that Jn lies in the principal direction corresponding to the principal curvature $-2\cot 2\phi$.

For any Z in a level hypersurface M_ϕ we draw a geodesic $\gamma(\phi)$ connecting the points A and Z (γ is unique and perpendicularly intersects M_ϕ, since the cut locus distance is $\frac{\pi}{2}$), then extend it to the focal variety $\mathbb{CP}^{n-1}$. This yields a unique intersection point $A' \in \mathbb{CP}^{n-1}$. These two points A and A' uniquely determine a complex projective line $\mathbb{CP}^1 = S^2(\frac{1}{2})$ which perpendicularly intersects the geodesic sphere M_ϕ at $S^1(\frac{1}{2}\sin 2\phi)$. Therefore, in $\mathbb{CP}^n$ there are geodesic polar coordinates (ϕ, α, Z'), where α is another coordinate on $\mathbb{CP}^1$ from A, and Z' is the coordinates on the focal variety $\mathbb{CP}^{n-1}$. Choosing a local orthonormal frame field in $\mathbb{CP}^{n-1}$ near A' then parallel translating it back to the Z along $\gamma(\phi)$, it can be proved that all of those lie in principal directions corresponding to the principal curvature $-cot\, \phi$ (see author's previous paper [**X1**]). Hence, we have

Proposition 2.1. *The geodesic sphere* $M_\phi = S^{2n-1}(\sin\phi) \times S^1(\cos\phi)/S^1$ *in* $\mathbb{CP}^n$ *has principal curvatures* $-\cot\phi$ *of multiplicity* $2n-2$ *and* $-2\cot 2\phi$.

3. Construction of Equivariant Maps into $\mathbb{CP}^2$.

Let (M, g) and (N, h) be Riemannian manifolds with metric tensors g and h, respectively. Harmonic maps are described as critical points of the following energy functional

$$E(f) = \frac{1}{2}\int_M e(f) * 1, \tag{3.1}$$

where $e(f)$ stands for the energy density. The Euler-Lagrange equation of the energy functional is

$$\tau(f) = 0, \tag{3.2}$$

where $\tau(f)$ is the tension field. In local coordinates

$$\tau(f) = \left(\Delta_M f^\alpha + g^{ij}\Gamma^\alpha_{\beta\gamma}\frac{\partial f^\beta}{\partial x^i}\frac{\partial f^\gamma}{\partial x^j}\right)\frac{\partial}{\partial y^\alpha}, \tag{3.3}$$

where $\Gamma^\alpha_{\beta\gamma}$ denotes the Christoffel symbols of the target manifold N. Here and in the sequal we use the summation convention. For more detail knowledge of harmonic maps please consult [**E-L2**].

Let $\pi_1 : M \to \bar{M}, \pi_2 : N \to \bar{N}$ be Riemannian submersions. If $f : M \to N$ is a fiber-preserving map, namely for the points x_1, $x_2 \in M$, $\pi_2(f(x_1)) = \pi_2(f(x_2))$ provided $\pi_1(x_1) = \pi_1(x_2)$, then f is called an equivariant map with respect to Riemannian submersions π_1 and π_2. Due to the structure of the Riemannian submersion there are vertical vector fields which are tangent to fiber submanifolds and horizontal vector fields which are orthogonal complements of the vertical vector fields. A map f is called horizontal if it maps any horizontal vector field to a horizontal one [**X1**].

Now we are going to define a concrete equivariant map from $\mathbb{B}^3$ into $\mathbb{CP}^2$ and apply the reduction theorem in [**X1**] to obtain a reduced harmonicity equation.

Let $\mathbb{B}^3$ be the open unit 3-dimensional ball in $\mathbb{R}^3$. For any $Z \in \mathbb{B}^3$ there exist $X \in S^1$ and $(r, z) \in D$, such that

$$Z = (rX, z),$$

where

$$D = \left\{(r, z) \in \mathbb{R}^2, \quad r^2 + z^2 < 1, \quad r > 0\right\}.$$

It can be verified that $\tilde{r} = (r, z) : \mathbb{B}^3 \to D$ is an isoparametric map of rank 2 with fiber submanifold $S^1(r)$. In D we define a usual flat metric $dr^2 + dz^2$ such that $\tilde{r} : \mathbb{B}^3 \setminus \{r = 0\} \to D$ is Riemannian submersion [**X2**].

On the other hand, on the target manifold $\mathbb{CP}^2$, as described in the last section, there is an isoparametric function ϕ with focal point A and the focal variety $\mathbb{CP}^1$.

Let $f_1 : S^1 \to \mathbb{CP}^1$ be a harmonic map with the constant energy density $\frac{\lambda_1}{2}$, $f_2 : S^1 \to S^1$ be a harmonic map with the constant energy density $\frac{\lambda_2}{2}$. Now we define a map $f : \mathbb{B}^3 \to \mathbb{CP}^2$ as follows. For any $Z = (rX, z) \in \mathbb{B}^3 \setminus \{r = 0\}$ we join A and $f_1(X) \in \mathbb{CP}^1$ by the unique complex projective line which intersects a level hypersurface $M_{\phi(r,z)}$ at a circle $S^1(\frac{1}{2}\sin 2\phi)$. By then using f_2 we have a point $f(Z) \in M_{\phi(r,z)} \in \mathbb{CP}^2$, where the smooth function $\phi(r, z)$ on D will be determined later by the harmonicity equation. It is easily seen that f is an equivariant map with respect to Riemannian submersions in both domain and target manifolds. It induces a harmonic map between fiber submanifolds. It is also a horizontal map.

Thus, we can use the reduction theorem in ([**X1**], pp. 273-275) to derive the harmonicity equation. Let B_2 be the second fundamental form of the fiber submanifold M_ϕ in $\mathbb{CP}^2$. Let $\{\frac{1}{r}e\}$ be a unit vector field of $S^1(\frac{1}{r})$. By a direct computation

$$B_2\left(f_*\frac{1}{r}e, f_*\frac{1}{r}e\right) = -\frac{\lambda_1 \sin\phi\cos\phi}{r^2} - \frac{\lambda_2 \sin 2\phi \cos 2\phi}{2r^2}. \tag{3.4}$$

Therefore, the reduced harmonicity equation follows

$$\frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi\cos 2\phi = 0, \tag{3.5}$$

$$(r, z) \in D, \qquad 0 < \phi < \frac{\pi}{2}.$$

If $\lim_{r\to 0}\phi(r,z) = 0$, then f can be continuously extended to whole $\mathbb{B}^3$. Furthermore, $f(\mathbb{B}^3)$ does not lie in a complex projective line $\mathbb{CP}^1$ for $\lambda_1 \neq 0$, since any complex projective line starting from A intersects the focal variety $\mathbb{CP}^1$ at only one point. If $\lambda_2 \neq 0$, then $f(\mathbb{B}^3)$ does not lie in a real projective plane $\mathbb{RP}^2$. We are interested in the general case when both λ_1 and λ_2 do not vanish. Our construction is essentially a generalization of that in [**Z**].

If the boundary data are also equivariant with respect to isoparametric map $\tilde{r}$ and the isoparametric function ϕ, then the boundary condition is also reduced to the boundary ∂D. Furthermore, suppose that the function $\psi = \phi|_{\partial D}$ satisfies the following conditions:

$$
\begin{aligned}
&1) \quad \psi = 0 \qquad \text{when} \quad r = 0;\\
&2) \quad \max_{\partial D} \psi \le \frac{\pi}{2}.
\end{aligned}
\tag{3.6}
$$

Any solution to the equation (3.5) with boundary conditions (3.6) supplies us a continuous map f from $\mathbb{B}^3$ into $\mathbb{CP}^2$, which is smooth harmonic on $\mathbb{B}^3 \setminus \{r = 0\}$. One can prove that the map is weakly harmonic on whole $\mathbb{B}^3$ by a cut-off function technique. Thus, by main regularity theorem for harmonic maps (see [**Hi**] or [**E-L2**, p. 397]), f is a smooth harmonic map.

4. Heat Flow.

Let us consider the following evolution problem:

$$
\frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi\cos 2\phi,
\tag{4.1}
$$

$$
\phi(\cdot,0) = \phi_0(r,z), \qquad 0 \le \phi_0 \le \frac{\pi}{2},
\tag{4.2}
$$

$$
\phi(\cdot,t)|_{\partial D} = \phi_0|_{\partial D} = \psi, \quad \psi(0,z) = 0,
\tag{4.3}
$$

where ϕ_0 is a regular function on $\bar{D}$ and is of order $O\left(r^{\sqrt{\lambda_1+\lambda_2}}\right)$ as $r \to 0$.

4.1. Short Time Existence. We first prove the short time existence for (4.1) - (4.3). As derived above (4.1) is the reduction equation of the general harmonicity equation from $\mathbb{B}^3$ into $\mathbb{CP}^2$. In $\mathbb{B}^3$ we choose axially symmetric coordinates (r,θ,z) and in $\mathbb{CP}^2$ we have geodesic polar coordinates $(\phi,\alpha,\beta,\gamma)$, where (β,γ) are the coordinates in the focal variety $\mathbb{CP}^1$ of the isoparametric function ϕ. We consider the initial-boundary value problem of the evolution equations for harmonic maps f from $\mathbb{B}^3$ into $\mathbb{CP}^2$ as follows.

$$\frac{\partial f}{\partial t} = \tau(f), \tag{4.4}$$

$$f(\cdot,0) = f_0(\cdot),\ f(\cdot,t)|_{S^2} = f_0(\cdot)|_{S^2}. \tag{4.5}$$

It is known that for regular f_0 there exists a unique regular soluton $f : \bar{\mathbb{B}}^3 \times [0,T) \to \mathbb{CP}^2$ to the problem (4-4) – (4.5), where $T \in (0,\infty]$ is the maximal existence time (see [**H**, p. 122]). In our case the initial-boundary conditions are equivariant. Besides (4.2) and (4.3) we also have the following conditions.

$$\begin{aligned} \alpha(\cdot,0) &= k_2\theta, & \alpha(\cdot,t)|_{S^2} &= k_2\theta, \\ \beta(\cdot,0) &= k_1\theta, & \beta(\cdot,t)|_{S^2} &= k_1\theta, \\ \gamma(\cdot,0) &= 0, & \gamma(\cdot,t)|_{S^2} &= 0, \end{aligned} \tag{4.6}$$

where $\lambda_1 = k_1^2$ and $\lambda_2 = k_2^2$.

If we can prove that the solution to the equations (4.4) with the equivariant conditions (4.2), (4.3) and (4.6) is also equivariant, then by uniqueness we will complete the proof of short time existence for (4.1) – (4.3). To do this we consider the tension field in the above coordinates. Notice that the concrete expression of the tension field in each component does not involve θ variable explicitly, and neither do the cofficients in equations (4.4). The solution to the equations (4.4) is invariant under translation of the θ variable. Due to the equivariant initial-boundary conditions and uniqueness of the solution, a priori we can assume that the solution has the following form:

$$\begin{aligned} \phi &= \phi(r,z,t), \\ \alpha &= k_2\theta + \bar{\alpha}(r,z,t), \\ \beta &= k_1\theta + \bar{\beta}(r,z,t), \\ \gamma &= \bar{\gamma}(r,z,t), \end{aligned} \tag{4.7}$$

where $\phi(r,z,0) = \phi_0(r,z)$, $\bar{\alpha}(r,z,0) = 0$, $\bar{\beta}(r,z,0) = 0$, $\bar{\gamma}(r,z,0) = 0$. Let h_{ij} be the metric tensor in the geodesic polar coordinates on the target manifold $\mathbb{CP}^2$ as described above and $h_{ij} = \mathrm{diag}(1,\ \sin^2\phi\cos^2\phi,\ \sin^2\phi,$

$\sin^2\phi \sin^2\beta \cos^2\beta)$. The equations (4.4) then become

(4.8)
$$\begin{aligned}
\phi_t &= \Delta\phi - \frac{1}{2}\frac{\partial h_{22}}{\partial \phi}\left(\bar\alpha_r^2 + \bar\alpha_z^2 + \frac{k_2^2}{r^2}\right) - \frac{1}{2}\frac{\partial h_{33}}{\partial \phi}\left(\bar\beta_r^2 + \bar\beta_z^2 + \frac{k_1^2}{r^2}\right)\\
&\quad - \frac{1}{2}\frac{\partial h_{44}}{\partial \phi}\left(\bar\gamma_r^2 + \bar\gamma_z^2\right),\\
&h_{22}(\bar\alpha_t - \Delta\bar\alpha) - \frac{\partial h_{22}}{\partial \phi}(\phi_r\bar\alpha_r + \phi_z\bar\alpha_z) = 0,\\
&h_{33}\left(\bar\beta_t - \Delta\bar\beta\right) - \frac{\partial h_{33}}{\partial \phi}\left(\phi_r\bar\beta_r + \phi_z\bar\beta_z\right) + \frac{1}{2}\frac{\partial h_{44}}{\partial \beta}\left(\bar\gamma_r^2 + \bar\gamma_z^2\right) = 0,\\
&h_{44}\left(\bar\gamma_t - \Delta\bar\gamma\right) - \frac{\partial h_{44}}{\partial \phi}\left(\phi_r\bar\gamma_r + \phi_z\bar\gamma_z\right) - \frac{\partial h_{44}}{\partial \beta}\left(\bar\beta_r\bar\gamma_r + \bar\beta_z\bar\gamma_z\right) = 0.
\end{aligned}$$

Let $v = \sqrt{h_{22}}\bar\alpha$. From (4.8) it follows that

$$\begin{aligned}
v_t - \Delta v = -\Bigg[&\frac{1}{4h_{22}}\left(\frac{\partial h_{22}}{\partial \phi}\right)^2\left(\bar\alpha_r^2 + \bar\alpha_z^2 + \frac{k_2^2}{r^2}\right)\\
&+ \frac{1}{4h_{22}}\frac{\partial h_{22}}{\partial \phi}\frac{\partial h_{33}}{\partial \phi}\left(\bar\beta_r^2 + \bar\beta_z^2 + \frac{k_1^2}{r^2}\right)\\
&+ \frac{1}{4h_{22}}\frac{\partial h_{22}}{\partial \phi}\frac{\partial h_{44}}{\partial \phi}\left(\bar\gamma_r^2 + \bar\gamma_z^2\right) + \frac{1}{\sqrt{h_{22}}}\frac{\partial^2\sqrt{h_{22}}}{\partial \phi^2}(\phi_r^2 + \phi_z^2)\Bigg]v,\\
&v(r,z,0) = 0, \quad v(r,z,t)|_{\partial D} = 0 \quad \text{for } 0 < t < T.
\end{aligned}$$

By the concrete expressions for the metric tensor h_{ij} we know that the coefficient of v is bounded from above on $\bar D \times [0,s]$ for $s \in (0,T)$. By using the maximum principle [**F**] we conclude $v \equiv 0$, and then $\alpha \equiv 0$. Similarly, after proving $\beta \equiv 0$ we also can prove $\gamma \equiv 0$. Hence, we have

Lemma 4.1. *The evolution problem* (4.1) – (4.3) *has a unique regular solution* $\phi(r,z,t)$ *on* $D \times [0,T)$ *for some* $T > 0$. *Furthermore, if* $\phi_0 \not\equiv 0$,

$$0 < \phi(r,z,t) < \frac{\pi}{2} \quad \textit{for} \quad (r,z,t) \in D \times (0,T). \tag{4.9}$$

Proof. It suffices to prove (4.9). We write equation (4.1) in the form

$$\frac{\partial \phi}{\partial t} = \frac{\partial^2\phi}{\partial r^2} + \frac{\partial^2\phi}{\partial z^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + q(r,z,t)\phi,$$

where

$$q(r,z,t) = -\frac{\lambda_1 \sin 2\phi + \lambda_2 \sin 2\phi \cos 2\phi}{2r^2\phi(r,z,t)},$$

which is bounded from above on $\bar{D} \times [0, s]$ for $s \in (0, T)$. Due to (4.2) and (4.3) we can employ the maximum principle to obtain $\phi(r, z, t) > 0$ unless it is identically 0. Similarly, let $\eta = \frac{\pi}{2} - \phi$. Then η satisfies

$$\frac{\partial \eta}{\partial t} = \frac{\partial^2 \eta}{\partial r^2} + \frac{\partial^2 \eta}{\partial z^2} + \frac{1}{r}\frac{\partial \eta}{\partial r} + p(r, z, t)\eta,$$

where

$$p(r, z, t) = \frac{\lambda_1 \sin 2\eta - \lambda_2 \sin 2\eta \cos 2\eta}{2r^2 \eta(r, z, t)}.$$

Since $\eta(o, z, t) = \frac{\pi}{2} > 0$ we may choose small ε such that $\eta > 0$ on $D_\varepsilon \times [0, s]$, where $s \in (0, T)$, and

$$D_\varepsilon = \{(r, z) \in D; \quad r < \varepsilon\}.$$

It is easily seen that $p(r, z, t)$ is bounded from above on $\{\bar{D} \setminus D_\varepsilon\} \times [0, s]$. Due to (4.2) – (4.3) we use the maximum principle again to conclude $\eta > 0$. $\square$

4.2. Barrier Functions. To analyze the blow-up phenomena let us consider the z-independent solutions of (3.5), which are solutions to the following ODE

$$\frac{d^2\phi}{dr^2} + \frac{1}{r}\frac{d\phi}{dr} - \frac{\lambda_1}{2r^2}\sin 2\phi - \frac{\lambda_2}{2r^2}\sin 2\phi \cos 2\phi = 0, \tag{4.10}$$

$$0 < r < 1, \qquad 0 < \phi < \frac{\pi}{2}.$$

To solve equation (4.10) with the condition $\lim_{r\to 0}\phi(r) = 0$ we make the change of variable $r = e^x$, $-\infty < x < 0$. Then (4.10) becomes

$$\frac{d^2\phi}{dx^2} - \frac{\lambda_1}{2}\sin 2\phi - \frac{\lambda_2}{2}\sin 2\phi \cos 2\phi = 0, \tag{4.11}$$

$$-\infty < x < 0, \quad 0 < \phi < \frac{\pi}{2}, \quad \lim_{x\to -\infty}\phi(x) = 0. \tag{4.12}$$

Multiplying (4.11) by $\frac{d\phi}{dx}$ and integrating, we obtain

$$\left(\frac{d\phi}{dx}\right)^2 - \lambda_1 \sin^2\phi - \frac{\lambda_2}{4}\sin^2 2\phi = c, \tag{4.13}$$

where c is a constant. Due to the conditions (4.12) the constant c has to be zero and (4.13) becomes

$$\frac{d\phi}{dx} = \pm\sqrt{\lambda_1 + \lambda_2 \cos^2\phi}\,\sin\phi.$$

Noting (4.12), the minus sign of the right hand side of the above equation is impossible. Therefore, (4.13) reduces to

$$\frac{d\phi}{dx} = \sqrt{\lambda_1 + \lambda_2 \cos^2\phi}\sin\phi. \tag{4.14}$$

For any initial condition $\phi(x_0) = \tau$, $-\infty < x_0 < \infty$, there is a unique solution $\phi(x)$ to (4.14), which can be extended to whole line $(-\infty, \infty)$ since the right hand side of (4.14) is bounded. Notice that the constant solutions of (4.14) are $\phi = k\pi$, $(k = 0, \pm 1, \cdots)$. Let us consider the solutions ϕ of (4.14) with initial condition $0 < \tau < \pi$. By uniqueness the solution curve on (x, ϕ) plane lies within two lines $\phi \equiv 0$ and $\phi \equiv \pi$. Thus $\lim_{x\to-\infty}\phi(x)$ exists, which implies that there exists a sequence of points $\{x_k\} \to -\infty$ such that $\frac{d\phi}{dx}(x_k) \to 0$. Considering the equation (4.14) on those points gives $\lim_{x\to-\infty}\phi(x) = 0$. Similarly, $\lim_{x\to\infty}\phi(x) = \pi$. In summarizing, we have

Lemma 4.2. *For any $\tau < \pi$ there exists a unique solution $\phi_\tau(x)$ of (4.14) satisfying the boundary conditions $\phi_\tau(0) = \tau$ and $\lim_{x\to-\infty}\phi(x) = 0$. Futhermore,*

$$\phi_{\tau_1}(x) < \phi_{\tau_2}(x),$$

where $\tau_1 < \tau_2$.

Lemma 4.3. *Let $\phi(r)$ be a solution to (4.10) satisfying the boundary conditions*

$$\lim_{r\to 0}\phi(r) = 0 \quad \text{and} \quad \phi(r_0) = 2\arctan\left(cr_0^{\sqrt{\lambda_1}}\right) \le \frac{\pi}{2}.$$

Then we have the estimates

$$2\arctan\left(cr^{\sqrt{\lambda_1+\lambda_2}}\right) \le \phi(r) \le 2\arctan\left(cr^{\sqrt{\lambda_1}}\right), \tag{4.15}$$

where $0 < r \le r_0 < 1$ and c is a positive constant.

Proof. Let

$$L_1(\psi) = \frac{d\psi}{dx} - \sqrt{\lambda_1}\sin\psi$$

for $-\infty < x \le \ln r_0$. It can be verified that $\psi = 2\arctan\left(c\exp\sqrt{\lambda_1}x\right)$ is a solution to the equation

$$L_1(\psi) = 0.$$

For a solution $\phi(x)$ to (4.14) with

$$\phi(\ln r_0) = 2\arctan\left(cr_0^{\sqrt{\lambda_1}}\right) \le \frac{\pi}{2}$$

$$\begin{aligned} L_1(\phi) &= \frac{d\phi}{dx} - \sqrt{\lambda_1}\sin\phi \\ &= \frac{d\phi}{dx} - \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi + \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi - \sqrt{\lambda_1}\sin\phi \\ &= \sqrt{\lambda_1 + \lambda_2\cos^2\phi}\sin\phi - \sqrt{\lambda_1}\sin\phi \geq 0. \end{aligned}$$

Notice that

$$\phi(\ln r_0) = \psi(\ln r_0), \quad \lim_{x\to-\infty}\phi(x) = \lim_{x\to-\infty}\psi(x) = 0.$$

If there exists a point $x \in (-\infty, \ln r_0)$ such that $(\phi - \psi)(x) > 0$, then there is a positive maximum point $x_0 \in (-\infty, \ln r_0)$ of $\phi - \psi$. We have

$$0 \leq L_1(\phi - \psi)(x_0) = \sqrt{\lambda_1}(\sin\psi(x_0) - \sin\phi(x_0)),$$

but the right hand side of the above expression is negative. The contradiction implies $\phi \leq \psi$ on $(-\infty, \ln r_0]$. Reversing back to the original variable gives

$$\phi(r) \leq 2\arctan\left(cr^{\sqrt{\lambda_1}}\right).$$

Let

$$L_2(\psi) = \frac{d\psi}{dx} - \sqrt{\lambda_1 + \lambda_2}\sin\psi.$$

Then any solution of (4.14) is a supersolution of $L_2(\psi) = 0$. By the similar argument as the above will obtain another inequality of (4.15) $\square$

Lemma 4.4. *There exists a solution $\bar{\phi}$ to equation* (4.10) *satisfying the condition $\lim_{r\to 0}\bar{\phi}(r) = 0$ such that $\bar{\phi} \geq \phi_0$ on $\bar{D}$, where ϕ_0 is the given initial-boundary data satisfying conditions* (4.2) *and* (4.3).

Proof. By the condition of ϕ_0 near $r = 0$ there are constants K and δ such that

$$\phi_0(r,z) \leq Kr^{\sqrt{\lambda_1+\lambda_2}} \quad \text{when} \quad r \leq \delta \;\text{ and }\; (r,z) \in \bar{D}.$$

On the other hand, there is $\delta_1 > 0$, such that

$$\phi_c = 2\arctan\left(cr^{\sqrt{\lambda_1+\lambda_2}}\right) \geq \sin\phi_c = \frac{2cr^{\sqrt{\lambda_1+\lambda_2}}}{1 + c^2r^{2\sqrt{\lambda_1+\lambda_2}}} \geq Kr^{\sqrt{\lambda_1+\lambda_2}},$$

when $c > \frac{K}{2}$ and $r \leq \delta_1$. For this c,

$$\phi_c \geq \phi_0 \quad \text{on} \quad \bar{D} \cap \{r < \delta_0\},$$

where $\delta_0 = \min(\delta, \delta_1)$. Furthermore, we can choose c sufficiently large such that on $\bar{D} \cap \{\delta_0 \leq r \leq 1\}$

$$\phi_c \geq 2\arctan\left(c\delta_0^{\sqrt{\lambda_1+\lambda_2}}\right) \geq \max_{\bar{D}}(\phi_0).$$

We have shown that there exists c_0 such that $\phi_{c_0} \geq \phi_0$ on $\bar{D}$.

Choose r_0 such that $c_0 r_0^{\sqrt{\lambda_1+\lambda_2}} = 1$. Denote by $\bar{\phi}$ the solution to (4.10) with the boundary conditions $\lim_{r\to 0} \bar{\phi}(r) = 0$ and $\bar{\phi}(r_0) = \frac{\pi}{2}$. By Lemma 4.3,

$$\bar{\phi} \geq \phi_{c_0} \geq \phi_0 \qquad \text{on} \quad \bar{D} \cap \{0 \leq r \leq r_0\}.$$

Notice that $\bar{\phi}$ is monotone increasing and

$$\bar{\phi}|_{[r_0,1]} \geq \frac{\pi}{2} \geq \phi_0.$$

□

We need the following comparison principle, as in [**G1**, Lemma 4.3]. For completeness we also include the proof here.

Lemma 4.5. *Let $\phi(r,z,t)$ be a regular solution to* (4.1) – (4.3) *on* $[0,T)$. *Let $\bar{\phi}$ be a regular solution to equation* (4.1). *Moreover, let ϕ and $\bar{\phi}$ satisfy the initial-boundary relations:*

$$\begin{aligned} &\bar{\phi}(r,z,0) \geq \phi_0(r,z) \qquad \text{on } D, \\ &\bar{\phi}|_{\partial D} \geq \phi_0|_{\partial D}, \quad \bar{\phi}(0,z,t) = \phi_0(0,z) = 0. \end{aligned} \tag{4.16}$$

Then $\bar{\phi} \geq \phi$ on $\bar{D} \times [0,T)$.

Proof. Let $\eta = \phi - \bar{\phi}$. By (4.16) $\eta \leq 0$ on $\bar{D} \times \{0\}$ and on $\partial D \times [0,T)$. By equation (4.1) η satisfies

$$\eta_t = \eta_{rr} + \eta_{zz} + \frac{\eta_r}{r} + p(r,z,t)\eta \qquad \text{on } D \times [0,T), \tag{4.17}$$

where

$$\begin{aligned} p(r,z,t) &= -\frac{\lambda_1 \left(\sin 2\bar{\phi} - \sin 2\phi\right)}{2\left(\bar{\phi} - \phi\right) r^2} - \frac{\lambda_2 \left(\sin 4\bar{\phi} - \sin 4\phi\right)}{4\left(\bar{\phi} - \phi\right) r^2} \\ &= -\frac{1}{r^2} \int_0^1 \cos 2\left[s\bar{\phi} + (1-s)\phi\right] ds - \frac{1}{r^2} \int_0^1 \cos 4\left[s\bar{\phi} + (1-s)\phi\right] ds. \end{aligned}$$

Since $\phi(0,z,t) = \bar{\phi}(0,z,t) = 0$, for each $t_0 \in (0,T)$ there exists $\varepsilon > 0$ such that $p < 0$ on $D_\varepsilon \times [0,t_0]$. Hence $p(r,z,t)$ is bounded from above on $\bar{D} \times [0,t_0]$. By using the maximum principle again we conclude that $\eta \leq 0$ on $\bar{D} \times [0,T)$. □

4.3. Proof of the Main Theorem. Now we are in a position to prove the theorem stated in the introduction by the standard method, as shown in [**G1**].

Let $\phi(r,z,t)$ be the unique solution to (4.1) – (4.3) on $D \times [0,T)$, where T is the maximum existence time. If T is finite, then ϕ must blow up at T, i.e. for some $(r,z) \in \bar{D}$

$$\lim_{t \to T} \sup |\nabla \phi(r,z,t)| = \infty.$$

By Lemma 4.4 we have the regular solution $\bar{\phi}$ to (4.1), which is independent of t and z. By Lemma 4.5, $\bar{\phi} \geq \phi$ on $\bar{D} \times [0,T)$. Therefore, from (4.15) it follows that

$$\frac{\phi(r,z,t) - \phi(0,z,t)}{r} \leq \frac{\bar{\phi}(r) - \bar{\phi}(0)}{r} \leq \frac{2\arctan\left(cr^{\sqrt{\lambda_1}}\right)}{r},$$

for sufficiently small r, and $|\phi_r|$ is bounded at $r = 0$ when $t \to T$.

Therefore, if blow-up first occurs on $r = 0$, then we have a sequence $(r_i, z_i, t_i) \to (0, z^*, T)$ for which $|\phi_z| \to \infty$. Then for all i sufficiently large, we have

$$|\phi_z(r_i, z_i, t_i)| > 1.$$

On the other hand, since ϕ is $C^{2+\alpha}(D \times \{t\})$ for $t < T$, there exists $\{a_i\}$ such that

$$\|\phi(\cdot, t_i)\|_{2+\alpha} < \frac{1}{a_i}. \tag{4.18}$$

We can choose $r_i < a_i$ and obtain

$$\|\phi(\cdot,t_i)\|_{2+\alpha} > \max|\phi_{zr}(\cdot,t_i)| \geq \frac{|\phi_z(r_i,z_i,t_i) - \phi_z(0,z_i.t_i)|}{r_i} > \frac{1}{r_i} > \frac{1}{a_i},$$

which contradicts (4.18). Thus we conclude that there is no blow-up on $r = 0$ and there exists $\varepsilon > 0$, such that

$$\sup_{D_\varepsilon \times [0,T)} \|\nabla \phi\|_\infty < \infty. \tag{4.19}$$

The solution ϕ can also be viewed as a bounded solution to the linear parabolic equation

$$\phi_t = \phi_{rr} + \phi_{zz} + \frac{1}{r}\phi_r + p(r,z,t) \qquad \text{on} \quad D \setminus D_\varepsilon \times [0,T),$$

where

$$p(r,z,t) = -\frac{\lambda_1 \sin 2\phi + \lambda_2 \sin 2\phi \cos 2\phi}{2r^2}.$$

It is easily seen that

$$|p(\phi, r)| \leq \frac{\lambda_1 + \lambda_2}{2\varepsilon^2} \qquad \text{on} \quad D \setminus D_\varepsilon.$$

Hence we can apply the estimates for linear parabolic equations to obtain

$$\|\phi\|_{C^{2+\alpha,1+\alpha}(D\setminus D_\varepsilon \times [0,T))} < C(\varepsilon), \tag{4.20}$$

where $C(\varepsilon)$ is a constant depending only on ε (see [**L-S-U**], pp. 351-355). Inequalitties (4.19) and (4.20) mean that there is no blow-up for the solution and ϕ is the global solution and

$$\sup_{t\in[0,\infty)} \|\nabla\phi(\cdot, t)\|_\infty < C.$$

It turns out that (4.20) holds for $T = \infty$.

We now study the convergence of $\phi(r, z, t)$ when t goes to infinity. By a direct computation the energy functional of the defined map f is

(4.21)

$$E(\phi) = \frac{1}{2} \int_D \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 + \frac{\lambda_1 \sin^2\phi}{r^2} + \frac{\lambda_2 \sin^2 2\phi}{4r^2} \right] r\, dr\, dz.$$

We have

$$\begin{aligned} \frac{dE}{dt} &= \int_D \left(\phi_r\phi_{rt} + \phi_z\phi_{zt} + \frac{\lambda_1 \sin 2\phi\, \phi_t}{2r^2} + \frac{\lambda_2 \sin 2\phi \cos 2\phi\, \phi_t}{2r^2} \right) r\, drdz \\ &= \int_D \left[\frac{d}{dr}(\phi_r\phi_t r) + \frac{d}{dz}(\phi_z\phi_t r) \right] drdz \\ &\quad - \int_D \left(\phi_{rr} + \phi_{zz} + \frac{1}{r}\phi_r - \frac{\lambda_1 \sin 2\phi}{2r^2} - \frac{\lambda_2 \sin 2\phi \cos 2\phi}{2r^2} \right) \phi_t r\, drdz. \end{aligned}$$

Using Stokes' theorem and the fact that $\phi_t|_{\partial D} = 0$, we see that the first term of the above expression vanishes. From (4.1) it follows

$$\frac{dE}{dt} = -\int_D (\phi_t)^2 r\, drdz. \tag{4.22}$$

Since

$$E(\phi(r, z, 0)) = E(\phi_0(r, z)) < \infty,$$

there exists a sequence of points $\{t_k\} \to \infty$, such that

$$\int_D \phi_t^2(x, z, t_k) r\, drdz \to 0.$$

Thus for any $\eta \in L^2(D)$

$$\int_D \phi_t(r, z, t_k)\eta r\, dr dz \to 0. \tag{4.23}$$

This means that $\phi_t(r, z, t_k)$ converges to zero weakly in $L^2(D)$ as t_k approaches infinity. From the estimate (4.20) we may choose a subsequence of $\{t_k\}$ (denoted by $\{t_k\}$ for simplicity), such that $\phi(x, z, t_k)$ converges to $\phi_\infty = \phi(r, z, \infty)$ strongly in $C^{2+\alpha}(D \setminus D_\varepsilon)$. Due to (4.23), ϕ_∞ is a weak solution to (3.5), and therefore a regular solution to (3.5). From the previous discussion

$$0 \le \phi(r, z, t) \le \bar{\phi}(r)$$

for any t. It follows that $\phi_\infty \to 0$ as $r \to 0$. Hence ϕ_∞ is a regular solution to (3.5)-(3.6) and by our previous discussion supplies an equivariant harmonic map from $\mathbb{B}^3$ into $\mathbb{CP}^2$. □

5. Final Remarks.

1. If the domain manifold is the unit disk $\mathbb{B}^2$ we consider the polar coordinates instead of the axially symmetric coordinates. By the similar but simpler discussion we have a corresponding theorem . As for target manifold being $\mathbb{CP}^n$ $(n > 2)$ we can also conclude a similar result.

2. It is natural to investigate the higher-dimensional cases. Let $\mathbb{B}^{n+1}$ be the open unit ball in $(n+1)$-dimensional Euclidean space. For any $Z \in \mathbb{B}^{n+1}$, there exist $X \in S^{n-1}$, $(r, z) \in D$ such that

$$Z = (rX, z),$$

where D is as defined in Section 1. One of the factors of our construction is the harmonic maps from S^{n-1} into S^1 in this case. When $n > 2$ such a harmonic map has to be constant, then the construction reduces to the special case. If let the target manifolds be quaternionic projective spaces, there appears S^3 instead of S^1. It seems that results could be obtained for maps from higher-dimensional ball into quaternionic projective spaces (see e.g. [**X2**]). The analogue of (4.10) will be more complicated in this case, and in particular will not admit a first integral. It is possible that the stability theory of ODE would provide some information.

References

[C-D] Kung-Ching Chang and Wei-Yue Ding, *A Result on the Global Existence for Heat Flows of Harmonic Maps from D^2 into S^2*, Nematics, J.M. Coron et al. ed. Kluwer Academic Publ., (1990), 37-48.

[C-D-Y] Kung-Ching Chang, Wei-Yue Ding and Rugang Ye, *Finite-Time Blow-up of the Heat Flow of Harmonic Maps from Surfaces*, J. Differential Geom., **36** (1992), 507-515.

[C-G] J.M. Coron and J.M. Ghidaglia, *Explosion en Temps pour le Flot des Applications Harmoniques*, C. R. Acad. Sci. Paris, **308(I)** (1989), 339-344.

[E-L1] J. Eells and L. Lemaire, *Examples of Harmonic Maps from Disks to Hemispheres*, Math. Z., **185** (1984), 517-519.

[E-L2] ———, *Another Report on Harmonic Maps*, Bull. London Math. Soc., **20** (1988), 385-524.

[F] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., 1964.

[G1] J.F. Grotowski, *Harmonic Map Heat Flow for Axially Symmetric Data*, Manus. Math., **73(2)** (1991), 207-228.

[G2] ———, *Finite Time Blow-up for the Harmonic Map Heat Flow*, Calc. Var., **1** (1993), 231-236.

[G3] ———, *Concentrated Boundary Data and Axially Symmetric Harmonic Maps*, J. Geom. Anal., **3(3)** (1993), 279-292.

[H] R.T. Hamilton, *Harmonic Maps of Manifolds with Boundary*, Lecture Notes Math., Springer-Verlag, **471** (1975).

[Ha] R. Hardt, *Axially Symmetric Harmonic Maps*, Proc. Workshop on 'Defects Singularity and Patterns in Nematic Liquid Crystals: Math. and Phys. Aspects', Université Paris-Sud, (1990), 179-188.

H-K-L1] R. Hardt, D. Kinderlehrer and F.H. Lin, *Existence and Partial Regularity of Static Liquid Configurations*, Comm. Math. Phys., **105** (1986), 547-570.

H-K-L2] ———, *The Variety of Configurations of Static Liquid Crystals*, Progress in Nonlinear Diffrential Equations and Their Appl., Birkhäuser, **4** (1990), 115-132.

[H-L-P] R. Hardt, F.H. Lin and C.C. Poon, *Axially Symmetric Harmonic Maps Minimizing a Relaxed Energy*, Comm. Pure Appl. Math., **XLV** (1992), 417-459.

[Hi] S. Hildebrandt, *Nonlinear Elliptic Systems and Harmonic Mappings*, Proc. Beijing Symp. on Differential Geometry and Differential Equations, Gordon and Breach, (1980), 481-615.

H-K-W] S. Hildebrandt, H. Kaul and K.O. Widman, *An Existence Theorem for Harmonic Mappings of Riemannian Manifolds*, Acta Math., **138** (1977), 1-16.

[J-K] M. Jäger and H. Kaul, *Rotationally Symmetric Harmonic Maps from a Ball into a Sphere and the Regularity Problem for Weak Solutions of Elliptic Systems*, J. Reine Angew. Math., **343** (1983), 146-161.

[L-S-U] O.A. Ladyzhenskaja, V.A. Solonikov and N.N. Ural'ceva, *Linear and Quasilinear Elliptic Equations*, Academic Press, New York, 1968.

[X1] Y.L. Xin, *Riemannian Submersions and Equivariant Harmonic Maps*, Proc. Diff. Geom. in Honour of Professor Su Buchin World Scientific Publ. Co., (1993), 272-287.

[X2] ———, *Equivariant Harmonic Maps into the Sphere via Isoparametric Maps*, Manus. Math., **79** (1993), 49-71.

[Z] D. Zhang, *The Existence of Nonminimal Regular Harmonic Maps from B^3 to S^2*, Ann. SNS Pisa, **16(IV)** (1989), 355-365.

Received November 18, 1994. This author's research partially supported by NNSFC and SFECC.

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