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INTEGRALS ON p-ADIC UPPER HALF PLANES AND HIDA FAMILIES OVER TOTALLY REAL FIELDS

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Abstract

Bertolini–Darmon and Mok proved a formula of the second derivative of the two-variable p-adic L-function of a modular elliptic curve over a totally real field along the Hida family in terms of the image of a global point by some p-adic logarithm map. The theory of p-adic indefinite integrals and p-adic multiplicative integrals on p-adic upper half planes plays an important role in their work. In this paper, we generalize these integrals for p-adic measures which are not necessarily $\mathbb{Z}$-valued, and prove a formula of the second derivative of the two-variable p-adic L-function of an abelian variety of GL(2)-type associated to a Hilbert modular form of weight 2.

Contents

1. Introduction

Bertolini–Darmon and Mok proved a formula of the second derivative of the two-variable p-adic L-function of a modular elliptic curve over a totally real field along the Hida family in terms of the image of a global point by some p-adic logarithm map

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([3], [14]). In this paper, we generalize their results to abelian varieties of GL(2)-type associated to Hilbert modular forms of weight 2.

Let F be a totally real field. We assume that an odd prime number p is inert in F and denote by $\mathfrak{p}$ the prime of F above p. Let f be a cuspidal Hilbert modular eigenform of parallel weight 2 over F. We assume that f is a newform of level $\Gamma_0(\mathfrak{n})$ (here, $\mathfrak{n}$ is a non-zero ideal of $\mathcal{O}_F$) and the sign ϵ_f of the functional equation of the complex L-function of f is equal to -1. Let $\mathbb{Q}(f)$ be the Hecke fields of f, which is a finite extension of $\mathbb{Q}$ generated by the Fourier coefficients of f.

Let A be an abelian variety of GL(2)-type over F associated to f. We assume that A has split multiplicative reduction at $\mathfrak{p}$ (in addition, if $[F : \mathbb{Q}]$ is odd, suppose that A is multiplicative at some prime other than $\mathfrak{p}$). We denote by $L_p(s, k)$ the *two variable p-adic L-function along the Hida family of f.*

Theorem 1.1 (see Theorem 4.11)**.**

$$\left.\frac{d^2}{dk^2} L_p(k/2, k)\right|_{k=2} = l \cdot (\log\mathrm{Norm}_p^A(P))^2 \tag{1}$$

where $l \in \mathbb{Q}(f)^\times$ and $P \in A(F) \otimes_{\mathbb{Z}} \mathbb{Q}$ is a global point. The map

$$\log\mathrm{Norm}_p^A \colon A(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{C}_p$$

is a p-adic logarithm map (*see* Definition 4.1).

Here, the Hecke character ψ in Theorem 4.11 is the trivial character.

Let us explain the outline of the proof. Let Φ be an automorphic form on the multiplicative group of a definite quaternion algebra B/F corresponding to f by the Jacquet–Langlands correspondence (we can find such a quaternion algebra B by the assumption of the reduction of the abelian variety A). The key notion for proving the formula (1) is the notion of *indefinite integrals* and *multiplicative integrals* associated to Φ. In fact, we prove the following equalities:

LHS of (1) = (indefinite integral) = (multiplicative integral) = RHS of (1).

The first equality is proved by using an explicit formula of L-values by Gross–Hatcher and Xue. In the work of Bertolini–Darmon and Mok, multiplicative integrals are defined only when A is an elliptic curve. We shall modify the definition of multiplicative integrals by following Dasgupta's method, and prove the second and third equalities. For the third equality, we use the theory of p-adic uniformization of Shimura curves by Manin–Drinfeld and Cerednik–Drinfeld (see Sections 2.4 and 4.1). For the second equality, we shall prove the following generalized formula of p-adic integrals on p-adic upper half planes $\mathcal{H}$:

Theorem 1.2 (see Theorem 3.35). *Let $\tau_1, \tau_2 \in \mathcal{H}(\bar{\mathbb{Q}}_p)$ be $\bar{\mathbb{Q}}_p$-valued points on p-adic upper half plane. We have*:

$$I_\Phi(\tau_1) - I_\Phi(\tau_2) = \iota\left\{((\mathrm{logNorm}_p + 2\alpha_\mathfrak{p} \cdot \alpha'_\mathfrak{p}(0) \cdot \mathrm{ord}_\mathfrak{p}) \otimes_\mathbb{Z} \mathrm{id}_{\mathcal{O}_{\mathbb{Q}(\Phi)}})\left(\times\!\!\!\!\!\!\int_{[\tau_1]-[\tau_2]} \omega_{\mu_\Phi}\right)\right\},$$

where I_Φ is the indefinite integral (*see* Section 3.5) *and $\alpha_\mathfrak{p}$* (*resp. an p-adic analytic function $\alpha_\mathfrak{p}(s)$*) *is the Hecke eigenvalue of $T(\mathfrak{p})$ of Φ* (*resp. of the Hida family associated to Φ*). *The symbol $\times\!\!\!\!\!\!\int$ is the multiplicative integral whose integrated values are in $\mathbb{C}_p \otimes_\mathbb{Z} \mathcal{O}_{\mathbb{Q}(\Phi)}$* (*see* Section 2.2). *The map ι is a natural multiplication map $\mathbb{C}_p \otimes_\mathbb{Z} \mathcal{O}_{\mathbb{Q}(\Phi)} \to \mathbb{C}_p$* (*we denote by $\mathbb{Q}(\Phi)$ the Hecke field of Φ*).

When A is an elliptic curve, this formula is proved by Bertolini–Darmon and Mok ([3], [14]). For the proof of Theorem 1.2, see Section 3.5.

I would like to thank Tetsushi Ito for his encouragement and help throughout the preparation of this paper.

NOTATION 1.3. For a number field (resp. valuation field) L, we denote the ring of integers of L (resp. valuation ring of L) by $\mathcal{O}_L$. We denote an algebraic closure of L by $\bar{L}$.

Throughout the paper, we fix an odd prime p and a totally real field F. We assume p is *inert* in F. Let $\mathfrak{p}$ be a unique prime ideal of $\mathcal{O}_F$ above p. Let $\mathbb{A}_F$ be the adèle ring of F and let $\mathbb{A}_{F,f}$ be the ring of finite adèles. Let $\mathbb{C}_p$ be the p-adic completion of $\bar{\mathbb{Q}}_p$. We fix embeddings $F \hookrightarrow \bar{\mathbb{Q}}$ and $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. We denote by ord_p, $\log_p$ the valuation map and the Iwasawa logarithm map respectively (they are normalized by $\mathrm{ord}_p(p) = 1$ and $\log_p(p) = 0$ respectively). We have a canonical decomposition

$$\mathbb{Q}_p^\times \cong p^\mathbb{Z} \times \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$$

and we denote by $\langle \cdot \rangle$ the projection $\mathbb{Q}_p^\times \to 1 + p\mathbb{Z}_p$.

2. Multiplicative integrals on p-adic upper half planes

In the first half of this section, we summarize basics on p-adic measures on projective lines over non-archimedean local fields. We introduce the Bruhat–Tits tree for PGL_2 and multiplicative integrals following [8]. They play a main role in this paper. In the second half, following [8], we review the p-adic uniformization theory for Mumford curves and their Jacobian varieties and define an important invariant, the L-invariants.

In this section, let K be a finite extention of $\mathbb{Q}_p$, $\mathcal{O}_K$ the valuation ring with a uniformizer π_K, and k_K the residue field. Recall that the p-adic completion of $\bar{\mathbb{Q}}_p$ is denoted by $\mathbb{C}_p$.

2.1. Basic notions of p-adic measures.

DEFINITION 2.1. The Bruhat–Tits tree $\mathcal{T}_K$ of $\mathrm{PGL}_2(K)$ is a graph whose vertex is the homothety class of $\mathcal{O}_K$-lattices in $K \oplus K$ and two vertices are connected by an edge if there exist representatives of them such that one contains another and the quotient is isomorphic to k_K. We often identify the $\mathcal{T}_K$ with a geometrical realization as a simplicial complex.

We denote by $\mathcal{V}(\mathcal{T}_K)$ (resp. $\mathcal{E}(\mathcal{T}_K)$) the set of the vertices (resp. the oriented edges) of $\mathcal{T}_K$. Let $v^*, w^* \in \mathcal{V}(\mathcal{T}_K)$ be the vertices which are the homothety class of $\mathcal{O}_K \oplus \mathcal{O}_K$, $\mathcal{O}_K \oplus \pi_K \mathcal{O}_K$ respectively, and let $e^* \in \mathcal{E}(\mathcal{T}_K)$ be the oriented edge from w^* to v^*, denoted by $e^* = (w^*, v^*)$. For any oriented edge $e \in \mathcal{E}(\mathcal{T}_K)$, we denote the vertex of source (resp. target) by s_e (resp. t_e), and we also denote by $\bar{e} \in \mathcal{E}(\mathcal{T}_K)$ the oppositely oriented edge of e.

For an oriented edge $e \in \mathcal{E}(\mathcal{T}_K)$, there exists $\gamma \in \mathrm{PGL}_2(K)$ such that $e = \gamma e^*$. Then we assign to e an open compact subset $\gamma \mathcal{O}_K \subset \mathbb{P}^1(K)$ (the action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1(K)$ is given by the Möbius transformation) and denote it by U_e. Note that U_e is well-defined and independent of the choice of γ. The set $\{U_e\}_{e \in \mathcal{E}(\mathcal{T})}$ is an open basis of $\mathbb{P}^1(K)$.

DEFINITION 2.2. An *end* of $\mathcal{T}_K$ is an equivalence class of sequences $\{v_n\}_{n \geq 0}$ of distinct vertices such that (v_n, v_{n+1}) is an oriented edge for all n. Here, two sequences $\{v_n\}_{n \geq 0}$ and $\{w_n\}_{n \geq 0}$ equivalent if there exists $n_0, k \in \mathbb{Z}$ such that $v_i = w_{i+k}$ for all $i \geq n_0$.

REMARK 2.3. There is a bijection between the set of ends of $\mathcal{T}_K$ and $\mathbb{P}^1(K)$ by the following correspondence:

$$\{v_n\} \mapsto \bigcap_n U_{(v_n, v_{n+1})}.$$

DEFINITION 2.4. Let H be an abelian group and let S be a non-empty subset of $\mathbb{P}^1(K)$. An *H-valued measure* μ on S is a map which assigns an element of H to each open compact subset of S with following two conditions:
1. $\mu(U \cup V) = \mu(U) + \mu(V)$ for disjoint open compact subsets $U, V \subset S$,
2. $\mu(S) = 0$.

We denote by $\mathrm{Meas}(S, H)$ the space of H-valued measures on S.

We put

$$E_{\mathcal{T}_K} := \bigoplus_e \mathbb{Z}e \Big/ \bigoplus_e \mathbb{Z}(e + \bar{e})$$

be a quotient of a free abelian group generated by oriented edges in $\mathcal{T}_K$ and let

$$V_{\mathcal{T}_K} := \bigoplus_{v \in \mathcal{V}(\mathcal{T}_K)} \mathbb{Z}v$$

be a free abelian group generated by vertices in $\mathcal{T}_K$. We define a homomorphism Tr by

$$\begin{array}{rccc} \mathrm{Tr}\colon & V_{\mathcal{T}_K} & \longrightarrow & E_{\mathcal{T}_K} \\ & \Cup & & \Cup \\ & v & \longmapsto & \displaystyle\sum_{s_e = v} e. \end{array}$$

Then we have

$$\mathrm{Meas}(\mathbb{P}^1(K), H) = \mathrm{Ker}(\mathrm{Tr}^*), \tag{2}$$

where $\mathrm{Tr}^*\colon \mathrm{Hom}_{\mathbb{Z}}(E_{\mathcal{T}_K}, H) \to \mathrm{Hom}_{\mathbb{Z}}(V_{\mathcal{T}_K}, H)$ denotes a homomorphism induced by Tr.

DEFINITION 2.5. We define a metric space $\mathcal{T}$ by

$$\mathcal{T} := \varinjlim_{L/K\colon \text{fin ext}} \mathcal{T}_L$$

where, for any finite extensions $L'/L/K$, $\mathcal{T}_L \to \mathcal{T}_{L'}$ is induced by $\otimes_{\mathcal{O}_L} \mathcal{O}_{L'}$. We define sets $\mathcal{V}(\mathcal{T})$ and $\mathcal{E}(\mathcal{T})$ as follows:

$$\mathcal{V}(\mathcal{T}) := \varinjlim_{L/K\colon \text{fin ext}} \mathcal{V}(\mathcal{T}_L) \subset \mathcal{T}, \tag{3}$$

$$\mathcal{E}(\mathcal{T}) := \bigcup_{L/K\colon \text{fin ext}} \mathcal{E}(\mathcal{T}_L). \tag{4}$$

For any two points $x, y \in \mathcal{T}$, we define a metric $d_{\mathcal{T}}$ as follows:

$$d_{\mathcal{T}}(x, y) := \lim_{e_{L/K} \to \infty} \frac{1}{e_{L/K}} \#\{v \in \mathcal{V}(\mathcal{T}_L) \mid v \text{ lies in the path from } x \text{ to } y\},$$

where $e_{L/K}$ is the ramification index of L/K.

REMARK 2.6. If both $x, y \in \mathcal{V}(\mathcal{T}_L)$ for some finite extension L/K, the distance $d_{\mathcal{T}}(x, y)$ is a rational number.

REMARK 2.7. As in Definition 2.2, we can similarly define ends in $\mathcal{T}$, which is a infinitely long path in the sense of the metric $d_{\mathcal{T}}$. There is a similar bijection as in Remark 2.3 between $\mathbb{P}^1(\mathbb{C}_p)$ and the set of ends of $\mathcal{T}$.

From Definition 2.5, we define two abelian groups $E_{\mathcal{T}}$ and $V_{\mathcal{T}}$:

$$E_{\mathcal{T}} := \varinjlim_{L/K:\,\text{fin ext}} E_{\mathcal{T}_L},$$
$$V_{\mathcal{T}} := \varinjlim_{L/K:\,\text{fin ext}} V_{\mathcal{T}_L}.$$

Let $\mathcal{H}$ be the p-adic upper half plane, which is a rigid analytic space over F. For any complete extension F over K inside $\mathbb{C}_p$, the set of F-valued points $\mathcal{H}(F)$ coincides with $\mathbb{P}^1(F) \setminus \mathbb{P}^1(K)$. In particular, we have

$$\mathcal{H}(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(K).$$

DEFINITION 2.8. We define the continuous map red_K by

$$\begin{array}{cccc} \mathrm{red}_K\colon \mathcal{H}(\mathbb{C}_p) & \longrightarrow & \mathcal{T}_K \subset \mathcal{T} \\ \Psi & & \Psi \\ [\tau] & \longmapsto & x_\tau, \end{array}$$

where, x_τ is unique element in $\mathcal{V}(\mathcal{T}) \cap \mathcal{T}_K$ such that $\{v_n\}_{n\geq 0}$ is a representative of the end corresponding to τ satisfying $v_0 = x_\tau$ and $v_n \notin \mathcal{T}_K$ for any $n > 0$. We call red_K the reduction map to K.

There exists a boundary map ∂:

$$\begin{array}{ccc} \partial\colon E_{\mathcal{T}} & \longrightarrow & V_{\mathcal{T}} \\ \Psi & & \Psi \\ e = (s_e, t_e) & \longrightarrow & t_e - s_e, \end{array}$$

and we have the following commutative diagram induced by red_K:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Div}_0(\mathcal{H}(\mathbb{C}_p)) \arrow[r] \arrow[d, "\mathrm{red}_K"] & \mathrm{Div}(\mathcal{H}(\mathbb{C}_p)) \arrow[r] \arrow[d, "\mathrm{red}_K"] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & E_{\mathcal{T}} \arrow[r, "\partial"] & V_{\mathcal{T}} \arrow[r] & \mathbb{Z} \arrow[r] & 0. \end{tikzcd}$$

Let H be a finitely generated free abelian group. Then we define an embedding

$$\mathrm{Hom}_{\mathbb{Z}}(V_{\mathcal{T}_K}, H) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}(\mathcal{H}(\mathbb{C}_p)), \mathbb{Q} \otimes_{\mathbb{Z}} H)$$

as follows: Let $\phi \in \mathrm{Hom}_{\mathbb{Z}}(V_{\mathcal{T}_K}, H)$, $\tau \in \mathcal{H}(\mathbb{C}_p)$ and e be an edge in $\mathcal{T}_K$ containing $\mathrm{red}_K(\tau)$. Then we define

$$\phi([\tau]) := d_{\mathcal{T}}(\mathrm{red}(\tau), s_e)\phi(t_e) + d_{\mathcal{T}}(\mathrm{red}(\tau), t_e)\phi(s_e).$$

Similarly we define an embedding

$$\mathrm{Hom}_{\mathbb{Z}}(E_{\mathcal{T}_K}, H) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}_0(\mathcal{H}(\mathbb{C}_p)), \mathbb{Q} \otimes_{\mathbb{Z}} H)$$

as follows: Let $\psi \in \mathrm{Hom}_{\mathbb{Z}}(E_{\mathcal{T}_K}, H)$. Then for any $\tilde{e} \in \mathcal{E}(\mathcal{T})$, if $\tilde{e} \cap \mathcal{T}_K = \emptyset$ as sets, we define $\psi(\tilde{e}) = 0$. If $\tilde{e} \cap \mathcal{T}_K \neq \emptyset$ as sets, there exists an oriented edge $e \in \mathcal{E}(\mathcal{T}_K)$ which contain $\tilde{e}$ as sets and same direction as e. Then we define

$$\phi(\tilde{e}) := d_{\mathcal{T}}(s_{\tilde{e}}, t_{\tilde{e}})\phi(e).$$

Via the reduction map, we regard ψ as an element in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}_0(\mathcal{H}(\mathbb{C}_p)), \mathbb{Q} \otimes_{\mathbb{Z}} H)$.

2.2. *p*-adic multiplicative integrals. In this section, we define multiplicative integrals for H-valued measures on $\mathbb{P}^1(K)$ following [8]. Proposition 2.11 in this section implies that the multiplicative integrals have more information than usual integrals.

DEFINITION 2.9. Let F/K be a complete extension contained in $\mathbb{C}_p$. Let H be a finitely generated free abelian group, S be a subset in $\mathbb{P}^1(K)$, $d \in \mathrm{Div}_0(\mathbb{P}^1(F) \setminus S)$ a divisor of degree 0, and let $\mu \in \mathrm{Meas}(S, H)$ be a H-valued additive measure on S. Then we define:

$$\begin{aligned} \times\!\!\!\!\!\!\int_d \omega_\mu &:= \times\!\!\!\!\!\!\int_d f_d(t)\, d\mu(t) \\ &:= \lim_{\|\mathcal{U}\| \to 0} \prod_{U \in \mathcal{U}} f_d(t_U) \otimes_{\mathbb{Z}} \mu(U) \in F^\times \otimes_{\mathbb{Z}} H \end{aligned}$$

where $\mathcal{U}$ is an open compact disjoint covering of S, $\|\mathcal{U}\|$ is the supremum of the diameter of U for $U \in \mathcal{U}$, f_d is a rational function on $\mathbb{P}^1(F)$ whose divisor is d, and t_U is an element in U.

REMARK 2.10. Because of the properties of μ in Definition 2.4, the middle term of the above formula is independent of the choice of f_d. So the definition is well-defined.

If S is invariant under the action of a subgroup $\Gamma \subset \mathrm{PGL}_2(K)$, we define the action of $\gamma \in \Gamma$ on $\mu \in \mathrm{Meas}(S, H)$ by

$$\gamma \cdot \mu(U) = \mu(\gamma^{-1}U)$$

for any open compact subset $U \subset S$.

Proposition 2.11. *Let H be a finitely generated free abelian group, and let μ be an H-valued measure on $\mathbb{P}^1(K)$. Let* $\mathrm{ord}\colon \mathbb{C}_p \to \mathbb{Q}$ *be the valuation such that* $\mathrm{ord}(\pi_K) = 1$.

Then for any $\tau_1, \tau_2 \in \mathcal{H}(\mathbb{C}_p)$

$$(\mathrm{ord} \otimes_{\mathbb{Z}} \mathrm{id}_H)\left(\times\!\!\!\!\!\!\int_{[\tau_1]-[\tau_2]} \omega_\mu \right) = \mu([\tau_1] - [\tau_2]),$$

where μ *is regarded as an element in* $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}_0(\mathcal{H}(\mathbb{C}_p)), H)$ *as in* Section 2.1 (*see the equality* (2) *and the end of* Section 2.1).

Proof. We may assume $\tau_1, \tau_2 \in \mathcal{H}(L)$ for some finite extension L/K. We denote $e_{L/K}$ the ramified index of L/K. By partition the section from τ_2 to τ_1 and by replacing μ with $\gamma\mu$ for suitable $\gamma \in \mathrm{PGL}_2(K)$, we may also assume both $\mathrm{red}_K(\tau_1)$ and $\mathrm{red}_K(\tau_2)$ lie in the edge e^*.

The elements τ_i have the following expansion:

$$\tau_i = \pi_L^{-a_i} u_i + x_i$$

where $u \in \mathcal{O}_L^\times$ such that if $a_i = 0$, the image in k_L is not contained in k_K, $x_i \in \mathcal{O}_L$ and $0 \le a_i < e_{L/K}$.

Then we have

$$\mathrm{ord}(x - \tau_i) = \begin{cases} \mathrm{ord}(x) & (x \in \mathbb{P}^1(K) \setminus \mathcal{O}_K), \\ -\dfrac{a_i}{e_{L/K}} & (x \in \mathcal{O}_K). \end{cases}$$

Thus

$$\mathrm{ord}\left(\frac{x - \tau_1}{x - \tau_2}\right) = \begin{cases} 0 & (x \in \mathbb{P}^1(K) \setminus \mathcal{O}_K), \\ \dfrac{a_2}{e_{L/K}} - \dfrac{a_1}{e_{L/K}} & (x \in \mathcal{O}_K). \end{cases}$$

Therefore

$$(\mathrm{ord} \otimes_{\mathbb{Z}} \mathrm{id}_H)\left(\times\!\!\!\!\!\!\int_{[\tau_1]-[\tau_2]} \omega_\mu \right) = \left(\frac{a_2}{e_{L/K}} - \frac{a_1}{e_{L/K}} \right) \mu(\mathcal{O}_K)$$

but the right hand side is $\mu([\tau_1] - [\tau_2])$. $\square$

2.3. The Manin–Drinfeld theorem. Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a discrete finitely generated subgroup without torsion, and let $\mathcal{L}_\Gamma$ be the set of the $Q \in \mathbb{P}^1(K)$ such that there exists $P \in \mathbb{P}^1(K)$ and a sequence $\{\gamma_n\}_{n \in \mathbb{N}} \subset \Gamma$ consisting of distinct elements such that $\gamma_n P \to Q$ $(n \to \infty)$.

DEFINITION 2.12. The Bruhat–Tits tree of Γ is the subtree $\mathcal{T}_\Gamma \subset \mathcal{T}_K$ defined by

$$\mathcal{T}_\Gamma := \bigcup_{\substack{\mathcal{L}_\Gamma \cap U_e \neq \emptyset \\ \mathcal{L}_\Gamma \setminus U_e \neq \emptyset}} e.$$

Denote by $\mathrm{Meas}(\mathcal{L}_\Gamma, H)^\Gamma$ the Γ-invariant part of $\mathrm{Meas}(\mathcal{L}_\Gamma, H)$. Then we have:

$$\begin{aligned}\mathrm{Meas}(\mathcal{L}_\Gamma, H)^\Gamma &= \mathrm{Ker}\big(\mathrm{Hom}_{\mathbb{Z}}(E_{\mathcal{T}_\Gamma}, H) \xrightarrow{\mathrm{Tr}^*} \mathrm{Hom}_{\mathbb{Z}}(V_{\mathcal{T}_\Gamma}, H)\big)^\Gamma \\ &= \mathrm{Ker}\big(\mathrm{Hom}_{\mathbb{Z}}((E_{\mathcal{T}_\Gamma})_\Gamma, H) \xrightarrow{\mathrm{Tr}^*} \mathrm{Hom}_{\mathbb{Z}}((V_{\mathcal{T}_\Gamma})_\Gamma, H)\big) \\ &= \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}(\mathrm{Tr})_\Gamma, H),\end{aligned}$$

where $(V_{\mathcal{T}_\Gamma})_\Gamma$, $(E_{\mathcal{T}_\Gamma})_\Gamma$ are the maximal Γ-invariant quotient of $V_{\mathcal{T}_\Gamma}$, $E_{\mathcal{T}_\Gamma}$ and they are equal to the corresponding abelian groups of the graph $\Gamma\backslash\mathcal{T}_\Gamma$.

REMARK 2.13. If μ is Γ-invariant, $⨏_{\gamma d}\, \omega_\mu = ⨏_d\, \omega_\mu$ for any $\gamma \in \Gamma$.

It is known that the the quotient $\Gamma\backslash\mathcal{T}_\Gamma$ is a finite graph, and $\mathrm{Coker}(\mathrm{Tr})_\Gamma$ is isomorphic to $H^1(\Gamma, \mathbb{Z})$ (see [8], Section 2.3).

We have $\mu \in \mathrm{Meas}(\mathcal{L}_\Gamma, H^1(\Gamma, \mathbb{Z}))$ corresponding to

$$\mathrm{id}_{H^1(\Gamma,\mathbb{Z})} \in \mathrm{Hom}(H^1(\Gamma, \mathbb{Z}), H^1(\Gamma, \mathbb{Z})).$$

This measure μ can be described explicitly: We fix a vertex $v \in \mathcal{T}_\Gamma$. Let $e_1, e_2, \dots, e_n$ be a set of edges of $\Gamma\backslash\mathcal{T}_\Gamma$ (we fix orients of them). For any $\gamma \in \Gamma$, let e_γ be an element in $E_{\mathcal{T}_\Gamma}$ such that $\partial(e_\gamma) = \gamma v - v$. Write $e_\gamma = m_1 e_1 + \cdots + m_n e_n$ and $e_\gamma^* := m_1 e_1^* + \cdots + m_n e_n^*$ where e_j^* is the dual of e_j. Then μ is as follows:

$$\mu(U_e)(\gamma) = e_\gamma^*(e).$$

Roughly speaking, the value of $\mu(U_e)(\gamma)$ is the number including orient of e lying in the path from v to γv modulo Γ.

Theorem 2.14 (Mumford)**.** *Let X be a Mumford curve over K (i.e. stable reduction of X contains only rational curves that intersect at normal crossing over k_K). Then there exists a subgroup $\Gamma \subset \mathrm{PGL}_2(K)$ and an $\mathrm{Aut}(\mathbb{C}_p/K)$-equivariant rigid analytic isomorphism*:

$$X(\mathbb{C}_p) \cong \Gamma\backslash\mathcal{H}_\Gamma(\mathbb{C}_p).$$

Moreover Γ is discrete, free of rank g (g is the genus of X) and unique up to conjugation in $\mathrm{PGL}_2(K)$.

REMARK 2.15. For the Mumford curves appearing in the p-adic uniformization of Shimura curves, we have $\mathcal{L}_\Gamma = \mathbb{P}^1(K)$.

Let Γ be as in Theorem 2.14. Then $H := \mathrm{Coker}(\mathrm{Tr})_\Gamma$ is finitely generated free abelian group. Let μ be the universal Γ-invariant measure in $\mathrm{Meas}(\mathcal{L}_\Gamma, H)^\Gamma$, corresponding to $\mathrm{id} \in \mathrm{End}_{\mathbb{Z}}(H)$.

At first, we construct a $\mathbb{Z}$-lattice Λ in $\mathbb{C}_p^\times \otimes_{\mathbb{Z}} H$. By the following exact sequence:

$$0 \to \mathrm{Div}_0(\mathcal{H}_\Gamma(\mathbb{C}_p)) \to \mathrm{Div}(\mathcal{H}_\Gamma(\mathbb{C}_p)) \to \mathbb{Z} \to 0,$$

we have a homomorphism

$$\delta\colon H_1(\Gamma, \mathbb{Z}) \to \mathrm{Div}_0(\mathcal{H}_\Gamma(\mathbb{C}_p))_\Gamma$$

by the long exact sequence of the group homology (we denote $H_0(\Gamma, \cdot)$ by $(\cdot)_\Gamma$). Since there is no stabilizers for an element of $\mathcal{H}(\mathbb{C}_p)$ (see [9], p. 7, Proposition (1.6.4)), we see that $H_1(\Gamma, \mathrm{Div}(\mathcal{H}_\Gamma(\mathbb{C}_p))) = 0$. Thus δ is injective.

On the other hand, the map

$$\mathrm{Div}_0(\mathcal{H}_\Gamma(\mathbb{C}_p))_\Gamma \ni d \mapsto \times\!\!\!\!\!\!\int_d \omega_\mu \in \mathbb{C}_p^\times \otimes H$$

is a well-defined homomorphism (see Remark 2.13). Therefore we define the lattice Λ by the image of $\times\!\!\!\!\!\!\int \omega_\mu \circ \delta$. Now we state the following theorem of Manin–Drinfeld describing the p-adic uniformization of the Jacobian variety of a Mumford curve.

Theorem 2.16 (Manin–Drinfeld)**.** *The morphism*

$$\begin{array}{ccc} J(X)(\mathbb{C}_p) & \longrightarrow & (\mathbb{C}_p^\times \otimes H)/\Lambda \\ \Psi & & \Psi \\ \tilde{x} - \tilde{y} & \longmapsto & \times\!\!\!\!\!\!\int_{\tilde{x}-\tilde{y}} \omega_\mu \end{array}$$

is an $\mathrm{Aut}(\mathbb{C}_p/K)$*-equivariant rigid analytic isomorphism.*

Proof. See [8], Theorem 2.5. □

2.4. *L*-invariants. In this section, we construct L-invariants associated to a homomorphism $\phi\colon \mathbb{C}_p^\times \to \mathbb{C}_p$ following [8]. Then for each ϕ, we define a map from $J(X)(\mathbb{C}_p)$ to $\mathrm{Hom}(\mathrm{Meas}(\mathcal{L}_\Gamma, \mathbb{C}_p)^\Gamma, \mathbb{C}_p)$ and determines the value in $\mathbb{C}_p$ for a Γ-invariant $\mathbb{C}_p$-valued measure.

We rewrite the statement of Theorem 2.16 as the following exact sequence:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(\Gamma, \mathbb{Z}) & \xrightarrow{j} & \mathbb{C}_p^\times \otimes \mathrm{Coker}(\mathrm{Tr})_\Gamma & \longrightarrow & J(X)(\mathbb{C}_p) & \longrightarrow & 0 \\ & & \wr\| & & \wr\| & & & & \\ & & \Gamma^{ab} & & \mathrm{Hom}_{\mathbb{Z}}(\Gamma^{ab}, \mathbb{C}_p^\times) & & & & \\ & & \Psi & & \Psi & & & & \\ & & \gamma & \longmapsto & \left[\gamma' \mapsto \times\!\!\!\!\!\!\int_{\gamma v - v} \omega_\mu(\gamma')\right] & & & & \end{array}$$

Fix a homomorphism $\phi\colon \mathbb{C}_p^\times \to \mathbb{C}_p$. Let ord: $\mathbb{C}_p^\times \to \mathbb{Q} \subset \mathbb{C}_p$ be the valuation map with $\mathrm{ord}(\pi_K) = 1$. From the explicit description of μ (see the end of Section 2.1) and Proposition 2.11, the pairing:

$$\begin{array}{ccc} \Gamma^{ab} \times \Gamma^{ab} & \longrightarrow & \mathbb{C}_p \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (\gamma, \gamma') & \longmapsto & \mathrm{ord}\left(\times\!\!\!\!\!\!\int_{\gamma v - v} \omega_\mu(\gamma') \right) \end{array}$$

is symmetric and non-degenerate. Therefore the composition of j in the above diagram and $\mathrm{ord} \otimes_{\mathbb{Z}} \mathrm{id}$ induces an isomorphism

$$H_1(\Gamma, \mathbb{C}_p) \xrightarrow{\cong} \mathbb{C}_p \otimes_{\mathbb{Q}} \mathrm{Coker}(\mathrm{Tr})_\Gamma.$$

The map $(\phi \otimes_{\mathbb{Z}} \mathrm{id}) \circ j$ induces a similar homomorphism.

DEFINITION 2.17. For any homomorphism $\phi\colon \mathbb{C}_p^\times \to \mathbb{C}_p$, the *L-invariant* is a unique endomorphism:

$$\mathscr{L}_\phi \in \mathrm{End}(\mathbb{C}_p \otimes_{\mathbb{Z}} \mathrm{Coker}(\mathrm{Tr})_\Gamma)$$

such that:

$$(\phi \otimes_{\mathbb{Z}} \mathrm{id} - \mathscr{L}_\phi \circ (\mathrm{ord} \otimes_{\mathbb{Z}} \mathrm{id})) \circ j = 0.$$

From the above exact sequence, we define a well-defined homomorphism ϕ^X:

$$\begin{array}{rcl} \phi^X\colon J(X)(\mathbb{C}_p) & \longrightarrow & \mathbb{C}_p \otimes_{\mathbb{Z}} \mathrm{Coker}(\mathrm{Tr})_\Gamma \cong \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}(\mathrm{Tr}), \mathbb{C}_p)^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ P & \longmapsto & \phi \otimes_{\mathbb{Z}} \mathrm{id}(\tilde{P}) - \mathscr{L}_\phi \circ (\mathrm{ord} \otimes_{\mathbb{Z}} \mathrm{id})(\tilde{P}), \end{array}$$

where $\tilde{P}$ is a lift in $\mathbb{C}_p^\times \otimes_{\mathbb{Z}} \mathrm{Coker}(\mathrm{Tr})_\Gamma$ and $*$ means the $\mathbb{C}_p$-linear dual.

The abelian group $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Coker}_\Gamma(\mathrm{Tr}), \mathbb{C}_p)$ is isomorphic to $\mathrm{Meas}(\mathcal{L}_\Gamma, \mathbb{C}_p)^\Gamma$ (see the equality (5) in Section 2.3). Thus for any $P \in J(X)$, $\phi^X(P)$ gives values in $\mathbb{C}_p$ for each Γ-invariant $\mathbb{C}_p$-valued measure.

3. Automorphic forms on definite quaternion algebras and Hida families

3.1. Basic definitions. For any algebraic group G over F, we denote by $G(\mathbb{A}_F)$ and $G(\mathbb{A}_{F,f})$ the $\mathbb{A}_F$ and $\mathbb{A}_{F,f}$-valued points respectively. For any $x \in G(\mathbb{A}_F)$, we denote by $x_v \in G(F_v)$ the v-component of x for any place v of F and for any subgroup $H = \prod H_v \subset G(\mathbb{A}_F)$ we denote H^v the subset in H which consists of $h \in H$ such that $h_v = 1$.

Let F be a totally real field and $\mathfrak{p}$ the unique prime ideal above the odd prime number p as in Notation 1.3 Let B be a definite quaternion algebra over F, which is ramified at all archimedean places. Let $\mathfrak{n}^-$ be the product of the finite prime ideal where B ramified at. We assume that B is split at $\mathfrak{p}$. We denote by $\hat{B}^\times$ the group $B^\times(\mathbb{A}_{F,f})$, which is the group of finite adèlic points of $B^\times$.

Let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{O}_F$ which is relatively prime to $\mathfrak{n}^-$. For any prime $\mathfrak{l}$, let $R(\mathfrak{a})_\mathfrak{l} \subset B_\mathfrak{l} := B \otimes_F F_\mathfrak{l}$ be as follows:

$$R(\mathfrak{a})_\mathfrak{l} := \begin{cases} \text{the unique maximal order of } B_\mathfrak{l} & \text{if } \ \mathfrak{l} \mid \mathfrak{n}^-, \\ \text{an Eichler order of level } \mathfrak{a}\mathcal{O}_{F_\mathfrak{l}} & \text{if } \ \mathfrak{l} \nmid \mathfrak{n}^-. \end{cases}$$

Let $\hat{R}(\mathfrak{a}) := \prod_\mathfrak{l} R(\mathfrak{a})_\mathfrak{l} \subset B \otimes \mathbb{A}_{F,f}$, and $R(\mathfrak{a}) := B \cap \hat{R}(\mathfrak{a})$, which is called an Eichler order of B of level $\mathfrak{a}$.

For each prime $\mathfrak{l} \nmid \mathfrak{n}^-$, we fix an isomorphism of $F_\mathfrak{l}$-algebras

$$\iota_\mathfrak{l}\colon B_\mathfrak{l} = B \otimes_F F_\mathfrak{l} \xrightarrow{\cong} M_2(F_\mathfrak{l})$$

(for any ring A, $\mathrm{M}_2(A)$ is the ring of 2×2 matrices with entries in A). The map $\iota_\mathfrak{l}$ induces an isomorphism between $B_\mathfrak{l}^\times$ and $\mathrm{GL}_2(F_\mathfrak{l})$. By exchanging $\iota_\mathfrak{l}$ for its conjugation, we may assume that the image of $R(\mathfrak{a})_\mathfrak{l}$ is

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathcal{O}_{F_\mathfrak{l}}) \;\middle|\; c \equiv 0 \bmod \mathfrak{a}\mathcal{O}_{F_\mathfrak{l}} \right\}.$$

DEFINITION 3.1. For any non-zero ideal $\mathfrak{a}$ of $\mathcal{O}_F$ which is relatively prime to $\mathfrak{n}^-$. Then we put:

$$\begin{aligned} \Sigma_0(\mathfrak{a}, \mathfrak{n}^-) &:= \hat{R}(\mathfrak{a})^\times, \\ \Sigma_1(\mathfrak{a}, \mathfrak{n}^-) &:= \left\{ u \in \Sigma_0(\mathfrak{a}, \mathfrak{n}^-) \;\middle|\; \iota_\mathfrak{l}(u) \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{l}}) \text{ for } \mathfrak{l} \nmid \mathfrak{n}^- \right\}, \\ \Delta_0(\mathfrak{a}, \mathfrak{n}^-) &:= \hat{R}(\mathfrak{a}), \\ \Delta_1(\mathfrak{a}, \mathfrak{n}^-) &:= \left\{ x \in \Delta_0(\mathfrak{a}, \mathfrak{n}^-) \;\middle|\; \iota_\mathfrak{l}(u) \equiv \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix} \bmod \mathfrak{a}\mathrm{M}_2(\mathcal{O}_{F_\mathfrak{l}}) \text{ for } \mathfrak{l} \nmid \mathfrak{n}^- \right\}. \end{aligned}$$

DEFINITION 3.2. Let Σ be an open compact subgroup of $\hat{B}^\times$ and let M be a $\mathbb{Z}_p$-module with $\iota_\mathfrak{p}(\Sigma_\mathfrak{p})$ action. An *M-valued automorphic form on $\hat{B}^\times$ of level* Σ is a function

$$\Phi\colon \hat{B}^\times \to M$$

such that

$$\Phi(\gamma b u) = \iota_\mathfrak{p}(u_\mathfrak{p})^{-1} \cdot \Phi(b)$$

for all $\gamma \in B^\times$, $b \in \hat{B}^\times$, $u \in \Sigma$. We denote by $S(\Sigma, M)$ the *space of M-valued automorphic forms on $\hat{B}^\times$ of level Σ.*

REMARK 3.3. Since $B^\times \backslash \hat{B}^\times / \Sigma$ is a finite set, Φ is determined by its values on a finite set of representatives of the double coset space.

DEFINITION 3.4. For each embedding $\sigma : F_p \to \bar{\mathbb{Q}}_p$, and any $n \geq 0$, let Sym^n be the $\mathbb{C}_p$-vector space of homogeneous polynomials of degree n in the indeterminates X_σ, Y_σ with coefficients in $\mathbb{C}_p$. We put

$$\mathcal{B}_n := \bigotimes_{\sigma : F_p \to \bar{\mathbb{Q}}_p} \mathrm{Sym}^{n_\sigma}.$$

We define a right action of $\mathrm{GL}_2(F_p)$ on $\mathcal{B}_n$:

$$\bigotimes_\sigma P^\sigma(X_\sigma, Y_\sigma)|\gamma := \bigotimes_\sigma P^\sigma(a^\sigma X_\sigma + b^\sigma Y_\sigma, c^\sigma X_\sigma + d^\sigma Y_\sigma)$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_\mathfrak{p})$ and $P^\sigma(X^\sigma, Y^\sigma) \in \mathrm{Sym}^n$. Then we put

$$V_n := \mathrm{Hom}_{\mathbb{C}_p}(\mathcal{B}_n, \mathbb{C}_p)$$

with the left action of $\mathrm{GL}_2(F_\mathfrak{p})$ induced by $\mathcal{B}_n$. For any $k \geq 2$, we call $S(\Sigma, V_{k-2})$ *the space of classical automorphic forms on $\hat{B}^\times$ of weight k, and the level Σ.*

We consider the following action of $\hat{F}^\times$ on $S(\Sigma, V_k)$ by

$$z \cdot \Phi(b) := \Phi(zb).$$

This action factors through the infinite idele class group

$$Z_F(\Sigma) := \hat{F}^\times / F_+^\times (\hat{\mathcal{O}}_F^\times \cap \Sigma)^\mathfrak{p}.$$

We have a natural surjection from $Z_F(\Sigma)$ to a finite idele class group $Cl(\Sigma)$

$$Z_F(\Sigma) \to Cl(\Sigma) := \hat{F}^\times / F_+^\times (\hat{\mathcal{O}}_F^\times \cap \Sigma)$$

whose kernel is given by the image of $\mathcal{O}_{F_\mathfrak{p}}^\times \cap \Sigma_\mathfrak{p}$ in $Z_F(\Sigma)$.

Let $\chi_{F,\mathrm{cycl}}$ be the restriction of the cyclotomic character to $\mathrm{Gal}(\bar{F}/F)$. By Definitions 3.2 and 3.4, the action of $\mathcal{O}_{F_\mathfrak{p}}^\times \cap \Sigma_\mathfrak{p}$ is given by multiplying $\chi_{F,\mathrm{cycl}}^{k-2}(z)$ for each $z \in \mathcal{O}_{F_\mathfrak{p}}^\times \cap \Sigma_\mathfrak{p}$.

DEFINITION 3.5. For each character η of $Cl(\Sigma)$, we define

$$\begin{aligned} &S(\Sigma, V_k, \eta) \\ &:= \{\Phi \in S(\Sigma, V_k) \mid \Phi(zb) = \chi_{F,\mathrm{cycl}}^{k-2}(z)\eta(z)^{-1}\Phi(b) \text{ for all } z \in \hat{F}^\times,\ b \in \hat{B}^\times\}. \end{aligned}$$

We have a decomposition:

$$S(\Sigma, V_k) = \bigoplus_{\eta} S(\Sigma, V_k, \eta),$$

where $\eta\colon Cl(\Sigma) \to \mathbb{C}_p^\times$ runs over the characters of $Cl(\Sigma)$.

We recall the definition of Hecke operators.

DEFINITION 3.6. Let $\mathfrak{n} \subset \mathcal{O}_F$ be a nonzero prime ideal which is relatively prime to $\mathfrak{n}^-$. There exist two kinds of operators $T(\mathfrak{a}), T(\mathfrak{a}, \mathfrak{a})$ for certain non-zero ideals $\mathfrak{a} \subset \mathcal{O}_F$ acting on the space of automorphic forms. Let $(\Delta, \Sigma) = (\Delta_0, \Sigma_0)$ or (Δ_1, Σ_1).

- (*Definition of* $T(\mathfrak{a})$): For any $(\mathfrak{a}, \mathfrak{n}) = 1$, given the right coset decomposition

$$\{x \in \Delta(\mathfrak{n}, \mathfrak{n}^-) \mid \mathrm{Nrd}_{B/F}(x)\mathcal{O}_F = \mathfrak{a}\} = \bigsqcup_i \sigma_i \Sigma(\mathfrak{n}, \mathfrak{n}^-). \tag{5}$$

Let M be a $\mathbb{Z}_p$-module as in Definition 3.2 with the action of $\Delta(\mathfrak{n}, \mathfrak{n}^-)$ which is compatible with that of $\iota_{\mathfrak{l}}(\Sigma(\mathfrak{n}, \mathfrak{n}^-))$. For any $\Phi \in S(\Sigma(\mathfrak{n}, \mathfrak{n}^-), M)$, we define

$$(T(\mathfrak{a})\Phi)(b) := \sum_i \sigma_i \Phi(b\sigma_i).$$

- (*Definition of* $T(\mathfrak{a}, \mathfrak{a})$): For any $(\mathfrak{a}, \mathfrak{n}\mathfrak{n}^-) = 1$, let $a \in \mathbb{A}_{F,f}$ be an element such that $a\mathcal{O}_F = \mathfrak{a}$. Let M be a $\mathbb{Z}_p$-module as above. For any $\Phi \in S(\Sigma(\mathfrak{n}, \mathfrak{n}^-), M)$, we define

$$(T(\mathfrak{a}, \mathfrak{a})\Phi)(b) := \Phi(ba).$$

REMARK 3.7. When $\mathfrak{a} = \mathfrak{p}$ and $\mathfrak{n}$ is prime to $\mathfrak{p}$, th Hecke operator $T(\mathfrak{p})$ is described explicitly. The right coset decomposition of (5) is given as follows (see [17], Proposition 3.36):

$$\text{if} \quad \Sigma = \Sigma_0, \quad \bigsqcup_{b \in \mathcal{O}_F/\mathfrak{p}} \begin{pmatrix} \pi_{\mathfrak{p}} & b \\ 0 & 1 \end{pmatrix} \Sigma_0(\mathfrak{n}, \mathfrak{n}^-) \sqcup \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}} \end{pmatrix} \Sigma_0(\mathfrak{n}, \mathfrak{n}^-), \tag{6}$$

$$\text{if} \quad \Sigma = \Sigma_1, \quad \bigsqcup_{c \in \mathcal{O}_F/\mathfrak{p}} \begin{pmatrix} 1 & 0 \\ \pi_{\mathfrak{p}} c & \pi_{\mathfrak{p}} \end{pmatrix} \Sigma_1(\mathfrak{p}\mathfrak{n}, \mathfrak{n}^-), \tag{7}$$

where $\pi_{\mathfrak{p}} \in \mathcal{O}_{F_{\mathfrak{p}}}$ is a uniformizer.

3.2. Quaternionic automorphic forms and the Bruhat–Tits tree. Let $\Sigma := \Sigma_0(\mathfrak{a}, \mathfrak{n}^-)$ for a non-zero ideal $\mathfrak{a}$ relatively prime to $\mathfrak{n}^-$. By the strong approximation theorem, the reduced norm map $\mathrm{Nrd}_{B/F}$ gives a bijection:

$$\mathrm{Nrd}_{B/F}\colon B^\times\backslash\hat{B}^\times/B_{\mathfrak{p}}^\times\Sigma \to F_+^\times\backslash\mathbb{A}_{F,f}^\times/\hat{\mathcal{O}}_F^\times =: Cl_F^+,$$

where Cl_F^+ is the narrow ideal class group. Note that since we have assumed p is inert in F, the image of $F_{\mathfrak{p}}^\times$ in Cl_F^+ is trivial. We have a decomposition

$$\hat{B}^\times = \bigsqcup_{i=1}^{h} B^\times x_i B_{\mathfrak{p}}^\times \Sigma, \tag{8}$$

where $h = \#Cl_F^+$ is the narrow class number of F and the elements $x_i \in \hat{B}^\times$ satisfies $(x_i)_{\mathfrak{p}} = 1$ and the images of $x_1, \dots, x_h$ by $\mathrm{Nrd}_{B/F}$ give a set of complete representatives of the finite group Cl_F^+.

For $i = 1, \dots, h$, we define

$$\tilde{\Gamma}_i = \tilde{\Gamma}_i(\mathfrak{a}, \mathfrak{n}^-) := B^\times \cap x_i \hat{B}_p^\times \Sigma x_i^{-1}, \tag{9}$$

$$\Gamma_i = \Gamma_i(\mathfrak{a}, \mathfrak{n}^-) := \{\gamma \in \tilde{\Gamma}_i \mid \mathrm{Nrd}_{B/F}(\gamma) \in U_{F,+}\}, \tag{10}$$

where $U_{F,+}$ is the set of totally positive units. Using (8), we have a bijection

$$\bigsqcup_{i=1}^{h} \tilde{\Gamma}_i\backslash\hat{B}_p^\times/\Sigma_p \xrightarrow{\cong} B^\times\backslash\hat{B}^\times/B_{\mathfrak{p}}^\times\Sigma, \tag{11}$$

which sends $g \in \tilde{\Gamma}_j\backslash\hat{B}_p/\Sigma_p$ to $x_j g$.

By (11), an M-valued automorphic form $\Phi \in S(\Sigma, M)$ (where $\Sigma \subset \hat{B}^\times$ is an open compact subgroup and M is a $\mathbb{Z}_p$-module as in Definition 3.2) can be defined as an h-tuple of functions $\phi^1, \dots, \phi^h$ on $\mathrm{GL}_2(F_{\mathfrak{p}})$ by the rule $\phi^i(g) = \Phi(x_i g)$ for $i = 1, \dots, h$. These functions ϕ^i satisfy

$$\phi^i(\gamma g u) = u^{-1}\phi^i(g) \tag{12}$$

for $\gamma \in \tilde{\Gamma}_i$, $g \in \mathrm{GL}_2(F_p)$, $u \in \Sigma_p$.

Now we give another description of quaternionic automorphic forms on $\hat{B}^\times$ in terms of latices and the Bruhat–Tits tree. Let $\Phi \in S(\Sigma, M)$ be an M-valued automorphic form, and let $(\phi^1, \dots, \phi^h)$ be an h-tuple attached to Φ as above. There exists a bijection

$$\begin{array}{rcl} \xi_{\mathfrak{a}}\colon \hat{B}_{\mathfrak{p}}^\times/\Sigma & \xrightarrow{1:1} & \mathcal{P}(\mathfrak{a}) := \{(L_1, L_2) \mid L_1, L_2 \subset F_{\mathfrak{p}}^2 \text{ are lattices s.t. } L_1/L_2 \cong \mathcal{O}_F/\mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}}\mathfrak{a}}\} \\ \Psi & & \Psi \\ g & \longmapsto & (g(\mathcal{O}_{F_p} \times \mathcal{O}_{F_p}), g(\mathcal{O}_{F_p} \times \mathfrak{a}\mathcal{O}_{F_p})). \end{array}$$

DEFINITION 3.8. For $i = 1, \ldots, h$ and for $(L_1, L_2) \in \mathcal{P}(\mathfrak{a})$ we define

$$c_{\phi^i}(L_1, L_2) := g\phi^i(g),$$

where $g(\mathcal{O}_{F_p} \times \mathcal{O}_{F_p}) = L_1$ and $g(\mathcal{O}_{F_p} \times \mathfrak{a}\mathcal{O}_{F_p}) = L_2$.

REMARK 3.9. By the formula (12), the function c_{ϕ^i} has the following property:

$$c_{\phi^i}(\gamma L_1, \gamma L_2) = \gamma c_{\phi^i}(L_1, L_2)$$

for all $\gamma \in \tilde{\Gamma}_i(\mathfrak{a}, \mathfrak{n}^-)$.

3.3. Measure valued automorphic forms. Fix a valuation ring $\mathcal{O} \subset \mathbb{C}_p$ finite flat over $\mathbb{Z}_p$ containing all conjugates of $\mathcal{O}_F$. Let $\mathfrak{n}^+$ be an ideal relatively prime to $\mathfrak{p}\mathfrak{n}^-$ and let $\Sigma := \Sigma(\mathfrak{n}^+, \mathfrak{n}^-)$ be an open compact subgroup of $\hat{B}^\times$. We denote by $Z_{F,0}$ the kernel of the homomorphism from $Z_F(\Sigma)$ to $Cl_F(\Sigma)$. Explicitly, $Z_{F,0} = \mathcal{O}_{F_\mathfrak{p}}^\times/\mathfrak{c}$, where $\mathfrak{c}$ is the closure of the set of totally positive units of $\mathcal{O}_F$ in $\mathcal{O}_{F_\mathfrak{p}}^\times$.

We define several rings as follows:

$$\begin{aligned} \tilde{\Lambda}_F &:= \mathcal{O}[[Z_{F,0}]], \\ \tilde{\Lambda}_\mathbb{Q} &:= \mathcal{O}[[\mathbb{Z}_p^\times]], \\ \Lambda &:= \mathcal{O}[[1 + \mathbb{Z}_p]], \\ \Lambda^\dagger &:= \mathbb{C}_p\langle\langle T - 2\rangle\rangle. \end{aligned}$$

Here, $\tilde{\Lambda}_F$, $\tilde{\Lambda}_\mathbb{Q}$, Λ are the completed group algebras and $\Lambda^\dagger$ is the ring of convergent power series with coefficients in $\mathbb{C}_p$. We regard $\tilde{\Lambda}_\mathbb{Q}$ (resp. $\Lambda^\dagger$) as $\tilde{\Lambda}_F$ (resp. Λ)-algebras via the homomorphism of $\mathcal{O}$-algebras induced by the following group homomorphisms:

$$Z_{F,0} \ni x \mapsto N_{F_p/\mathbb{Q}_p}(x) \in \mathbb{Z}_p$$
$$(\text{resp. } 1 + \mathbb{Z}_p \ni x \mapsto x^{T-2} \in \Lambda^\dagger).$$

DEFINITION 3.10. For an $\mathcal{O}_{F_\mathfrak{p}}$-lattice $L \subset F_\mathfrak{p}^2$, the *primitive part* of L is $L \setminus \mathfrak{p}L$ and we denote it by L'

We define several spaces:

$$\begin{aligned} X &:= \mathfrak{c}\backslash(\mathcal{O}_{F_p}^2)', \\ X' &:= \mathfrak{c}\backslash(\mathcal{O}_{F_\mathfrak{p}}^\times) \times \mathfrak{p}\mathcal{O}_{F_\mathfrak{p}}, \\ \mathcal{W} &:= \mathfrak{c}\backslash(F_\mathfrak{p}^2 - \{(0, 0)\}). \end{aligned}$$

We define the spaces of compactly supported measures on them

$$\mathcal{D}_* := \{\text{compactly supported measures on } X\},$$
$$\mathcal{D}'_* := \{\text{compactly supported measures on } X'\},$$
$$\mathcal{D} := \{\text{compactly supported measures on } \mathcal{W}\}.$$

Via the zero extension, we have natural inclusions:

$$\mathcal{D}'_* \subset \mathcal{D}_* \subset \mathcal{D}$$

For any function f on $\mathcal{W}$, we define an action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F_{\mathfrak{p}})$ by

$$f|\gamma(x, y) := f(ax + by, cx + dy).$$

Then we define the action of $\mathrm{GL}_2(F_{\mathfrak{p}})$ on $\mathcal{D}$ by

$$\int_S f\, d(g \cdot \mu) := \int_{g^{-1}(S)} f|g\, d\mu,$$

where $\mu \in \mathcal{D}$, $g \in \mathrm{GL}_2(F_p)$ and S a compact subset of $\mathcal{W}$.

DEFINITION 3.11. Let $\mathrm{pr}\colon \mathcal{D} \to \mathcal{D}_*$ (resp. $\mathrm{pr}'\colon \mathcal{D} \to \mathcal{D}'_*$) be a natural projection via restrictions. We define the action of $g \in \Delta_0(\mathfrak{n}^+, \mathfrak{n}^-)$ (resp. $\Delta_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-)$) on $\mu \in \mathcal{D}_*$ (resp. $\mathcal{D}'_*$) by

$$(g, \mu) \mapsto \mathrm{pr}(g \cdot \mu)$$
$$(\text{resp. } (g, \mu) \mapsto \mathrm{pr}'(g \cdot \mu)).$$

DEFINITION 3.12. The set of weight characters $\mathcal{X}_F$ is

$$\mathcal{X}_F := \mathrm{Hom}_{\mathrm{cont}}(Z_{F,0}, \mathbb{C}_p).$$

DEFINITION 3.13. A function f on X is said to be homogeneous with respect to the weight character $\phi \in \mathcal{X}_F$ if

$$f\left|\begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\right. = \phi(\bar{c}) f$$

for any $c \in \mathcal{O}_{F_p}^\times$ and $\bar{c}$ is an image in $Z_{F,0}$.

DEFINITION 3.14. For any $k \geq 2$, We define a weight character P_k by $\langle \chi_{F,\mathrm{cycl}} \rangle^{k-2}$.

DEFINITION 3.15. We put

$$\mathcal{D}^{\mathrm{cycl}} := \mathcal{D} \otimes_{\tilde{\Lambda}_F} \tilde{\Lambda}_{\mathbb{Q}},$$
$$\mathcal{D}^{\mathrm{cycl},\dagger} := \mathcal{D}^{\mathrm{cycl}} \otimes_{\mathbb{C}_p \hat{\otimes} \Lambda} \Lambda^{\dagger},$$

where $\mathbb{C}_p \hat{\otimes} \Lambda := \mathbb{C}_p[[1 + p\mathbb{Z}_p]]$. Similarly we define $\mathcal{D}_*^{\mathrm{cycl}}$, $\mathcal{D}_*^{\mathrm{cycl},\dagger}$, $(\mathcal{D}'_*)^{\mathrm{cycl}}$ and $(\mathcal{D}'_*)^{\mathrm{cycl},\dagger}$.

REMARK 3.16. For $\mu \in \mathcal{D}^{\mathrm{cycl}}$, we can consider the integration $\int f\, d\mu$ for only homogeneous functions: Let $\mu = \sum_{i=1}^{r} \mu_i \otimes_{\Lambda_F} \lambda_i$ with $\mu \in \mathcal{D}$ and $\lambda_i \in \tilde{\Lambda}_{\mathbb{Q}}$. Then for any homogeneous function with the weight character $\langle \chi_{F,\mathrm{cycl}} \rangle^{s-2}$, we define

$$\int f\, d\mu := \sum_{i=1}^{r} \langle \lambda_i(s) \rangle \int f\, d\mu_i,$$

where $s \mapsto \langle \lambda(s) \rangle$ is induced by the composition $\mathbb{Z}_p^{\times} \to 1 + p\mathbb{Z}_p \to \Lambda^{\dagger}$. Note that the definition is well-defined. Similarly, we can define an integral of an element of $\mathcal{D}^{\mathrm{cycl},\dagger}$ when the s is sufficiently close to 2.

DEFINITION 3.17. For any $k \geq 2$ and for any $\eta \in \mathcal{B}_k$, we define the function $\tilde{\eta}$ on $\mathcal{W}$

$$\tilde{\eta}(x, y) := \omega_F(x)^{2-k} \eta(x, y),$$

where $\omega_F(x) := \chi_{F,\mathrm{cycl}}(x) / \langle \chi_{F,\mathrm{cycl}}(x) \rangle$.

REMARK 3.18. This $\tilde{\eta}$ is homogeneous with the weight character $P_{n,v}$.

DEFINITION 3.19. For any $k \geq 2$, we define the specialization map to weight k by

$$\begin{array}{cccc} \rho_k \colon & \mathcal{D}_* & \longrightarrow & V_k \\ & \Psi & & \Psi \\ & \mu & \longmapsto & \left[\eta \mapsto \displaystyle\int_{X'} \tilde{\eta}(x, y)\, d\mu(x, y) \right]. \end{array}$$

The map ρ_k is $\Delta_1(\mathfrak{p}, \mathfrak{n}^-)$-equivariant. By the same formula, we define the specialization map ρ_k on $\mathcal{D}_*^{\mathrm{cycl}}$ and on $\mathcal{D}_*^{\mathrm{cycl},\dagger}$ if k is sufficiently close to 2 p-adically.

REMARK 3.20. If $k \equiv 2 \bmod p - 1$, the specialization map ρ_k is $\Delta_0(\mathfrak{p}, \mathfrak{n}^-)$-equivariant. More precisely, we have

$$\rho_k(u \cdot \mu) = \omega_F^{2-k}(u) u \cdot \rho_k(\mu)$$

for any $\mu \in \mathcal{D}_*$ and $u \in \Sigma_0(\mathfrak{p}, \mathfrak{n}^-)$.

This specialization map induces

$$(\rho_k)_*\colon S(\Sigma_0(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*) \to S(\Sigma_1(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-),V_k)$$

(as in Remark 3.20, we can replace the Σ_1 of right hand side with Σ_0 if $k \equiv 2 \bmod p-1$).

Similarly we also define specialization maps $(\rho_{n,v})_*$ on the spaces $S(\Sigma(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*^{\mathrm{cycl}})$ and on $S(\Sigma(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*^{\mathrm{cycl},\dagger})$ if k close to 2 p-adically (note that since the set of double coset $B^\times\backslash\hat{B}^\times/\Sigma(\mathfrak{n}^+,\mathfrak{n}^-)$ is finite, the specialization map is defined for all $k \geq 2$ such that k is sufficiently close to 2).

By definition, the specialization map commutes with the action of Hecke operators (in Definition 3.6)

$$\rho_k \circ T(\mathfrak{a}) = T(\mathfrak{a}) \circ \rho_k,$$
$$\rho_k \circ T(\mathfrak{b},\mathfrak{b}) = T(\mathfrak{b},\mathfrak{b}) \circ \rho_k$$

for all nonzero ideals $\mathfrak{a}$ prime to $\mathfrak{p}\mathfrak{n}^-$ and $\mathfrak{b}$ prime to $\mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$.

On the other hand, the action of $T(\mathfrak{p})$ on $S(\Sigma(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*)$ is also transferred to the action of $T(\mathfrak{p})$ on $S(\Sigma(\mathfrak{n}^+,\mathfrak{n}^-),V_k)$:

Proposition 3.21. *For any $k \geq 2$, we have*

$$\rho_k \circ T(\mathfrak{p}) = T(\mathfrak{p}) \circ \rho_k.$$

Proof. This follows from a simple computation by using the formulae (6) and (7) of Definition 3.6. □

3.4. Hida deformation of measure valued forms. In the previous section, we have defined the specialization maps ρ_k. These maps give a p-adic family of quaternionic automorphic forms. By Hida's theory, there exists a p-adic family of Hecke eigenforms.

Let $\mathbf{T}$ be the free polynomial algebra over $\mathbb{Z}$ in the symbols $\{T(\mathfrak{a})\}$ for ideals $\mathfrak{a}$ such that $(\mathfrak{a},\mathfrak{n}^-)=1$ and $\{T(\mathfrak{b},\mathfrak{b})\}$ for ideals $\mathfrak{b}$ such that $(\mathfrak{b},\mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-)=1$. Then $\mathbf{T}$ acts on the spaces $S(\Sigma_0(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*^{\mathrm{cycl},\dagger})$ and $S(\Sigma_1(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-),V_k)$ as Hecke operators.

We call $\Phi_k \in S(\Sigma_1(\mathfrak{p}\mathfrak{n}^+,\mathfrak{n}^-),V_k)$ a *Hecke eigenform* if it is an eigenvector for the action of $\mathbf{T}$. When Φ_k is a Hecke eigenform, we denote by $a(\mathfrak{a},\Phi_k)$ the eigenvalue for $T(\mathfrak{a})$ for an ideal $\mathfrak{a}$ is prime to $\mathfrak{n}^+$. We call Φ_k is $\mathfrak{p}$-*ordinary* if $a(\mathfrak{p},\Phi)$ is a p-adic unit.

Similarly, we call $\Phi_\infty \in S(\Sigma_0(\mathfrak{n}^+,\mathfrak{n}^-),\mathcal{D}_*^{\mathrm{cycl},\dagger})$ a *Hecke eigenform* if it is an eigenvector for the action of $\mathbf{T}\otimes_{\mathbb{Z}}\Lambda^\dagger$, and we denote by $a(\mathfrak{a},\Phi_\infty)(T) \in \Lambda^\dagger$ the eigenvalue of $T(\mathfrak{a})$ of Φ_∞. We can show that there exists a positive radius such that $a(\mathfrak{a},\Phi_\infty)(T)$ can be defined for all $\mathfrak{a}$.

Now we state Hida's theory of lifting a Hecke eigenform to a p-adic family, in the style of Greenberg–Stevens in [10].

Theorem 3.22. *Let $\Phi = \Phi_2 \in S(\Sigma_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-), V_2)$ be a $\mathfrak{p}$-ordinary Hecke eigenform, and new at primes dividing $\mathfrak{p}\mathfrak{n}^+$. Then there exists a Hecke eigenform*

$$\Phi_\infty \in S(\Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-), \mathcal{D}_*^{\mathrm{cycl},\dagger})$$

such that $\rho_2(\Phi_\infty) = \Phi_2$.

The following natural map:

$$S(\Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-), \mathcal{D}_*) \xrightarrow{\cong} S(\Sigma_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-), \mathcal{D}'_*) \tag{13}$$

is an isomorphism (see [14], Appendix II). Moreover this isomorphism commutes with the action of Hecke operators $T(\mathfrak{a})$ ($\mathfrak{a}$ is a non-zero ideal relatively prime to $\mathfrak{n}^-$) and $T(\mathfrak{b}, \mathfrak{b})$ ($\mathfrak{b}$ is a non-zero ideal relatively prime to $\mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$). This isomorphism induces an isomorphism

$$S(\Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-), \mathcal{D}_*^{\mathrm{cycl},\dagger}) \xrightarrow{\cong} S(\Sigma_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-), (\mathcal{D}'_*)^{\mathrm{cycl},\dagger}).$$

We denote by $\bar{\Phi}_\infty$ the image of Φ_∞ in Theorem 3.22 via the above isomorphism.

3.5. The indefinite integrals. As in Theorem 3.22, let

$$\Phi \in S(\Sigma_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-), V_2)$$

be an automorphic form of weight 2, and let

$$\Phi_\infty \in S(\Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-), \mathcal{D}_*^{\mathrm{cycl},\dagger})$$

be a measure valued form. The forms Φ and Φ_∞ are Hecke eigenforms. Let $(\phi^1, \dots, \phi^h)$ and $(\phi^1_\infty, \dots, \phi^h_\infty)$ be h-tuples corresponding to Φ and Φ_∞ respectively. We denote by $\alpha_\mathfrak{p}$ and $\alpha_\mathfrak{p}(s)$ the eigenvalue of $T(\mathfrak{p})$ on Φ and Φ_∞ respectively (note that $\alpha_\mathfrak{p}(0) = \alpha_\mathfrak{p}$).

Let $\mathbb{Q}(\Phi)$ be the Hecke field of the automorphic form Φ, which is a number field generated over $\mathbb{Q}$ by all of the Hecke eigenvalues of Φ. We can regard Φ as a $\mathbb{C}_p$-valued function on a finite set. Recall that we have fixed an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$

Let $\bar{\Phi}_\infty \in S(\Sigma_0(\mathfrak{p}\mathfrak{n}^+, \mathfrak{n}^-), (\mathcal{D}'_*)^{\mathrm{cycl},\dagger})$ be the image of Φ by the isomorphism as in the end of Section 3.4 (which is also a Hecke eigenform) and let $(\bar{\phi}^1_\infty, \dots, \bar{\phi}^h_\infty)$ be the h-tuple corresponding to $\bar{\Phi}_\infty$. We regard Φ_∞ and $\bar{\Phi}_\infty$ as $\mathcal{D}^{\mathrm{cycl},\dagger}$-valued functions by the canonical injection $\mathcal{D}_*^{\mathrm{cycl},\dagger} \hookrightarrow \mathcal{D}^{\mathrm{cycl},\dagger}$ and $(\mathcal{D}'_*)^{\mathrm{cycl},\dagger} \hookrightarrow \mathcal{D}_*^{\mathrm{cycl},\dagger}$ via the zero extension. Let U be a p-adic neighborhood of 2 such that for $k \geq 2$ with $k \in U$ the weight k specialization Φ_k of Φ_∞ is defined.

Fix a point $\tau \in \mathcal{H}(\bar{\mathbb{Q}}_p)$ in the p-adic upper half plane. Let $K/F_\mathfrak{p}$ be a finite extension over $F_\mathfrak{p}$ containing the element τ. Then we define a function $F_s^\tau(x, y)$ on $\mathcal{W}$:

$$F_s^\tau(x, y) := \langle N_{K/\mathbb{Q}_p}(x_\mathfrak{p} - \tau_\mathfrak{p} y_\mathfrak{p})\rangle^{(s-2)/[K:F_\mathfrak{p}]}.$$

Note that this function is defined on an open compact subset of $\mathcal{W}$ if s is sufficiently close to 2. The function F_s^τ does not depend on the choices of K.

DEFINITION 3.23. Let $\tau \in \mathcal{H}(\bar{\mathbb{Q}}_p)$. We define the functions $\theta^\tau_{\phi^i}$ and $\bar{\theta}^\tau_{\phi^i}$ as follows:

$$\theta^\tau_{\phi^i}(s; L) := \alpha_{\mathfrak{p}}(s)^{-\operatorname{ord}_p(\det(\xi^{-1}_{\mathfrak{n}^+}(L)))} c_{\phi^i}(L, L)(F_s^\tau) \quad \text{for } L \text{ such that } (L, L) \in \mathcal{P}(\mathfrak{n}^+),$$
$$\bar{\theta}^\tau_{\phi^i}(s; L_1, L_2) := \alpha_{\mathfrak{p}}(s)^{-\operatorname{ord}_p(\det(\xi^{-1}_{\mathfrak{p}\mathfrak{n}^+}(L_1, L_2)))} c_{\bar{\phi}^i}(L_1, L_2)(F_s^\tau) \quad \text{for } (L_1, L_2) \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+),$$

where c_{ϕ^i}, $c_{\bar{\phi}^i}$, ξ and $\mathcal{P}$ are as in Section 3.2. These functions are defined if s is sufficiently close to 2 p-adically.

The functions $\theta^\tau_{\phi^i}(s; \cdot)$ and $\bar{\theta}^\tau_{\phi^i}(s; \cdot)$ are analytic in the variable s. So we can consider the derivative with respect to s and define new functions on $\mathcal{P}(\mathfrak{n}^+)$, $\mathcal{P}(\mathfrak{p}\mathfrak{n}^+)$ as follows:

DEFINITION 3.24. Let $\tau \in \mathcal{H}(\bar{\mathbb{Q}}_p)$. We define functions I^τ and $\bar{I}^\tau$ as follows:

$$I^\tau_{\phi^i}(L) := \left.\frac{d\theta^\tau_{\phi^i}(s; L)}{ds}\right|_{s=2} \quad \text{for } L \text{ such that } (L, L) \in \mathcal{P}(\mathfrak{n}^+),$$
$$\bar{I}^\tau_{\phi^i}(L_1, L_2) := \left.\frac{d\bar{\theta}^\tau_{\phi^i}(L_1, L_2)}{ds}\right|_{s=2} \quad \text{for } (L_1, L_2) \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+).$$

Lemma 3.25. *For $\theta^\tau_{\phi^i}$, $\bar{\theta}^\tau_{\phi^i}$, $I^\tau_{\phi^i}$ and $\bar{I}^\tau_{\phi^i}$, we have the following formulae*:

$$\theta^\tau_{\phi^i}(s; L) = \sum_{\{L' \mid (L, L') \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+)\}} \bar{\theta}^\tau_{\phi^i}(s; L, L') \tag{14}$$
$$= \bar{\theta}^\tau_{\phi^i}(s; L, L') + \bar{\theta}^\tau_{\phi^i}\left(s; \frac{1}{p}L', L\right) \quad \textit{for } (L, L') \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+), \tag{15}$$
$$\theta^\tau_{\phi^i}(s; pL) = \alpha_{\mathfrak{p}}(s)^{-2}\theta^\tau_{\phi^i}(s; L), \tag{16}$$
$$\bar{\theta}^\tau_{\phi^i}(s; pL_1, pL_2) = \alpha_{\mathfrak{p}}(s)^{-2}\bar{\theta}^\tau_{\phi^i}(s; L_1, L_2), \tag{17}$$
$$I^\tau_{\phi^i}(L) = \sum_{\{L' \mid (L, L') \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+)\}} \bar{I}^\tau_{\phi^i}(L, L') \tag{18}$$
$$= \bar{I}^\tau_{\phi^i}(L, L') + \bar{I}^\tau_{\phi^i}\left(\frac{1}{p}L', L\right) \quad \textit{for } (L, L') \in \mathcal{P}(\mathfrak{p}\mathfrak{n}^+), \tag{19}$$
$$I^\tau_{\phi^i}(pL) = \alpha_{\mathfrak{p}}^{-2} I^\tau(L), \tag{20}$$
$$\bar{I}^\tau_{\phi^i}(pL_1, pL_2) = -2\alpha'_{\mathfrak{p}}(0)\alpha_{\mathfrak{p}}^{-3}\bar{\theta}^\tau_{\phi^i}(0; L_1, L_2) + \alpha_{\mathfrak{p}}^{-2}\bar{I}^\tau_{\phi^i}(L_1, L_2). \tag{21}$$

Proof. The equalities (16), (17) are easily proved by simple computations. For the equalities (14) and (15), take $g \in \mathrm{GL}_2(F_{\mathfrak{p}})$ satisfying $g(\mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{F_{\mathfrak{p}}}) = L$. Then we have

$$\begin{aligned} c_{\phi^i}(L, L)(F_s^\tau) &= \int_X F_s^\tau|g \, d\phi_\infty^i(g) \\ &= \int_{\mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{F_{\mathfrak{p}}}^\times} F_s^\tau|g \, d\phi_\infty^i(g) + \int_{\mathcal{O}_{F_{\mathfrak{p}}}^\times \times \mathcal{O}_{F_{\mathfrak{p}}}} F_s^\tau|g \, d\phi_\infty^i(g). \end{aligned}$$

For the first term, let $L' := g(\mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{F_{\mathfrak{p}}})$. Then by definition

$$\int_{\mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}} \times \mathcal{O}_{F_{\mathfrak{p}}}^\times} F_s^\tau|g \, d\phi_\infty^i(g) = c_{\bar{\phi}^i}(L, L')(F_s^\tau). \tag{22}$$

For the second term, let $\sigma_\infty := \begin{pmatrix} 1 & 0 \\ 0 & \pi_{\mathfrak{p}} \end{pmatrix}$ and for any $c \in \mathcal{O}_{F_{\mathfrak{p}}}$, let $\sigma_c = \begin{pmatrix} 1 & 0 \\ \pi_{\mathfrak{p}} c & \pi_{\mathfrak{p}} \end{pmatrix}$ ($\pi_{\mathfrak{p}}$ is a uniformizer in $\mathcal{O}_{F_{\mathfrak{p}}}$). Then we have

$$\begin{aligned} \int_{\mathcal{O}_{F_{\mathfrak{p}}}^\times \times \mathcal{O}_{F_{\mathfrak{p}}}} F_s^\tau|g \, d\phi_\infty^i(g) &= \int_{\mathcal{O}_{F_{\mathfrak{p}}}^\times \times \mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}}} F_s^\tau|g\sigma_\infty^{-1}(\sigma_\infty \cdot d\phi_\infty^i)(g) \\ &= \sum_{c \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\sigma_c(\mathcal{O}_{F_{\mathfrak{p}}}^\times \times \mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}})} F_s^\tau|g\sigma_\infty^{-1}(\sigma_\infty \cdot d\phi_\infty^i)(g) \\ &= \sum_{c \in \mathcal{O}_{F_{\mathfrak{p}}}/\mathfrak{p}} \int_{\mathcal{O}_{F_{\mathfrak{p}}}^\times \times \mathfrak{p}\mathcal{O}_{F_{\mathfrak{p}}}} F_s^\tau|g\sigma_\infty^{-1}\sigma_c \, d\bar{\phi}_\infty^i(g\sigma_\infty^{-1}\sigma_c) \qquad (23) \\ &= (T(\mathfrak{p})\bar{\phi}_\infty^i)(g\sigma_\infty^{-1})(F_s^\tau) \\ &= \alpha_{\mathfrak{p}}(s) c_{\bar{\phi}_\infty^i}\left(\frac{1}{p}L', L\right)(F_s^\tau). \qquad (24) \end{aligned}$$

The equalities (22), (23) imply (14), and the equalities (22), (24) imply (15).

The equalities (18), (19) and (21) easily follows by differentiating (14), (15) and (17) respectively. For (20), since Φ is a newform, $\theta_{\phi^i}^\tau|_{s=2} = 0$ by the formula (14), it also follows by differentiating the equality (16). $\square$

From now on, we assume the following condition:

ASSUMPTION 3.26. $\alpha_{\mathfrak{p}}^2 = 1$.

REMARK 3.27. The above assumption holds if Φ_2 comes from a modular elliptic curve over F with multiplicative reduction at $\mathfrak{p}$.

As in the proof of Lemma 3.25, we have $\theta_{\phi^i}^\tau|_{s=2} = 0$. Thus by the assumption (3.26) and the equality (14), we regard $\bar{\theta}_{\phi^i}^\tau|_{s=2}$ as an element in $\mathrm{Meas}(\mathbb{P}^1(F_{\mathfrak{p}}), \mathcal{O}_{\mathbb{Q}(\Phi)})$. By definition, $\bar{\theta}^{\tau}_{\phi^i}|_{s=2}$ does not depend on the choice of τ. So we denote it by μ_{ϕ^i}. Then the measure μ_{ϕ^i} is the same as one defined in [3] Section 2.9, namely

$$\mu_{\phi^i} = \mathrm{proj}_*(c_{\phi^i}(\mathcal{O}^2_{F_p}, \mathcal{O}^2_{F_p})), \tag{25}$$

where proj is the following map:

$$\begin{array}{rccc} \mathrm{proj}\colon & \mathcal{W} & \longrightarrow & \mathbb{P}^1(F_p) \\ & \Psi & & \Psi \\ & (x, y) & \longmapsto & \dfrac{x}{y}. \end{array}$$

The function $I^{\tau}_{\phi^i}$ is a function on the set of vertices in the Bruhat–Tits tree $\mathcal{T}_{F_{\mathfrak{p}}}$ since we have $I^{\tau}_{\phi^i}(L) = I^{\tau}_{\phi^i}(pL)$ by Assumption 3.26 (see the formula (20)). We regard $I^{\tau}_{\phi^i}$ as an element in $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{Div}(\mathcal{H}(\mathbb{C}_p)), \mathbb{C}_p)$ as in the end of Section 2.1.

DEFINITION 3.28. For any $x \in \bar{\mathbb{Q}}_p$, we define

$$\mathrm{logNorm}_p(x) := \frac{1}{[L : \mathbb{Q}_p]} \log_p N_{L/\mathbb{Q}_p}(x),$$

where L is a finite extension over $\mathbb{Q}_p$ containing x. Recall that $\log_p\colon \mathbb{Q}_p^{\times} \to \mathbb{Q}_p$ is the Iwasawa logarithm map satisfying $\log_p(p) = 0$.

Since Φ is a function on a finite set, the image of μ_{ϕ^i} is a finitely generated abelian subgroup of $\mathbb{C}_p$. Thus as in Section 2.2, we consider the multiplicative integral attached to μ_{ϕ^i}. Then we have the following two lemmas:

Lemma 3.29. *For $x, y \in \mathcal{H}(\mathbb{C}_p)$ and $\tau \in \mathcal{H}(\bar{\mathbb{Q}}_p)$, we have*

$$I^{\tau}_{\phi^i}(x) - I^{\tau}_{\phi^i}(y) = 2\alpha_{\mathfrak{p}}\alpha'_{\mathfrak{p}}(0)\mu_{\phi^i}([x] - [y]) \tag{26}$$

$$= 2\alpha_{\mathfrak{p}}\alpha'_{\mathfrak{p}}(0) \cdot \iota\left\{\mathrm{ord}_p \otimes_{\mathbb{Z}} \mathrm{id}_{\mathcal{O}_{\mathbb{Q}(\Phi)}}\left(⨍_{[x]-[y]} \omega_{\mu_{\phi^i}}\right)\right\}, \tag{27}$$

where $\iota\colon \mathbb{C}_p \otimes_{\mathbb{Z}} \mathbb{C}_p \to \mathbb{C}_p$ is a natural multiplication map.

Proof. The equality (27) follows from Proposition 2.11. For the equality (26), we may assume $\mathrm{red}_{F_{\mathfrak{p}}}(x)$ and $\mathrm{red}_{F_{\mathfrak{p}}}(y)$ are vertices of $\mathcal{T}_{F_{\mathfrak{p}}}$. Then it follows from the formulae (19) and (21). □

Lemma 3.30. *For $x \in \mathcal{H}(\mathbb{C}_p)$ and $\tau_1, \tau_2 \in \mathcal{H}(\bar{\mathbb{Q}}_p)$, we have*

$$I^{\tau_1}_{\phi^i}(x) - I^{\tau_2}_{\phi^i}(x) = \int_{\mathbb{P}^1(F_{\mathfrak{p}})} \mathrm{logNorm}_p\left(\frac{t-\tau_1}{t-\tau_2}\right) d\mu_{\phi^i}(t) \tag{28}$$

$$= \iota\left\{\mathrm{logNorm}_p \otimes_{\mathbb{Z}} \mathrm{id}_{\mathcal{O}_{\mathbb{Q}(\Phi)}}\left(\times\!\!\!\!\!\!\int_{[\tau_1]-[\tau_2]} \omega_{\mu_{\phi^i}}\right)\right\} \tag{29}$$

where $\iota\colon \mathbb{C}_p \otimes_{\mathbb{Z}} \mathbb{C}_p \to \mathbb{C}_p$ is a natural multiplication map.

Proof. The equality (29) follows by definition. For (28), by Lemma 3.29, the difference $I^{\tau_1} - I^{\tau_2}$ is a constant function. Thus we may assume $\mathrm{red}_{F_{\mathfrak{p}}}(x)$ is the class of lattice $\mathcal{O}^2_{F_{\mathfrak{p}}}$. Then by definition,

$$\begin{aligned} I^{\tau_1}_{\phi^i}(x) - I^{\tau_2}_{\phi^i}(x) &= \frac{d}{ds}\phi_\infty(1)(F^{\tau_1}_s - F^{\tau_2}_s)\Big|_{s=2} \\ &= \int_X \mathrm{logNorm}_p\left(\frac{x-\tau_1 y}{x-\tau_2 y}\right) d\phi_\infty(1) \\ &= \int_{\mathbb{P}^1(F_{\mathfrak{p}})} \mathrm{logNorm}_p\left(\frac{t-\tau_1}{t-\tau_2}\right) d\mu_{\phi^i}(t). \end{aligned}$$

The last equality follows from (25). □

DEFINITION 3.31. We define the *indefinite integral* I_{ϕ^i} attached to ϕ^i as follows:

$$\begin{array}{ccc} I_{\phi^i} : \mathcal{H}(\bar{\mathbb{Q}}_p) & \longrightarrow & \mathbb{C}_p \\ \Psi & & \Psi \\ \tau & \longmapsto & I_{\phi^i}(\tau) := I^{\tau}_{\phi^i}(\tau). \end{array}$$

Proposition 3.32. *The discrete subgroup $\tilde{\Gamma}_i \subset B^\times$ acts on the indefinite integral I_{ϕ^i} as follows*:

$$I_{\phi^i}(\gamma\tau) = \alpha_{\mathfrak{p}}^{-\operatorname{ord}_{\mathfrak{p}}(\det(\gamma))} I_{\phi^i}(\tau)$$

for any $\gamma \in \tilde{\Gamma}_i$. In particular, the indefinite integrals are Γ_i-equivariant.

Proof. We have the following formula

$$I^{\tau}_{\phi^i}(\gamma x) = \alpha_{\mathfrak{p}}^{-\operatorname{ord}_{\mathfrak{p}}(\det(\gamma))} I^{\gamma^{-1}\tau}_{\phi^i}(x),$$

by a simple computation. The assertion of Proposition 3.32 follows from this by putting $x = \gamma^{-1}\tau$. □

DEFINITION 3.33. Let Φ be a quaternionic automorphic form satisfying the conditions as in the beginning of this section. We define *the indefinite integral I_Φ attached to Φ* by

$$\begin{array}{ccc} I_\Phi : \bigoplus_{i=1}^{h} \mathrm{Div}(\Gamma_i \backslash \mathcal{H}(\bar{\mathbb{Q}}_p)) & \longrightarrow & \mathbb{C}_p \\ \Downarrow & & \Downarrow \\ x := (x_i)_{i=1}^{h} & \longmapsto & I_\Phi(x) := \sum_{i=1}^{h} I_{\phi^i}(x_i). \end{array}$$

Proposition 3.34. *Let Φ be a quaternionic automorphic form satisfying the conditions as in the beginning of this section. The indefinite integral attached to Φ respects the action of Hecke operators. Namely, for all non-zero ideal $\mathfrak{a}$, we have*

$$I_\Phi(T(\mathfrak{a})z) = \alpha(\mathfrak{a}, \Phi) I_\Phi(z).$$

Proof. Let $\mathfrak{a}$ be an ideal in $\mathcal{O}_F$ prime to $\mathfrak{n}^-$. At first, we explicitly describe the action of $T(\mathfrak{a})$. As in Definition 3.6, we set

$$\{x \in \Delta_0(\mathfrak{n}^+, \mathfrak{n}^-) \mid \mathrm{Nrd}_{B/F}(x)\mathcal{O}_F = \mathfrak{a}\} = \bigsqcup_m \sigma_m \Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-),$$

and as in Section 3.2, we set

$$\hat{B}^\times = \bigsqcup_{i=1}^{h} B^\times x_i B_{\mathfrak{p}}^\times \Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-).$$

For any x_i and σ_m, there exist elements $b_{i,m} \in B^\times$, $g_{i,m} \in \mathrm{GL}_2(F_\mathfrak{p})$, $g'_{i,m} \in \Sigma_0(\mathfrak{n}^+, \mathfrak{n}^-)^\mathfrak{p}$ and a number $1 \le l_{i,m} \le h$ such that

$$x_i \sigma_m = b_{i,m} x_{l_{i,m}} g_{i,m} g'_{i,m}.$$

We note that there exists $\gamma_i \in \tilde{\Gamma}_i$ such that $\mathrm{ord}_\mathfrak{p}(\det(\gamma_i)) = 1$ (see [18], Corollary 5.9). Then the actions of $T(\mathfrak{a})$ on an automorphic form ϕ^i and an element $(z_i) \in \bigoplus_{i=1}^{h} \mathrm{Div}(\Gamma_i \backslash \mathcal{H}(\bar{\mathbb{Q}}_p))$ are given as follows:

$$(T(\mathfrak{a})\phi^i)(g) = \sum_m \phi^{l_{i,m}}(g_{i,m} g), \tag{30}$$
$$T(\mathfrak{a}) z_i = \sum_m (\gamma_{l_{i,m}}^{-v_{i,m}} g_{i,m} z_i)_{l_{i,m}},$$

where $v_{i,m} := \mathrm{ord}_\mathfrak{p}(\det(g_{i,m}))$.

Now we get back to the proof of Proposition 3.34. We put

$$z = (z_i)_{i=1}^h \in \bigoplus_{i=1}^h \mathrm{Div}(\Gamma_i \backslash \mathcal{H}(\bar{\mathbb{Q}}_p)).$$

Then we may assume x_i is a class of $\tau_i \in \mathcal{H}(\bar{\mathbb{Q}}_p)$ and $\mathrm{red}_{F_\mathfrak{p}}(\tau_i)$ is a class of a lattice $L_i \in \mathcal{P}(\mathfrak{n}^+)$. Take an element $g \in \mathrm{GL}_2(F_\mathfrak{p})$ satisfying $g(\mathcal{O}_{F_\mathfrak{p}}^2) = L_i$. Then the assertion follows from the following calculation.

$$\begin{aligned}
&I_\Phi(T(\mathfrak{a})z_i)\\
&= \sum_m I_{\phi^{l_{i,m}}}(\gamma_i^{-v_{i,m}} g_{i,m} z_i)\\
&\underset{\text{Proposition 3.32}}{=} \sum_m \alpha_p^{v_{i,m}} \frac{d}{ds} \alpha_\mathfrak{p}(s)^{-v_{i,m}} \alpha(s)^{-\operatorname{ord}_\mathfrak{p}(\det(g))} \int_X F_s^{g_{i,m} g\tau} |g_{i,m} g \, d\phi_\infty^{l_{i,m}}(g_{i,m} g) \Big|_{s=2}\\
&= \frac{d}{ds} \alpha_\mathfrak{p}(s)^{-\operatorname{ord}_\mathfrak{p}(\det(g))} \int_X F_s^\tau |g \, d \sum_m \phi^{l_{i,m}}(g_{i,m} g) \Big|_{s=2}\\
&\underset{(30)}{=} \alpha_\mathfrak{p}(\mathfrak{a}, \Phi) \frac{d}{ds} \alpha_\mathfrak{p}(s)^{-\operatorname{ord}_\mathfrak{p}(\det(g))} \int_X F_s^\tau |g \, d\phi^i(g) \Big|_{s=2}\\
&= \alpha_\mathfrak{p}(\mathfrak{a}, \Phi) I_\Phi(z_i).
\end{aligned}$$

□

Theorem 3.35. *As in the beginning of* Section 3.5, *let* Φ *be a quaternionic automorphic form satisfying* Assumption 3.26. *Let* $(\phi^1, \dots, \phi^h)$ *be an* h*-tuple corresponding to* Φ. *Let* μ_{ϕ^i} *be the measure attached to* ϕ^i *(see the equation* (25)*). For any* $\tau_1 = (\tau_{1,i})_{i=1}^h$ *and* $\tau_2 = (\tau_{2,i})_{i=1}^h \in \prod_{i=1}^h \mathcal{H}(\bar{\mathbb{Q}}_p)$, *we have*

$$\begin{aligned}
&I_\Phi(\tau_1) - I_\Phi(\tau_2)\\
&= \iota\left\{ ((\log\mathrm{Norm}_p + 2\alpha_\mathfrak{p}\alpha_\mathfrak{p}'(0) \cdot \mathrm{ord}_p) \otimes_\mathbb{Z} \mathrm{id}_{\mathcal{O}_{\mathbb{Q}(\Phi)}}) \left(\prod_{i=1}^h \times\!\!\!\!\!\!\int_{[\tau_{1,i}]-[\tau_{2,i}]} \omega_{\mu_{\phi^i}} \right) \right\},
\end{aligned}$$

where $\iota\colon \mathbb{C}_p \otimes_\mathbb{Z} \mathbb{C}_p \to \mathbb{C}_p$ *is a natural multiplication map.*

Proof. The assertion follows from Lemmata 3.29 and 3.30. □

4. Main results

4.1. *p*-adic uniformization of Shimura curves. We use the same notation as in Section 3. Let F be a totally real field and B a definite quaternion algebra over F as in Section 3.1. Fix an archimedean place ∞_0 of F. We denote by $\mathcal{B}$ the definite quaternion algebra over F obtained from B/F by switching the invariants at ∞_0 and $\mathfrak{p}$, namely

$$\begin{aligned}
&\mathrm{inv}_{\infty_0} \mathcal{B} = \mathrm{inv}_{\mathfrak{p}} B = 0,\\
&\mathrm{inv}_{\mathfrak{p}} \mathcal{B} = \mathrm{inv}_{\infty_0} B = 1/2,\\
&\mathrm{inv}_v \mathcal{B} = \mathrm{inv}_v B \quad \text{for any } v \neq \infty_0, \mathfrak{p}.
\end{aligned}$$

We fix an isomorphism $\mathcal{B} \otimes_F (F_{\infty_0}) \xrightarrow{\cong} M_2(\mathbb{R})$. Let $\mathcal{R}$ be an Eichler order of level $\mathfrak{n}^+$. By Shimura's theory, there exists a Shimura curve $X_{\mathcal{B}}(\mathfrak{n}^+)$, which is a proper smooth curve defined over F, whose $\mathbb{C}$-valued points are given by the double coset:

$$X_{\mathcal{B}}(\mathfrak{n}^+)(\mathbb{C}) = \mathcal{B}^\times \backslash \mathcal{B}^\times(\mathbb{A}_F)/\hat{\mathcal{R}}^\times \cdot \mathrm{SO}_2(\mathbb{R}),$$

where $\hat{\mathcal{R}} := \mathcal{R} \otimes_{\mathcal{O}_F} \hat{\mathcal{O}}_F$. Let $\mathcal{B}_+^\times$ be the subgroup of $\mathcal{B}^\times$ consisting of elements whose images by the reduced norm are totally positive elements. We have a double coset decomposition of the following form:

$$\mathcal{B}^\times(\mathbb{A}_{F,f}) = \bigsqcup_{i=1}^{h} \mathcal{B}_+^\times y_i \hat{\mathcal{R}}^\times,$$

where $h = \# Cl_F^+$ is the narrow class number of F. We define a subgroup $\Delta_i \subset \mathcal{B}_+^\times$ by

$$\Delta_i := \mathcal{B}_+^\times \cap y_i \hat{\mathcal{R}}^\times y_i^{-1}.$$

The set of $\mathbb{C}$-valued points of the Shimura curve $X_{\mathcal{B}}(\mathfrak{n}^+)$ can be written as

$$X_{\mathcal{B}}(\mathfrak{n}^+)(\mathbb{C}) \cong \bigsqcup_{i=1}^{h} \Delta_i \backslash \mathbb{H},$$

where $\mathbb{H} := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ is the Poincaré upper half plane.

We recall the p-adic uniformization of the Shimura curve $X_{\mathcal{B}}(\mathfrak{n}^+)$. By the theorem of Cerednik–Drinfeld (see [2] and [6]), we have a rigid analytic isomorphism

$$X_{\mathcal{B}}(\mathfrak{n}^+)(\mathbb{C}_p) = \bigsqcup_{i=1}^{h} \Gamma_i \backslash \mathcal{H}(\mathbb{C}_p), \tag{31}$$

where $\Gamma_i = \Gamma_i(\mathfrak{n}^+, \mathfrak{n}^-) \subset B_{\mathfrak{p}}^\times \cong \mathrm{GL}_2(F_{\mathfrak{p}})$ as in (10) of Section 3.2. Let

$$X_i = \Gamma_i \backslash \mathcal{H}(\mathbb{C}_p) \quad (i = 1, \dots, h)$$

be the connected components of $X_{\mathcal{B}}(\mathfrak{n}^+)(\mathbb{C}_p)$.

Let Φ and h-tuple $(\phi^1, \dots, \phi^h)$ be automorphic forms as in the beginning of Section 3.5. Let μ_{ϕ^i} be the measure attached to ϕ^i on $\mathbb{P}^1(F_{\mathfrak{p}})$ constructed as in (25).

DEFINITION 4.1. Let A be the abelian variety of Φ-component in $\bigoplus_{i=1}^{h} J(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then we define

$$\mathrm{logNorm}_p^A \colon A(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{C}_p$$

by

$$\mathrm{logNorm}_p^A((P_i)_i) := \sum_{i=1}^{h} \mathrm{logNorm}_p^{X_i}(P_i)(\mu_{\phi^i}),$$

for any $(P_i)_i \in A(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q} \subset \bigoplus_{i=1}^{h} J(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Here, $\mathrm{logNorm}_p^{X_i}$ is constructed in the end of Section 2.4.

Now we state one of the main results of this paper.

Theorem 4.2. *Let Φ, X_i and ϕ^i be as above. Let $J(X_i)$ be the Jacobian variety of X_i. Let A be the abelian variety of Φ-component in $\bigoplus_{i=1}^{h} J(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, for any $P = (P_i)_{i=1}^{h} \in A(\mathbb{C}_p) \otimes_{\mathbb{Z}} \mathbb{Q}$, we have*

$$I_\Phi\left(\sum_{i=1}^{h} \tilde{P}_i\right) = \mathrm{logNorm}_p^A(P),$$

where $\tilde{P}_i$ is a lift of P_i in $\mathrm{Div}_0(X_i)$.

Proof. Let μ_i be the universal measure associated to X_i as in Theorem 2.16. We have the decomposition into Hecke eigenspaces:

$$\mathbb{C}_p \otimes_{\mathbb{Z}} \left(\bigoplus_{i=1}^{h} \mathrm{Coker}(\mathrm{Tr})_{\Gamma_i}\right) = \bigoplus_j V_j.$$

Let V_0 be an eigenspace corresponding to Φ. Put $P := \sum_i P_i$ and $\tilde{P} := \sum_i \tilde{P}_i$. Then by Theorem 3.35 and as in Section 2.4, we have:

$$\begin{aligned}
&I_{\phi^i}(\tilde{P}_i)\\
&= \iota\left[\sum_i \{\mathrm{logNorm}_p \otimes_{\mathbb{Z}} \mu_{\phi^i} + 2\alpha'(0)(\mathrm{id}_{\mathbb{C}_p} \otimes_{\mathbb{Z}} \mu_{\phi^i}) \circ (\mathrm{ord}_p \otimes_{\mathbb{Z}} \mathrm{id})\}|_{V_0}\left(\times\!\!\!\!\!\!\int_{\tilde{P}_i} \omega_{\oplus_i \mu_i}\right)\right],\\
&\mathrm{logNorm}_p^{X_i}(P_i)(\mu_{\phi^i})\\
&= \iota\left[\sum_i \{\mathrm{logNorm}_p \otimes_{\mathbb{Z}} \mu_{\phi^i} - (\mathrm{id}_{\mathbb{C}_p} \otimes_{\mathbb{Z}} \mu_{\phi^i}) \circ \mathcal{L} \circ (\mathrm{ord}_p \otimes_{\mathbb{Z}} \mathrm{id})\}|_{V_0}\left(\times\!\!\!\!\!\!\int_{\tilde{P}_i} \omega_{\mu_i}\right)\right],
\end{aligned}$$

where $\iota\colon \mathbb{C}_p \otimes_{\mathbb{Z}} \mathbb{C}_p \to \mathbb{C}_p$ is the national multiplication map and $\mathcal{L} := \mathcal{L}_{\mathrm{logNorm}_p}$ is the L-invariant attached to the homomorphism $\mathrm{logNorm}_p$. By similar arguments as in Section 2.4, we have

$$-2\alpha'(0)\sum_i (\mathrm{id}\otimes_{\mathbb{Z}} \mu_{\phi^i})\Big|_{V_0} = \sum_i (\mathrm{id}\otimes_{\mathbb{Z}} \mu_{\phi^i})\circ \mathcal{L}|_{V_0}. \qquad \square$$

4.2. Heegner points and p-adic integrals. As in Section 4.1, let B be a definite quaternion algebra over F, which is ramified at all archimedean places and the finite places associated to prime ideals dividing $\mathfrak{n}^-$. Let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{O}_{F_\mathfrak{p}}$ relatively prime to $\mathfrak{n}^-$ and let R an Eichler order of B of level $\mathfrak{a}$.

DEFINITION 4.3. Let K be a quadratic extension over F. An *optimal embedding of level* $\mathfrak{a}$ from K into B is a pair $(\Psi, b) \in \mathrm{Hom}_{F\text{-alg}}(K, B) \times (\hat{B}^\times/\hat{R}^\times)$ satisfying

$$\Psi(\mathcal{O}_K) = b\hat{R}b^{-1} \cap \Psi(K).$$

For an optimal embedding (Ψ, b) of level $\mathfrak{a}$, we define

$$R_b := B \cap b\hat{R}b^{-1}.$$

Then R_b is an Eichler order of level $\mathfrak{a}$, and Ψ gives an embedding of $\mathcal{O}_K$ into R_b. We define an action of $g \in \hat{B}^\times$ on an optimal embedding (Ψ, b) by conjugation:

$$g\cdot(\Psi, b) := (g\Psi g^{-1}, gb).$$

We denote by $[\Psi, b]$ the orbit of (Ψ, b) associated with the subgroup $B^\times \subset \hat{B}^\times$. The set of the orbits of optimal embeddings of level $\mathfrak{a}$ is denoted by $\mathrm{Emb}_F(K, B, \mathfrak{a})$. We also define an action of the ideal class group $\mathrm{Pic}(\mathcal{O}_K)$ of K by identifying it with $\hat{K}^\times/K^\times\hat{\mathcal{O}}_K^\times$ and for any $\rho \in \mathrm{Pic}(\mathcal{O}_K)$ we often denote $\rho\cdot[\Psi, b]$ by $[\Psi^\rho, b^\rho]$.

The set $\mathrm{Emb}_F(K, B, \mathfrak{a})$ is described as follows. Let $\Sigma = \Sigma_0(\mathfrak{a}, \mathfrak{n}^-)$. As in Section 3.2, there is a bijection

$$\mathrm{Nrd}_{B/F}\colon B^\times\backslash\hat{B}^\times/B_\mathfrak{p}^\times\Sigma \xrightarrow{1:1} F_+^\times\backslash\mathbb{A}_{F,f}^\times/\hat{\mathcal{O}}_F^\times =: Cl_{F,+}.$$

Then we have a decomposition

$$\hat{B}^\times = \bigsqcup_{i=1}^{h} B^\times x_i B_\mathfrak{p}^\times\Sigma \tag{32}$$

where $h = \#Cl_F^+$ is the narrow class number of F, the element $x_i \in \hat{B}^\times$ satisfies $(x_i)_\mathfrak{p} = 1$

and the images of $x_1, \dots, x_h$ by $\mathrm{Nrd}_{B/F}$ gives a set of complete representatives of the finite group Cl_F^+. Via the bijection $\xi_{\mathfrak{a}}$ as in Section 3.2, we have a natural inclusion:

$$\mathrm{Emb}_F(K, B, \mathfrak{a}) \hookrightarrow \coprod_{i=1}^{h} \tilde{\Gamma}_i \backslash (\mathrm{Hom}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_K[1/p], R_i[1/p]) \times \mathcal{P}(\mathfrak{a})). \tag{33}$$

We always regard the left hand side as a subset of right hand side. For an element $[\Psi, b] \in \mathrm{Emb}_F(K, B, \mathfrak{a})$, if $[\Psi, b]$ belongs to the i-th component, we denote it by $[\Psi_i, L_i]$, which is the class of $(\Psi_i, L_i) \in \mathrm{Hom}_{\mathcal{O}_F\text{-alg}}(\mathcal{O}_K[1/p], R_i[1/p])$.

From now on, we suppose the following three conditions are satisfied:

ASSUMPTION 4.4. 1. All prime ideals $\mathfrak{q}$ of $\mathcal{O}_K$ dividing $\mathfrak{a}$ split in K.
2. All prime ideals $\mathfrak{q}$ of $\mathcal{O}_K$ dividing $\mathfrak{n}^-$ are inert in K.
3. The prime ideal $\mathfrak{p}$ dose not split in K.

REMARK 4.5. The above assumptions 1 and 2 are called the *Heegner condition*. Under these two assumptions, an optimal embedding of level $\mathfrak{a}$ exists. (see [18], Theorems 3.1 and 3.2).

Fix an isomorphism $\iota_p\colon B_{\mathfrak{p}}^\times \xrightarrow{\cong} \mathrm{GL}_2(F_{\mathfrak{p}})$. For an optimal embedding (Ψ, b), let τ_Ψ be the fixed point of $\iota_p(\Psi(K_{\mathfrak{p}}))$ in $\mathcal{H}(\bar{\mathbb{Q}}_p)$ satisfying:

$$\iota_p(\Psi(\alpha)) \cdot \begin{pmatrix} \tau_\psi \\ 1 \end{pmatrix} = \alpha \begin{pmatrix} \tau_\psi \\ 1 \end{pmatrix},$$

for any $\alpha \in K_{\mathfrak{p}}$.

Let $\psi_K : \mathrm{Pic}(\mathcal{O}_K) \to \{\pm 1\}$ be an unramified quadratic character over K. Via the isomorphism (31), let

$$(\xi_i)_{i=1}^h \in \bigoplus_{i=1}^{h} \mathrm{Div}(\Gamma_i \backslash \mathcal{H}(\mathbb{C}_p)) \otimes_{\mathbb{Z}} \mathbb{Q}$$

be the element corresponding to the Hodge class $\xi \in \mathrm{Pic}(X_{\mathcal{B}})(F) \otimes_{\mathbb{Z}} \mathbb{Q}$, which has degree one on each geometric component, and satisfies the relation

$$T(\mathfrak{l})\xi = (N_{F/\mathbb{Q}}(\mathfrak{l}) + 1)\xi$$

for any prime ideal $\mathfrak{l}$ of $\mathcal{O}_F$ prime to $\mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$.

DEFINITION 4.6. For an unramified quadratic character $\psi_K\colon \mathrm{Pic}(\mathcal{O}_K) \to \{\pm 1\}$, we define a point $\mathbf{P}_{\psi_K}$ belonging to $\bigoplus_{i=1}^h J(X_i) \otimes_{\mathbb{Z}} \mathbb{Q}$ as follows:

$$\mathbf{P}_{\psi_K} := \sum_{\rho \in \mathrm{Pic}(\mathcal{O}_K)} \psi_K(\rho)([\tau_{\Psi^\rho}]_{i_{\Psi^\rho}} - \xi_{i_{\Psi^\rho}}).$$

REMARK 4.7. The image of $\mathbf{P}_{\psi_K}$ by the projection to the Φ-part can be described as

$$(N_{F/\mathbb{Q}}(\mathfrak{q}) + 1 - \alpha(\mathfrak{q}, \Phi))^{-1}(N_{F/\mathbb{Q}}(\mathfrak{q}) + 1 - T(\mathfrak{q})) \sum_{\rho \in \mathrm{Pic}(\mathcal{O}_K)} \psi_K(\rho)[\tau_{\Psi^\rho}]_{i_{\Psi^\rho}}$$

where $\mathfrak{q}$ is a prime ideal of $\mathcal{O}_F$ prime to $\mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$ satisfying $N_{F/\mathbb{Q}}(\mathfrak{q}) + 1 \neq \alpha(\mathfrak{q}, \Phi)$.

REMARK 4.8. When $\mathfrak{p}$ is inert in K, by the complex multiplication theory, the point $\mathbf{P}_{\psi_K}$ is global point and belongs to Jacobian $\bigoplus_{i=1}^h J(X_i)(K_{\psi_K}) \otimes_{\mathbb{Z}} \mathbb{Q}$, where K_{ψ_K} is the quadratic field over K cut out by ψ_K. For details, see [14], Sections 4.2 and 4.3.

DEFINITION 4.9. Denote $K = F(\lambda)$ where $\lambda \in K$ is the element such that $\lambda^2 \in F$ and is totally negative. The *partial p-adic L-function* $\mathcal{L}_p$ associated to Φ and a class optimal embedding $[\Psi, b] = [\Psi_i, L_i]$ of level $\mathfrak{n}^+$ is defined by

$$\mathcal{L}_p(\Phi, \Psi, s) := \begin{cases} \langle C_i \cdot |\mathrm{Nrd}_{B/F}(x_i)|_{\mathbb{A}_{F,f}}\rangle^{s/2} \theta_{\phi^i}^{\tau_\Psi}(s; L_i) & \text{if } \mathfrak{p} \text{ is inert in } K, \\ \langle C_i \cdot |\mathrm{Nrd}_{B/F}(x_i)|_{\mathbb{A}_{F,f}}\rangle^{s/2} \dfrac{1}{2}(\theta_{\phi^i}^{\tau_\Psi}(s; L_i) + \theta_{\phi^i}^{\tau_\Psi}(s; \iota_{\mathfrak{p}}(\Psi_i(\lambda))L_i)) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } \mathfrak{p} \text{ is ramified in } K, & \end{cases}$$

where $C_i = c/\sqrt{|N_{F/\mathbb{Q}}(-\mathrm{Nrd}_{B/F}(\lambda))|_{\mathbb{R}}}$.

We also define, for any unramified quadratic character $\psi_K\colon \mathrm{Pic}(\mathcal{O}_K) \to \{\pm 1\}$,

$$\mathcal{L}_p(\Phi, \psi_K, s) := \sum_{\rho \in \mathrm{Pic}(\mathcal{O}_K)} \psi_K(\rho)\mathcal{L}_p(\Phi, \Psi^\rho, s).$$

Proposition 4.10. *The following equalities hold*:

$$\begin{aligned} \frac{d}{ds}\mathcal{L}_p(\Phi, \psi_K, s)\bigg|_{s=0} &= I_\Phi(\tilde{\mathbf{P}}_{\psi_K}) \\ &= \sum_{i=1}^h \mathrm{logNorm}_p^{X_i}(\mathbf{P}_{\psi_K,i})(\mu_{\phi^i}) \end{aligned}$$

where $\tilde{\mathbf{P}}_{\psi_K}$ is a lift of $\mathbf{P}_{\psi_K}$ in $\bigoplus_{i=1}^h \mathrm{Div}_0(\Gamma_i\backslash\mathcal{H}(\bar{\mathbb{Q}}_p)) \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\mathbf{P}_{\psi_K,i}$ is the image of the projection to $J(X_i)$.

Proof. The first equality follows from the definition of the indefinite integrals and Proposition 3.34 by using Remark 4.7. The second equality follows from Theorem 4.2. □

4.3. *p*-adic *L*-functions of Hilbert modular forms. Let A be a modular abelian variety of GL(2)-type over F which is multiplicative at $\mathfrak{p}$ (in addition, if $[F:\mathbb{Q}]$ is odd, suppose that A is multiplicative at some prime other than $\mathfrak{p}$). Let f be the Hilbert modular eigenform corresponding to A of weight 2, level $\mathfrak{n} := \mathfrak{p}\mathfrak{n}^+\mathfrak{n}^-$ and trivial character (here, $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are non-zero ideals prime to $\mathfrak{p}$, and $\mathfrak{n}^-$ is a square free ideal such that $\#\{\mathfrak{q}\colon \text{prime dividing } \mathfrak{n}^-\} \equiv [F:\mathbb{Q}] \bmod 2$). Let $\mathbf{f}_\infty$ be the Hida family of f and we denote by f_k, $f_k^\#$ the weight k specialization of $\mathbf{f}_\infty$ and its p-stabilization respectively. Let Φ be a quaternionic automorphic form of weight 2 and level $\mathfrak{p}\mathfrak{n}^+$ corresponding to f by the Jacquet–Langlands correspondence.

We briefly recall the p-adic L-functions and the two-variable p-adic L-functions of Hilbert modular forms ([7], [14]). We shall explain them in a general situation. Let g be a cuspidal Hilbert modular eigenform of parallel weight k and ordinary at $\mathfrak{p}$, and let χ (resp. ψ) be a finite order Hecke character of F unramified outside $\mathfrak{p}$ and infinite places (resp. at the conductor of g). There exists *p-adic L-function* $L_p(s, g, \chi\psi)$ defined on $s \in \mathbb{Z}_p$ such that

$$\begin{aligned} &L_p(r, g, \chi\psi) \\ &= \left(1 - \frac{\chi\psi\omega_F^{1-r}(\mathfrak{p})\mathcal{N}(\mathfrak{p})^{r-1}}{\alpha_1(\mathfrak{p}, g)}\right)\left(1 - \frac{(\chi\psi\omega_F^{1-r})^{-1}(\mathfrak{p})\alpha_2(\mathfrak{p}, g)}{\mathcal{N}(\mathfrak{p})^r}\right) \cdot \frac{L(r, g, (\chi\psi\omega_F^{1-r})^{-1})}{\Omega_{g,\chi\psi,r}} \end{aligned}$$

for any $r = 1, \dots, k-1$. Here, $L(s, g, \chi\psi\omega_F^{1-r})$ is the complex L-function of g, $\alpha_1(\mathfrak{p}, g)$ (resp. $\alpha_2(\mathfrak{p}, g)$) is the p-adic unit root (resp. p-adic non-unit root) of the Hecke polynomial

$$X^2 - a(\mathfrak{p}, g)X + \epsilon_\mathfrak{p}\mathcal{N}(\mathfrak{p})^{k-1}$$

with

$$\epsilon_\mathfrak{p} = \begin{cases} 0 & \mathfrak{p} | (\text{the conductor of } g), \\ 1 & \text{otherwise}, \end{cases}$$

and $\Omega_{g,\chi\psi,r} \in \mathbb{C}^\times$ is a complex number such that

$$\frac{L(m, g, (\chi\psi\omega_F^{1-r})^{-1})}{\Omega_{g,\chi\psi,r}} \in \bar{\mathbb{Q}}.$$

We always regard an element in $\bar{\mathbb{Q}}$ as in $\mathbb{C}_p$ through the fixed embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Note that $\Omega_{g,\chi\psi,r}$ can be described as the product of *period* of f and some non-zero constants (see [14], Section 5.1).

4.4. Proof of the main theorems. Let $L_p(s, f, \psi)$ be the p-adic L-function of f with $\chi = 1$. Clearly, when $\psi(\mathfrak{p}) = \alpha_{\mathfrak{p}}$, we have trivial equality

$$L_p(1, f, \psi) = 0,$$

and we call it *exceptional zero*.

According to [15], Theorem 6.8, we can construct the *two-variable p-adic L-function* $L_p(s, k, \psi)$ such that

$$L_p(s, m, \psi) = L_p(s, f_m^{\#}, \psi)$$

for any $m \in U \cap \mathbb{Z}$ and $m \equiv 2 \bmod 2(p-1)$, where $U \in \mathbb{Z}_p$ is an open neighborhood of 2.

We prove the main theorem of this paper which is a generalization of the main results of [3] and [14].

Theorem 4.11. *Let ψ be a quadratic Hecke character of F of conductor prime to $\mathfrak{n}$. We assume that the following conditions are satisfied*:

$$\psi(\mathfrak{p}) = \alpha_{\mathfrak{p}},$$
$$\epsilon(f, \psi) = -1,$$

where $\alpha_{\mathfrak{p}}$ is the Hecke eigenvalue of $T(\mathfrak{p})$ and $\epsilon(f, \psi)$ is the sign of the functional equation of the complex L-function $L(s, f, \psi)$. Let A be the abelian variety of GL(2)-*type associated to f as in beginning of the previous section. Then we have*

(1) *There exists a global point $P_\psi \in A(F^\psi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $l \in \mathbb{Q}(f)^\times$ such that*

$$\frac{d^2}{dk^2} L_p(k/2, k, \psi)\bigg|_{k=2} = l \cdot (\log\mathrm{Norm}_p^A(P_\psi))^2,$$

where F^ϕ is a quadratic extension corresponding to ϕ.

(2) *The element P_ψ is of infinite order if and only if the derivative of the complex L-function is nonzero $L'(1, A/F, \psi) \neq 0$, where $L(s, A/F, \psi)$ is the L-function of A. In that case,*

$$\dim_{\mathbb{Q}(f)}(A(F^\psi) \otimes_{\mathbb{Z}} \mathbb{Q})_\psi = 1.$$

Proof. We can choose a quadratic Hecke character ψ' which is unramified at the primes dividing $\mathfrak{n}$ and the conductor is prime to that of ψ such that

$$L_p(1, 2, \psi') \in \mathbb{Q}(f)^\times$$

and the quadratic extension over F associated with $\psi\psi'$ is a CM extension with $\mathfrak{p}$ inert.

By [14], Propositions 5.1 and 5.3, there exists an open neighborhood $U \in \mathbb{Z}_p$ and a p-adic analytic function η on U such that

$$\begin{aligned}(34)\qquad &\mathcal{L}_p(\Phi, \psi_K, s)^2 = \eta(s+2)L_p((s+2)/2, s+2, \psi)L_p((s+2)/2, s+2, \psi'),\\ &\eta(2) \in \mathbb{Q}(f)^\times,\end{aligned}$$

where ψ_K is the Hecke character of K associated with the composition of the fields F_ψ and $F_{\psi'}$, which are the quadratic extensions over F associated with ψ and ψ'.

Since $L_p(k/2, k, \psi)$ has trivial zero and $\epsilon(f, \psi) = -1$, we see from the functional equation that the order of vanishing of $L_p(k/2, k, \psi)$ is at least 2:

$$L_p(1, 2, \psi) = L'_p(1, 2\psi) = 0.$$

Thus by calculating the second derivative at $s = 0$ of (34), we have

$$2\left(\frac{d}{ds}\mathcal{L}_p(\Phi, \psi_K, s)\bigg|_{s=0}\right)^2 = \eta(2)L_p(1, 2, \psi')\frac{d^2}{dk^2}L_p(k/2, k, \psi)\bigg|_{k=2}.$$

Put $l := 2\eta(2)^{-1}L_p(1, 2, \psi')^{-1} \in \mathbb{Q}(f)^\times$, we have

$$\frac{d^2}{dk^2}L_p(k/2, k, \psi)\bigg|_{k=2} = l\left(\frac{d}{ds}\mathcal{L}_p(\Phi, \psi_K, s)\bigg|_{s=0}\right)^2.$$

On the other hand, by Proposition 4.10, the right hand side is equal to

$$l\left(\sum_{i=1}^{h} \mathrm{logNorm}_p^{X_i}(\mathbf{P}_{\psi_K,i})(\mu_{\phi^i})\right)^2,$$

where $\mathbf{P}_{\psi_K}$ is a global point and belongs to the Jacobian $\bigoplus_{i=1}^h J(X_i)(K_{\psi_K})\otimes_{\mathbb{Z}}\mathbb{Q}$, where $\coprod_i X_i$ is the Shimura curve as in Section 4.1 (see Remark 4.8).

By [19] and the proof of [14], Corollary 4.2, we have

$$\bar{\mathbf{P}}_{\psi_K} \in A(F^\phi) \otimes_{\mathbb{Z}} \mathbb{Q},$$

where $\bar{\mathbf{P}}_{\psi_K}$ is the image of $\mathbf{P}_{\psi_K}$ by the projection from $\bigoplus_{i=1}^h J(X_i)(K_{\psi_K})$ to $A(K_{\psi_K})$. We have (1) by putting $P_\psi := \bar{\mathbf{P}}_{\psi_K}$.

For (2), by the results of Zhang ([20]), we have

$$P_\psi \text{ is of infinite order} \Leftrightarrow L'(1, f/K, \psi_K) \neq 0$$

where $L(s, f/K, \psi_K)$ is the Rankin–Selberg convolution L-function of f and ψ_K. On the other hand, we have

$$L(s, f/K, \psi_K) = L(s, f, \psi) \cdot L(s, f, \psi').$$

Thus, by the choice of ψ and ψ', we have

$$L'(1, f/K, \psi_K) \neq 0 \Leftrightarrow L'(1, f, \psi) \neq 0 \Leftrightarrow L'(1, A/F, \psi) \neq 0.$$

The second equivalence follows from [19], Theorem A. Therefore, we have the first assertion of (2). The second assertion of (2) is obtained by the theorem of Kolyvagin–Logachev [13]. □

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