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STRUCTURE OF HEREDITARY ORDERS OVER LOCAL RINGS

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Let R be a noetherian integral domain and K its quotient field, and Σ a semi-simple K-algebra with finite degree over K. If Λ is a subring in Σ which is finitely generated R-module and $\Lambda K=\Sigma$, then we call it an order. If Λ is a hereditary ring, we call it a hereditary order (briefly h-order).

This order was defined in [1], and the author has substantially studied properties of h-orders in [5], and shown that we may restrict ourselves to the case where R is a Dedekind domain, and Σ is a central simple K-algebra.

In this note, we shall obtain further results when R is a discrete rank one valuation ring. Let R be such a ring, and Ω a maximal order with radical $\mathfrak{N}$, and $\Omega/\mathfrak{N}=\Delta_n$; Δ division ring. Then we shall show the following results: 1) Every h-order contains minimal h-orders Λ such that $\Lambda/N(\Lambda)\approx\Sigma\oplus\Delta$, where $N(\Lambda)$ is the radical of Λ, (Section 3); 2) The length of maximal chains for h-order is equal to n, and we can decide all chains which pass a given h-order, (Section 5); 3) For two h-orders Γ_1 and Γ_2 they are isomorphic if and only if they are of same form, (see definition in Section 4); 4) The number of h-orders in a nonminimal h-order is finite if and only if $R/\mathfrak{p}$ is a finite field, where $\mathfrak{p}$ is a maximal ideal in R, (Section 6).

In order to obtain those results we shall use a fundamental property of maximal two-sided ideals in Λ; $\{\mathfrak{M}, \mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N}, \mathfrak{N}^{-2}\mathfrak{M}\mathfrak{N}^2, \cdots, \mathfrak{N}^{-r+1}\mathfrak{M}\mathfrak{N}^{r-1}\}$ gives a complete set of maximal two-sided ideals in Λ, where $\mathfrak{N}=N(\Lambda)$, (Section 2).

H. Higikata has also determined h-orders over local ring in [8] by direct computation and the author owes his suggestions to rewrite this paper, (Section 6). However, in this note we shall decide h-orders as a ring, namely by making use of properties of idempotent ideals and radical.

We only consider h-orders over local ring in this paper, except Section 1, and problems in the global case will be discussed in [7] and in a special case, where Σ is the field of quaternions, we will be discussed in [6].

1. Notations and preliminary lemmas.

Throughout this note, we shall always assume that R is a discrete rank one valuation ring and K is the quotient field of R, and that Λ, Γ, Ω are h-orders over R in a central simple K-algebra Σ.

For two orders Λ, Γ, the left Γ- and right Λ-module $C_\Lambda(\Gamma)=\{x|\in\Sigma, \Gamma x\subseteq\Lambda\}$ is called "*(right) conductor of* Γ *over* Λ". By [5], Theorem 1.7, we obtain a one-to-one correspondence between order $\Gamma\,(\supseteq\Lambda)$ and two-sided idempotent ideal $\mathfrak{A}$ in Λ as follows:

$$\Gamma = \mathrm{Hom}^r_\Lambda(\mathfrak{A}, \mathfrak{A}) \quad \text{and} \quad C_\Lambda(\Gamma) = \mathfrak{A}\,.$$

Furthermore, we have a one-to-one correspondence between two-sided idempotent ideals $\mathfrak{A}$ and two-sided ideals $\mathfrak{M}$ containing the radical $\mathfrak{N}$ of an order Λ by [5], Lemma 2.4:

$$\mathfrak{A}+\mathfrak{N} = \mathfrak{M}\,.$$

Let $\Lambda/\mathfrak{N}=\Lambda/\mathfrak{M}_1\oplus\cdots\oplus\Lambda/\mathfrak{M}_n$, where the $\mathfrak{M}_i$'s are maximal two-sided ideals in Λ. Then $\mathfrak{M}$ is written uniquely as an intersection of some $\mathfrak{M}_i$'s, say $\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \cdots, \mathfrak{M}_{i_r}$. We shall denote those relations by

$$\mathfrak{A} = I(\mathfrak{M}) = I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \cdots, \mathfrak{M}_{i_r})\,.$$

Let $\Lambda/\mathfrak{M}_i=(\Delta_i)_{ni}$; Δ_i division ring. Then by [5], Theorem 4.6, we know that the Δ_i's depend only on Σ, and we shall denote it by Δ. For any order Γ, we denote the radical of Γ by $N(\Gamma)$. Let $\Gamma\supseteq\Lambda$ be h-orders, and $C(\Gamma)=I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \cdots, \mathfrak{M}_{i_r})$. Then $C(\Gamma)/C(\Gamma)\mathfrak{N}\approx\Lambda/\mathfrak{M}_{j_1}\oplus\cdots\oplus\Lambda/\mathfrak{M}_{j_{n-r}}$ $\oplus C(\Gamma)\frown\mathfrak{N}/C(\Gamma)\mathfrak{N}$ as a right Λ-module; $(i_1, i_2, \cdots, i_r, j_1, j_2, \cdots, j_{n-r})\equiv(1, 2, \cdots, n)$. By [5], Theorem 4.6 and its proof, we have

LEMMA 1.1. $\Gamma/N(\Gamma)\approx\mathrm{Hom}^r_{\Lambda/\mathfrak{N}}(C(\Gamma)/C(\Gamma)\mathfrak{N}, C(\Gamma)/C(\Gamma)\mathfrak{N})$, *and every simple component of* $C(\Gamma)\frown\mathfrak{N}/C(\Gamma)\mathfrak{N}$ *appears in some* $\Lambda/\mathfrak{M}_{j_t}$, $t=1, \cdots, n-r$.

Let $\hat{R}$ be the completion of R with respect to the maximal ideal $\mathfrak{p}$ in R, and $\hat{K}$ its quotient field. Then $\hat{\Sigma}=\Sigma\otimes\hat{K}$ is also central simple $\hat{K}$-algebra and $\hat{\Lambda}=\Lambda\otimes\hat{R}$ is an order over $\hat{R}$ in $\hat{\Sigma}$. If Ω is a maximal order in Σ, then $\hat{\Omega}$ is also maximal in $\hat{\Sigma}$ by [1], Proposition 2.5. Let Γ' be any order in $\hat{\Omega}$, then we can find some n such that $\hat{\Omega}\mathfrak{p}^n\subseteq\Gamma'$. Since $\Omega/\mathfrak{p}^n\Omega\approx\hat{\Omega}/\mathfrak{p}^n\hat{\Omega}$ as a ring, there exists an order Γ in Ω such that $\hat{\Gamma}=\Gamma'$. Furthermore, since $\otimes\hat{R}$ is an exact functor, we have

PROPOSITION 1.1. *Let* Ω *be a maximal order in* Σ. *Then there is a* *one-to-one correspondence between orders* Γ *in* Ω *and order* $\hat{\Gamma}$ *in* $\hat{\Omega}$.

If Λ is an h-order then $\mathfrak{N}$ is Λ-projective, and hence, $\hat{\mathfrak{N}}$ is $\hat{\Lambda}$-projective. Therefore, by usual argument (cf. [2], p. 123, Exer. 11, and [5], Lemma 3.6), we have

COROLLARY. *By the above correspondence h-orders in* Ω *correspond to those in* $\hat{\Omega}$.

PROPOSITION 1.2. *Let* Λ, Γ, *and* Ω *be as above. If* $\Lambda=\alpha'\Gamma\alpha'^{-1}$ *for a unit* α' *in* $\hat{\Omega}$, *then* $\Lambda=\alpha\Gamma\alpha^{-1}$, *and* α *is unit in* Ω.

Proof. Since $\hat{\Omega}/\mathfrak{p}^n\hat{\Omega}\approx\Omega/\mathfrak{p}^n\Omega$ for some n, and $\mathfrak{p}^n\Omega$ is contained in $N(\Omega)$, it is clear.

From those propositions many results in h-orders over R are obtained from those in h-orders in the ring of matrices of maximal order $\mathfrak{O}$ in a division ring Δ' over a complete field. Furthermore, all h-orders in $\mathfrak{O}_n$ are decided by Higikata [8]. However, in this note, we shall discuss properties of h-orders as a hereditary ring, namely, by means of idempotent ideals and radical, except the following lemma and the last section.

Let $\mathfrak{O}$ be as above. Then $\mathfrak{O}$ contains a unique maximal ideal (π), and every left or right ideal is two-sided and is equal to (π^n) by [3], p. 100, Satz 12. In $\Sigma=\Delta'_2$, we know by [6] that $\Lambda=\{(a_{i,j})\mid\in\Sigma,\ a_{i,j}\in R$, and $a_{1,2}\in(\pi)\}$ is an h-order in Σ. Analogously, we have

LEMMA 1.2. *Let* $\Sigma=(\Delta')_n$. *Then* $\Lambda=\{(a_{i,j})\mid\in\Sigma,\ a_{i,j}\in\mathfrak{O},\ a_{i,j}\in(\pi)$ *for* $i<j\}$ *is an h-order in* Σ, *and there exist no h-orders under* Λ.

Proof. Let $\mathfrak{N}=\{(a_{i,j})\mid\in\Lambda,\ a_{i,i}\in(\pi)\}$. It is clear that $\mathfrak{N}$ is a two-sided ideal in Λ. Furthermore, we can easily check that $\mathfrak{N}/(\pi)$ is nilpotent, and $\Lambda/\mathfrak{N}\approx\Sigma\oplus\mathfrak{O}/(\pi)$. Hence, $\mathfrak{N}$ is the radical of Λ. Let $\mathfrak{N}^{-1}=\{(a_{i,j})\mid\in\Sigma$, $(a_{i,j})\mathfrak{N}\subseteq\Lambda\}$. From the definition of $\mathfrak{N}$, we have $\mathfrak{N}^{-1}\ni e_{i,i+1}$, where the $e_{i,j}$'s are matrix units in Σ. Since $\mathfrak{N}^{-1}\mathfrak{N}\ni e_{1,2}e_{2,1}+\cdots+e_{n-1,n}e_{n,n-1}+(1/\pi)e_{n,1}\pi e_{1,n}=1\in\mathfrak{N}\mathfrak{N}^{-1}$, $\mathfrak{N}^{-1}\mathfrak{N}=\mathfrak{N}\mathfrak{N}^{-1}=\Lambda$. Therefore, Λ is hereditary by [2], p. 132, Proposition 3.2, and [5], Lemma 3.6. Since $\Lambda/\mathfrak{N}=\Sigma\oplus\mathfrak{O}/(\pi)$, the second part is clear by [5], Theorem 4.6.

We shall call such an h-order Λ "*minimal h-order*", namely there exist no h-orders contained in Λ and $\Lambda/N(\Lambda)=\Sigma\oplus\Delta$.

THEOREM 1.1. *In the central simple K-algebra* Σ, *there exists always a minimal h-order.*

In Sections 3, and 4 we shall show that every h-order contains a minimal h-order, and all minimal h-orders are isomorphic.

Finally we shall consider the converse of [4], Theorem 7.2.

THEOREM 1.2. *Let R be a Dedekind domain and P a finite set of primes in R, and Ω a maximal order over R in Σ. For any given h-order $\Lambda(\mathfrak{p})$ in Ω_p, $\mathfrak{p}\in P$, there exists a unique h-order Λ in Ω such that $\Lambda_p=\Lambda(\mathfrak{p})$ for $\mathfrak{p}\in P$, and $\Lambda_q=\Omega_q$ for $\mathfrak{q}\notin P$.*

Proof. First, we assume $P=\{\mathfrak{p}\}$. By [4], Theorem 3.3, $\Lambda(\mathfrak{p})=\Omega_p\cap\Omega'_2\cap\cdots\cap\Omega'_t:\Omega'_i$ maximal order over R_p. Let $\mathfrak{C}'_i=C_{\Omega_p}(\Omega'_i)$, then $\Omega'_i=\mathrm{Hom}^r_{\Omega_p}(\mathfrak{C}'_i,\mathfrak{C}'_i)$ where $\Omega'_1=\Omega_p$. Furthermore, $\mathfrak{C}'_i\supseteqq p^n\Omega_p$. Let $\mathfrak{C}_i=\mathfrak{C}'_i\cap\Omega$, then $\mathfrak{C}_{i_p}=\mathfrak{C}'_i$, and $\mathfrak{C}_{i_q}=\Omega_q$ since $\mathfrak{C}_i\supseteqq\mathfrak{p}^n$. Put $\Omega_i=\mathrm{Hom}^r_\Omega(\mathfrak{C}_i,\mathfrak{C}_i)$ and $\Lambda=\bigcap\Omega_i$. Then $\Lambda_p=\bigcap\mathrm{Hom}^r_{\Omega_p}(\mathfrak{C}_{i_p},\mathfrak{C}_{i_p})=\bigcap\Omega'_i=\Lambda(\mathfrak{p})$, and $\Lambda_q=\bigcap\mathrm{Hom}^r_{\Omega_q}(\mathfrak{C}_{i_q},\mathfrak{C}_{i_q})=\Omega_q$ if $\mathfrak{p}\neq\mathfrak{q}$. Hence, Λ is a desired h-order. Let Λ^{q_i} be such an h-order as above for $\mathfrak{p}=\mathfrak{q}_i$. Then $\Lambda=\bigcap_i\Lambda^{q_i}$ has a property in the theorem.

By virtue of this theorem we shall study, in this paper, h-orders over a valuation ring.

2. Normal sequence.

Let Λ be an h-order and $\mathfrak{N}$ the radical of Λ. Let $\{\mathfrak{M}_i\}$; $i=1,\cdots,n$, be the set of maximal two-sided ideals in Λ. Since $\mathfrak{N}^{-1}\mathfrak{N}=\mathfrak{N}\mathfrak{N}^{-1}=\Lambda$ by [5], Theorem 6.1, $\mathfrak{A}\to\mathfrak{A}^{\mathfrak{N}}=\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{N}$ gives a one-to-one correspondence among two-sided ideals $\mathfrak{A}$ in Λ, which preserves inclusion by [5], Proposition 4.1.

THEOREM 2.1. *Let Λ be an h-order with radical $\mathfrak{N}$ such that $\Lambda/\mathfrak{N}\approx\Delta_{m_1}\oplus\Delta_{m_2}\oplus\cdots\oplus\Delta_{m_n}$. For any maximal two-sided ideal $\mathfrak{M}$ in Λ, $\{\mathfrak{M},\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N},\mathfrak{N}^{-2}\mathfrak{M}\mathfrak{N}^2,\cdots,\mathfrak{N}^{-n+1}\mathfrak{M}\mathfrak{N}^{n-1}\}$ gives a complete set of maximal two-sided ideals in Λ.*

Proof. We may assume that $\mathfrak{N}^{-i}\mathfrak{M}\mathfrak{N}^i=\mathfrak{M}$. If $i<n$, there exists an h-order Ω such that $C(\Omega)=I(\mathfrak{M},\mathfrak{N}^{-1}\mathfrak{M}\mathfrak{N},\cdots,\mathfrak{N}^{-i+1}\mathfrak{M}\mathfrak{N}^{i-1})$. Let $\mathfrak{C}=C(\Omega)$ and $\mathfrak{M}_j=\mathfrak{N}^{-j+1}\mathfrak{M}\mathfrak{N}^{j-1}$, $\mathfrak{M}_i=\mathfrak{M}$. $\mathfrak{N}^{-1}(\bigcap_{j=1}^{i}\mathfrak{M}_j)\mathfrak{N}=\bigcap_{j=1}^{i}\mathfrak{M}_j$, and $\mathfrak{N}^{-1}\mathfrak{C}\mathfrak{N}=\mathfrak{C}$ by the observation in Section 1. Since $\mathfrak{C}\cap\mathfrak{N}/\mathfrak{C}\mathfrak{N}=\mathfrak{C}\cap\mathfrak{N}/\mathfrak{N}\mathfrak{C}$, $\mathfrak{C}+\mathfrak{N}/\mathfrak{N}$ is contained in the annihilator of $\mathfrak{C}\cap\mathfrak{N}/\mathfrak{C}\mathfrak{N}$ on $\Lambda/\mathfrak{N}$. However, by Lemma 1.1 $\mathfrak{C}\cap\mathfrak{N}/\mathfrak{C}\mathfrak{N}$ contains only simple components which appear in $\mathfrak{C}+\mathfrak{N}/\mathfrak{N}\approx\Lambda/\mathfrak{M}_{j_1}\oplus\cdots\oplus\Lambda/\mathfrak{M}_{j_{n-i}}$ as a right Λ-module, which is a contradiction.

From this theorem we can find a sequence of maximal two-sided ideals $\{\mathfrak{M}_i\}_{i=1,\cdots,n}$ in Λ such that $\mathfrak{N}^{-1}\mathfrak{M}_i\mathfrak{N}=\mathfrak{M}_{i+1}$, $\mathfrak{M}_{n+1}=\mathfrak{M}_1$ for all i. We shall call such a sequence $\{\mathfrak{M}_i\}$ "a normal sequence".

LEMMA 2.1. *Let Λ be an h-order with radical $\mathfrak{N}$. If Ω is an order containing properly Λ, then $\mathfrak{N}^{-1}\Omega\mathfrak{N}$ contains Λ and is not equal to Ω.*

Proof. Let $\mathfrak{C}=C(\Omega)$. It is clear that $\mathfrak{N}^{-1}\Omega\mathfrak{N}$ is an order containing Λ, and that $C(\mathfrak{N}^{-1}\Omega\mathfrak{N})=\mathfrak{N}^{-1}\mathfrak{C}\mathfrak{N}$. Since $\mathfrak{C}\neq\Lambda$, $\mathfrak{N}^{-1}\mathfrak{C}\mathfrak{N}\neq\mathfrak{C}$ by Theorem 2.1 and the observation in Section 1.

PROPOSITION 2.1. *Let Λ, $\mathfrak{N}$ be as above. For a two-sided ideal $\mathfrak{A}$ in Λ $\mathfrak{A}$ is inversible*[1] *in Λ if and only if $\mathfrak{A}\mathfrak{N}=\mathfrak{N}\mathfrak{A}$.*

Proof. If $\mathfrak{A}$ is inversible, then $\mathfrak{A}=\mathfrak{N}^t$ by [5], Theorem 6.1, and hence $\mathfrak{A}\mathfrak{N}=\mathfrak{N}\mathfrak{A}$. Conversely, let $\mathfrak{A}\mathfrak{N}=\mathfrak{N}\mathfrak{A}$, and $\Omega=\mathrm{Hom}^r_\Lambda(\mathfrak{A},\mathfrak{A})=\mathrm{Hom}^r_\Lambda(\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{N},\mathfrak{N}^{-1}\mathfrak{A}\mathfrak{N})\supseteq\mathfrak{N}^{-1}\Omega\mathfrak{N}$. Since Ω, $\mathfrak{N}^{-1}\Omega\mathfrak{N}$ contain same number of maximal two-sided ideals, $\Omega=\mathfrak{N}^{-1}\Omega\mathfrak{N}$. Therefore, $\Omega=\Lambda$ by Lemma 2.1, and hence $\mathfrak{A}$ is inversible by [5], Section 2.

LEMMA 2.2. *Let Λ be an h-order, and $\{\mathfrak{M}_i\}$ $i=1,\cdots,n$ the complete set of maximal two-sided ideals and $\mathfrak{A}$ a two-sided ideal in Λ. If $\mathfrak{A}\mathfrak{M}_i=\mathfrak{M}_i\mathfrak{A}$ for all i, then $\mathfrak{A}$ is principal, i.e., $\mathfrak{A}=\alpha\Lambda=\Lambda\alpha$.*

Proof. Since $\mathfrak{N}=\bigcap\mathfrak{M}_i=\sum_{i_1,i_2,\cdots,i_n}\mathfrak{M}_{i_1}\mathfrak{M}_{i_2}\cdots\mathfrak{M}_{i_n}$, $\mathfrak{A}\mathfrak{N}=\mathfrak{N}\mathfrak{A}$. Hence $\mathfrak{A}$ is inversible by Proposition 2.1, and $\Lambda=\mathrm{Hom}^r_\Lambda(\mathfrak{A},\mathfrak{A})$. Since $\mathfrak{A}$ is Λ-projective, we have a two-sided Λ-epimorphism $\psi:\Lambda\to\mathrm{Hom}^r_{\Lambda/\mathfrak{M}_i}(\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i,\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i)\to 0$. Since $\psi^{-1}(0)\supseteq\mathfrak{M}_i$, we obtain $\Lambda/\mathfrak{M}_i\approx\mathrm{Hom}^r_{\Lambda/\mathfrak{M}_i}(\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i,\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i)$. Hence, $\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i\approx\Lambda/\mathfrak{M}_i$ as a right Λ-module. Since $\mathfrak{A}$ is inversible, $\mathfrak{A}/\mathfrak{A}\mathfrak{N}\approx\Sigma\oplus\mathfrak{A}/\mathfrak{A}\mathfrak{M}_i\approx\Lambda/\mathfrak{N}$ as a right Λ-module. Therefore, $\mathfrak{A}=\alpha\Lambda$, and $\Lambda=\mathrm{Hom}^r_\Lambda(\alpha\Lambda,\alpha\Lambda)=\alpha\Lambda\alpha^{-1}$.

In any h-order Λ, we have $N(\Lambda)^m=\mathfrak{p}\Lambda$ for some m, we call m "*the ramification index of* Λ", and Λ "*unramified*" if $m=1$.

THEOREM 2.2. *Let Λ be an h-order with radical $\mathfrak{N}$, and $\{\mathfrak{M}_i\}$ $i=1,\cdots,n$ the set of maximal two-sided ideals. Then $\mathfrak{N}^n$ is principal. For a two-sided ideal $\mathfrak{A}$, $\mathfrak{A}\mathfrak{M}_i=\mathfrak{M}_i\mathfrak{A}$ for all i if and only if $\mathfrak{A}=\mathfrak{N}^{nr}$ for some r. Let Ω be an order containing Λ, and s, t are ramification indices of Ω and Λ, respectively. Then $n|t$, and $t|sn$. Therefore, if Ω is unramified, then $n=t$, and $\mathfrak{A}\mathfrak{M}_i=\mathfrak{M}_i\mathfrak{A}$ for all i if and only if $\mathfrak{A}=\mathfrak{p}^l\Lambda$ for some l,* (cf. Proposition 6.2).

Proof. The first part is clear by Theorem 2.1 and Lemma 2.2. Let $\mathfrak{N}^n=\alpha\Lambda=\Lambda\alpha$. Since $\alpha^{-1}\mathfrak{M}_i\alpha=\mathfrak{M}_i$ for all i and $\mathfrak{C}=C(\Omega)=I(\mathfrak{M}_{i_1},\cdots,\mathfrak{M}_{i_r})$, $\alpha^{-1}\mathfrak{C}\alpha=\mathfrak{C}$. Therefore, $\Omega=\mathrm{Hom}^r_\Lambda(\mathfrak{C},\mathfrak{C})=\mathrm{Hom}^r_{\alpha^{-1}\Lambda\alpha}(\alpha^{-1}\mathfrak{C}\alpha,\alpha^{-1}\mathfrak{C}\alpha)=\alpha^{-1}\Omega\alpha$. Thus $\alpha\Omega=\Omega\alpha$ is an inversible two-sided ideal in Ω, and hence, $\alpha\Omega=N(\Omega)^l$ by [5], Theorem 6.1. It is clear by Theorem 2.1 that $n|t$.

1) We call $\mathfrak{A}$ inversible in Λ if $\mathfrak{A}\mathfrak{A}^{-1}=\mathfrak{A}^{-1}\mathfrak{A}=\Lambda$; $\mathfrak{A}^{-1}=\{x|\in\Sigma,\ \mathfrak{A}x\mathfrak{A}\subseteq\Lambda\}$.

Furthermore, $\mathfrak{N}^t=(\mathfrak{N}^n)^{t/n}=\alpha^{t/n}\Lambda=\mathfrak{p}\Lambda$. Therefore, $\alpha^{t/n}\Omega=N(\Omega)^{l\cdot(t/n)}=\mathfrak{p}\Omega$, and hence, $l\cdot(t/n)=s$.

As an analogy to Lemma 2.2,

PROPOSITION 2.2. *Let α be a non-zero divisor in Λ. If $\alpha^{-1}\mathfrak{M}\alpha$ is a maximal ideal in Λ for a maximal ideal $\mathfrak{M}$, then $\Lambda\alpha\Lambda$ is principal ideal in Λ.*

Proof. Let $\alpha^{-1}\mathfrak{M}\alpha=\mathfrak{M}'$, then $\mathfrak{M}\alpha=\alpha\mathfrak{M}'$, and $\alpha^{-1}\mathfrak{M}=\mathfrak{M}'\alpha^{-1}$. If we set $\mathfrak{A}=\Lambda\alpha\Lambda$, $\mathfrak{A}'=\Lambda\alpha^{-1}\Lambda$, then $\mathfrak{M}\mathfrak{A}=\mathfrak{A}\mathfrak{M}'$ and $\mathfrak{A}'\mathfrak{M}=\mathfrak{M}'\mathfrak{A}'$. Since $\mathfrak{M}\mathfrak{A}\mathfrak{A}'$ $=\mathfrak{M}\alpha\mathfrak{A}'=\alpha\mathfrak{M}'\mathfrak{A}'=\alpha\mathfrak{M}'\alpha^{-1}\Lambda=\mathfrak{M}$, $\mathfrak{A}\mathfrak{A}'\subseteqq \mathrm{Hom}^l_\Lambda(\mathfrak{M},\mathfrak{M})$. Similarly, we obtain $\mathfrak{A}\mathfrak{A}'\subseteqq\mathrm{Hom}^r_\Lambda(\mathfrak{M},\mathfrak{M})$. Therefore, $\mathfrak{A}\mathfrak{A}'\subseteqq\mathrm{Hom}^l_\Lambda(\mathfrak{M},\mathfrak{M})\frown\mathrm{Hom}^r_\Lambda(\mathfrak{M},\mathfrak{M})=\Lambda$ by [5], Corollary 1.9 and Theorem 3.3. It is clear that $\mathfrak{A}\mathfrak{A}'\geqq\Lambda$, and hence $\mathfrak{A}\mathfrak{A}'=\Lambda$. Since $\mathfrak{A}\alpha^{-1}\subseteqq\mathfrak{A}\mathfrak{A}'=\Lambda$, $\mathfrak{A}\leqq\Lambda\alpha$, which implies $\mathfrak{A}=\alpha\Lambda=\Lambda\alpha$.

Next, we shall consider normal sequences of h-orders Γ and Λ $(\leqq\Gamma)$. Before discussing that, we shall quote the following notations. Let $\{\mathfrak{M}_i\}$ $i=1,\cdots,n$ be the normal sequence of Λ. We divide $S=\{\mathfrak{M}_i\}$ to the subsets $S'_1,\cdots,S'_r$, such that $\bigcup_i S'_i=S$, $S'_i\frown S'_j=\phi$, and for any $\mathfrak{M}_l\in S'_i$, $M_t\in S'_j$, $l<t$ if $i<j$. Let $S'_i=\{\mathfrak{M}_{t_i},\mathfrak{M}_{t_i+1},\cdots,\mathfrak{M}_{t_i+m_i-1}\}$. $S_i=S'_i-\{\mathfrak{M}_{t_i+m_i-1}\}$. Then we call m_i the length of S_i or S'_i. Let Γ be h-order containing Λ. Then $C(\Gamma)=I(\mathfrak{M}_{i_1},\cdots,\mathfrak{M}_{i_l})$, and by the above definition, $C(\Gamma)$ corresponds uniquely to $S_1,\cdots,S_r$; for example if $C(\Gamma)=I(\mathfrak{M}_1,\mathfrak{M}_2,\mathfrak{M}_3,\mathfrak{M}_6)$, then $S_1=\{\mathfrak{M}_1,\mathfrak{M}_2,\mathfrak{M}_3\}$, $S_2=\phi$, $S_3=\{\mathfrak{M}_6\}$, $S_i=\phi$, for $i>3$. Let $\mathfrak{C}_i=I(S_1,S_2,\cdots,S_{i-1},S'_i\bigcup S_{i+1},\cdots,S_r)$. Then $\Omega_i=\mathrm{Hom}^r_\Lambda(\mathfrak{C}_i,\mathfrak{C}_i)$ is an order such that there exist no orders between Ω_i and Γ by [5], Theorem 3.3.

LEMMA 2.3. *Let Γ, Λ, $\mathfrak{C}_i$, and S_i be as above, then $\{\mathfrak{C}_i\Gamma\}$ $i=1,\cdots,r$ is the set of maximal two-sided ideals in Γ if Γ is not maximal.*

Proof. Since $\mathfrak{C}_i\Gamma=C_\Gamma(\Omega_i)$ by [5], Proposition 3.1, we may prove by [5], Theorem 1.7 that every maximal two-sided ideal $\mathfrak{L}$ in Γ is idempotent. Since $\mathfrak{L}\neq N(\Gamma)$, $\mathfrak{L}$ is not inversible, and hence, $\tau^l_\Gamma(\mathfrak{L})$[2] $=\mathfrak{L}$ by [5], Section 2. Therefore, $\mathfrak{L}$ is idempotent by [5], Lemma 1.5.

By Lemma 1.1, we obtain that $\mathfrak{C}/\mathfrak{C}\mathfrak{N}\approx\mathfrak{N}_1\oplus\mathfrak{N}_2\cdots\oplus\mathfrak{N}_r$ as a right Λ-module, where $\mathfrak{N}_i$ is a direct sum of simple components in $\Lambda/\mathfrak{M}_{t_i+m_i-1}$.

LEMMA 2.4. *Let Λ, Γ, $\mathfrak{C}_i$ and $\mathfrak{C}/\mathfrak{C}\mathfrak{N}$ be as above. Then by the isomorphism φ in Lemma 1.1: $\Gamma/N(\Gamma)\approx\mathrm{Hom}^r_{\Lambda/\mathfrak{N}}(\mathfrak{C}/\mathfrak{C}\mathfrak{N},\mathfrak{C}/\mathfrak{C}\mathfrak{N})$ the maximal ideal $\mathfrak{C}_i\Gamma/N(\Gamma)$ corresponds to $\mathrm{Hom}^r_{\Lambda/\mathfrak{N}}(\sum_j\mathfrak{N}_j,\sum_{i\neq j}\mathfrak{N}_j)$.*

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2) $\tau^l_\Gamma(\mathfrak{L})$ means the two-sided ideal in Γ generated images of f; $f\in\mathrm{Hom}^l_\Gamma(\mathfrak{L},\Gamma)$.

Proof. Since $\mathfrak{C}_i\Gamma/N(\Gamma)$ is a maximal two-sided ideal in $\Gamma/N(\Gamma)$, $\mathfrak{C}_i\Gamma/N(\Gamma)$ is characterized by the image of $\mathfrak{C}/\mathfrak{C}\mathfrak{N}$ by $\varphi(\mathfrak{C}_i\Gamma/N(\Gamma))$. $\mathfrak{C}/\mathfrak{C}\mathfrak{N}=\Lambda/\mathfrak{M}_{t_1+m_1-1}\oplus\cdots\oplus\Lambda/\mathfrak{M}_{t_r+m_r-1}\oplus\mathfrak{C}\cap\mathfrak{N}/\mathfrak{C}\mathfrak{N}$, and $\mathfrak{C}_i\Gamma(\mathfrak{C}/\mathfrak{C}\mathfrak{N})=\mathfrak{C}_i\mathfrak{C}+\mathfrak{C}\mathfrak{N}/\mathfrak{C}\mathfrak{N}=\mathfrak{C}_i+\mathfrak{C}\mathfrak{N}/\mathfrak{C}\mathfrak{N}\geqq\Lambda/\mathfrak{M}_{t_1+m_1-1}\oplus\cdots\overset{\overset{i}{\vee}\,3)}{\oplus}\cdots\oplus\cdots\Lambda/\mathfrak{M}_{r_r+m_r-1}$, which implies the lemma.

LEMMA 2.5. *Let Λ be an h-order with radical $\mathfrak{N}$ and normal sequence $\{\mathfrak{M}_i\}$ $i=1,\cdots,r$. Then $\mathfrak{M}_i/\mathfrak{M}_i\mathfrak{N}\approx\Lambda/\mathfrak{M}_1\oplus\cdots\overset{\overset{i}{\vee}}{\oplus}\cdots\oplus\cdots\Lambda/\mathfrak{M}_r\oplus\mathfrak{N}_{i+1}$ as a right Λ-module. Hence, $\Omega_i/N(\Omega_i)=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_{i-1}}\oplus\Delta_{m_i+m_{i+1}}\oplus\Delta_{m_{i+2}}\cdots\oplus\Delta_{m_r}$, where $\mathfrak{N}_{i+1}$ is a direct sum of m_i simple components of $\Lambda/\mathfrak{M}_{i+1}$, and $\Lambda/\mathfrak{M}_i=\Delta_{m_i}$, and $\Omega_i=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_i,\mathfrak{M}_i)$.*

Proof. We obtain similarly to the proof of Lemma 2.2 that $\Lambda/\mathfrak{M}_i\approx\mathrm{Hom}^r_{\Lambda/\mathfrak{M}_{i+1}}(\mathfrak{N}/\mathfrak{N}\mathfrak{M}_{i+1},\mathfrak{N}/\mathfrak{N}\mathfrak{M}_{i+1})$, since $\Lambda=\mathrm{Hom}^r_\Lambda(\mathfrak{N},\mathfrak{N})$ and $\mathfrak{M}_i\mathfrak{N}=\mathfrak{N}\mathfrak{M}_{i+1}$. Furtheremore, since $\mathfrak{M}_i=C(\mathrm{Hom}^r_\Lambda(\mathfrak{M}_i,\mathfrak{M}_i))$, and $\mathfrak{N}/\mathfrak{M}_i\mathfrak{N}=\mathfrak{N}/\mathfrak{N}\mathfrak{M}_{i+1}$, we have the lemma by Lemma 1.1.

COROLLARY. *Let Λ be an h-order with radical $\mathfrak{N}$ such that $\Lambda/\mathfrak{N}\approx\sum_{i=1}^r\Delta_{m_i}$, then $\sum_{i=1}^r m_i$ does not depend on Λ, and the length of maximal chain for h-orders in Σ does not exceed $n=\sum_{i=1}^r m_i$.*

Proof. Since, every maximal order is isomorphic, Σm_i does not depend on Λ. Since $n=\Sigma m_i\geqq r$, the second part is clear by [5], Theorem 3.3.

REMARK. We shall show that every length of maximal chain is equal to n in the following section.

Before proving one of the main theorems in this section we shall consider a special situation of Lemma 2.3. Let $\Gamma=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_1,\mathfrak{M}_1)$. Then $\mathfrak{C}_i=I(\mathfrak{M}_1,\mathfrak{M}_i)$.

LEMMA 2.6. *Let Γ, Λ and $\mathfrak{C}_i$ be as above. Then $\{\mathfrak{C}_i\Gamma\}$ $i=2,\cdots,r$ is the normal sequence in Γ.*

Proof. Let $\mathfrak{L}_i=\mathfrak{C}_i\Gamma$. Then $\Omega=\mathrm{Hom}^r_\Lambda(\mathfrak{C}_2,\mathfrak{C}_2)=\mathrm{Hom}^r_\Gamma(\mathfrak{L}_2,\mathfrak{L}_2)$. If Ω is maximal, then Γ contains only two maximal ideals, and hence, we have nothing to prove. Thus, we may assume $r\geqq4$. We denote $N(\Gamma)$, $N(\Omega)$, $N(\Lambda)$ by $\mathfrak{N}$, $\mathfrak{N}'$, $\mathfrak{N}''$, respectively. Let $\Gamma_1=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_2,\mathfrak{M}_2)\subseteq\Omega$. Then $\mathfrak{M}_2/\mathfrak{M}_2\mathfrak{N}''=\Lambda/\mathfrak{M}_1\oplus\Lambda/\mathfrak{M}_3\oplus\mathfrak{N}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r$ and $\mathfrak{C}_2+\mathfrak{N}''/\mathfrak{N}''=\Lambda/\mathfrak{M}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r$ and $\mathfrak{C}_2\Gamma_1/N(\Gamma_1)$ is a maximal two-sided ideal, we obtain $\mathfrak{C}_2+$

3) $\overset{i}{\vee}$ means that we omit ith component.

$\mathfrak{M}_2\mathfrak{N}''/\mathfrak{M}_2\mathfrak{N}''=\Lambda/\mathfrak{M}_3\oplus\mathfrak{N}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r$. We consider a natural right Λ-homomorphism $\varphi:\mathfrak{E}_2/\mathfrak{E}_2\mathfrak{N}''\to\mathfrak{M}_2/\mathfrak{M}_2\mathfrak{N}''$. Then $\varphi(\mathfrak{E}_2/\mathfrak{E}_2\mathfrak{N}'')=\mathfrak{E}_2+\mathfrak{M}_2\mathfrak{N}_2''/\mathfrak{M}_2\mathfrak{N}''=\Lambda/\mathfrak{M}_3\oplus\mathfrak{N}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r$. On the other hand $\mathfrak{E}_2/\mathfrak{E}_2\mathfrak{N}''\simeq\Lambda/\mathfrak{M}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r\oplus\mathfrak{E}_2\cap\mathfrak{N}''/\mathfrak{E}_2\mathfrak{N}''$. Hence, $\mathfrak{E}_2\cap\mathfrak{N}''/\mathfrak{E}_2\mathfrak{N}''$ contains a directsum $\mathfrak{N}_3'$ of simple components which appear in $\Lambda/\mathfrak{M}_3$. Let $\{\mathfrak{D}_i=I(\mathfrak{L}_2,\mathfrak{L}_i)\Omega\}$ $i=3,\cdots,r$ be the set of maximal ideals in Ω. Since $\Omega=\mathrm{Hom}^r_\Gamma(\mathfrak{L}_2,\mathfrak{L}_2)$, we obtain by Lemmas 2.4, 2.5, $\mathfrak{D}_i/\mathfrak{N}'\approx\Gamma/\mathfrak{L}_i\approx\Lambda/\mathfrak{M}_i$ as a ring for $i\geqslant 3$ except one k of indices i. However, we have shown that $\mathfrak{E}_2/\mathfrak{E}_2\mathfrak{N}''\supseteq\Lambda/\mathfrak{M}_3\oplus\mathfrak{N}_3'$, and hence, we know $k=3$. Therefore, by Lemma 2.5 we obtain $\mathfrak{N}^{-1}\mathfrak{L}_2\mathfrak{N}=\mathfrak{L}_3$. Similarly, we can prove $\mathfrak{N}^{-1}\mathfrak{L}_i\mathfrak{N}=\mathfrak{L}_{i+1}$ for $i\leq n-1$. Therefore, we have proved the lemma by Theorem 2.1.

Now, we can prove the following theorem.

THEOREM 2.3. *Let Λ be an h-order with normal sequence $\{\mathfrak{M}_i\}$ $i=1,\cdots,n$. Then for an order Γ corresponding to a sequence $\{S_i\}$ $i=1,\cdots,r$, $\{\mathfrak{E}_i\Gamma\}$ $i=1,\cdots,r$ is the normal sequence in Γ. Furthermore, $C(\Gamma)/C(\Gamma)\mathfrak{N}\approx\mathfrak{N}_1^{l_1}\oplus\mathfrak{N}_2^{l_2}\oplus\cdots\oplus\mathfrak{N}_r^{l_r}$.[4] Hence, $\Gamma/N(\Gamma)\approx\Delta_{l_1}\oplus\cdots\oplus\Delta_{l_r}$, where $\mathfrak{N}_i$ is a simple component in $\Lambda/\mathfrak{M}_{t_i+m_i-1}$, and $l_i=\sum_{j=t_i}^{t_i+m_i-1}s_i$, and $\Lambda/\mathfrak{M}_i=\Delta_{s_i}$, $\mathfrak{E}_i=I(S_1,\cdots,S_i',\cdots,S_r)\Gamma$.*

Proof. We shall prove the theorem by induction on the number r of maximal two-sided ideals in Γ. If $r=n$, then $\Lambda=\Gamma$. If $r=n-1$, then the theorem is true by Lemma 2.6. We assume $r<n-1$. Let Γ' be an order between Λ and Γ such that $C(\Gamma')=I\{\bar{S}_0,\bar{S}_1,\cdots,\bar{S}_r\}$, and $\{\bar{S}_0',\bar{S}_1\}=S_1$, $\bar{S}_i=S_i$ for $i\geqslant 2$. Then $\{I(\bar{S}_0,\cdots,\bar{S}_i',\cdots,\bar{S}_r)\Gamma'\}$ $i=0,\cdots,r$ is the normal sequence in Γ' by induction hypothesis. Let $\mathfrak{L}_i=I(\bar{S}_0,\cdots,\bar{S}_i',\cdots,\bar{S}_r)\Gamma'$. Since $\mathfrak{L}_0=C(\Gamma)\Gamma'$, $\Gamma=\mathrm{Hom}^r_{\Gamma'}(\mathfrak{L}_0,\mathfrak{L}_0)$. Therefore, by Lemma 2.6, $\{I_{\Gamma'}(\mathfrak{L}_0,\mathfrak{L}_i)\Gamma\}$ $i=1,\cdots,r$ is the normal sequence in Γ. Since $S_1=\{\bar{S}_0',\bar{S}_1\}$, $I_{\Gamma'}(\mathfrak{L}_0,\mathfrak{L}_i)\Gamma=I(S_1,\cdots,S_i',\cdots,S_r)\Gamma$. Furthermore, $\Gamma/N(\Gamma)\approx\Delta_{l_0'+l_1}\oplus\Delta_{l_2'}\cdots\oplus\Delta_{l_r'}$, where $\Gamma'/N(\Gamma')\approx\Delta_{l_0'}\oplus\Delta_{l_1'}\oplus\Delta_{l_2'}\oplus\cdots\oplus\Delta_{l_r'}$; $l_i'=l_i$ for $i\geqslant 2$. Since $\sum_{i=0}^{r}l_i'=\sum_{i=1}^{r}l_i$, $l_1=l_0'+l_1'$. Thus we have proved the second part by Lemma 2.4.

Let Λ be an h-order with $\{\mathfrak{M}_i\}$ $i=1,\cdots,r$. If $\Lambda/\mathfrak{M}_i=\Delta_{m_i}$, then $(m_1,\cdots,m_r)$ is uniquely determined by Λ up to cyclic permutation. We call it a *form of* Λ. Furthermore, we know that $(m_1,\cdots,m_r)$ is a nonzero integral solution of

$$(1)\qquad \sum_{i=1}^{r}X_i=n\,.$$

---

4) For any right Λ-module $\mathfrak{M}$, $\mathfrak{M}^t$-means a direct sum of t copies of $\mathfrak{M}$.

COROLLARY. *If* Λ *is a minimal h-order in* Σ *with normal sequence* $\{\mathfrak{M}_i\}$ $i=1, \cdots, n$ *then for any nonzero integral solution* $(m_1, \cdots, m_r)$ *of* (1) *there exists an h-order* Γ, *whose form is* $(m_1, \cdots, m_r)$.

Proof. We associate a solution $(m_1, \cdots, m_r)$ to a set $\{S_1', \cdots, S_r'\}$, $S_1' = \{\mathfrak{M}_{t_i}, \cdots, \mathfrak{M}_{t_i+m_i-1}\}$, where $t_i = m_1 + \cdots + m_{i-1}$, $m_0 = 1$. Then $\Gamma = \mathrm{Hom}_\Lambda^r(I(S_1, \cdots, S_r), I(S_1, \cdots, S_r))$ is a desired order by the theorem.

3. Minimal *h*-orders.

By Theorem 1.1, we know that there exist minimal h-orders Λ in the central simple K-algebra, namely $\Lambda/N(\Lambda) = \Delta \oplus \cdots \oplus \Delta$. In this section, we shall show that every h-order contains minimal h-orders.

LEMMA 3.1. *Let* Γ *be an h-order and* Λ, Λ' *be h-orders in* Γ *such that there exist no orders between* Γ *and* Λ, Λ', *respectively. If* $C_\Lambda(\Gamma)/\mathfrak{N} \approx C_{\Lambda'}(\Gamma)/\mathfrak{N}$, *then* Λ *is isomorphic to* Λ' *by an inner-automorphism of unit element in* Γ, *where* $\mathfrak{N} = N(\Gamma)$.

Proof. Let $\mathfrak{C} = C_\Lambda(\Gamma)$, $\mathfrak{C}' = C_{\Lambda'}(\Gamma)$. Since $\mathfrak{C}/\mathfrak{N} \approx \mathfrak{C}'/\mathfrak{N}$, there exists a unit element ε in Γ such that $\mathfrak{C} = C'\varepsilon = \varepsilon^{-1}C'\varepsilon$. $\Gamma' = \mathrm{Hom}_\Lambda^l(\mathfrak{C}, \mathfrak{C}) = \mathrm{Hom}_\Lambda^l(\varepsilon^{-1}\mathfrak{C}'\varepsilon, \varepsilon^{-1}\mathfrak{C}'\varepsilon) \supseteqq \varepsilon^{-1}\,\mathrm{Hom}_\Lambda^l(\mathfrak{C}', \mathfrak{C}')\varepsilon = \varepsilon^{-1}\Gamma''\varepsilon$, where $\Gamma'' = \mathrm{Hom}_\Lambda^l(\mathfrak{C}', \mathfrak{C}')$. On the other hand, by Theorem 2.3, we obtain that Γ' and Γ'' contains the same number of maximal two-sided ideals as those in Γ. Hence, $\Gamma' = \varepsilon^{-1}\Gamma''\varepsilon$ by [5], Theorem 3.3. Furthermore, $\Lambda = \Gamma \frown \Gamma' = \Gamma \frown \varepsilon^{-1}\Gamma''\varepsilon = \varepsilon^{-1}(\Gamma \frown \Gamma'') = \varepsilon^{-1}\Lambda'\varepsilon$.

LEMMA 3.2. *Let* $\Gamma \supseteqq \Lambda$ *be h-orders, then* $N(\Lambda) \supseteq N(\Gamma)$.

Proof. Let $\mathfrak{N} = N(\Lambda)$, and $\mathfrak{N}' = \mathfrak{N}(\Gamma)$. We may assume that there are no orders between Λ and Γ. Then $\mathfrak{C}_\Lambda(\Gamma) = \mathfrak{M}$ is a maximal two-sided ideal in Λ by Lemma 2.4. Hence, we obtain by Lemma 1.1 that $\mathfrak{N}'\mathfrak{N} \subseteq \mathfrak{N}'\mathfrak{M} \subseteq \mathfrak{M}\mathfrak{N} \subseteq \mathfrak{N}$. Therefore, $\mathfrak{N}' = \mathfrak{N}'\Lambda \subseteq \mathfrak{M}\Lambda = \mathfrak{M}$. For any maximal two-sided ideal $\mathfrak{M}' \neq \mathfrak{M}$ in Λ, we have $\mathfrak{N}' = \mathfrak{N}'(\mathfrak{M} + \mathfrak{M}') \subseteq \mathfrak{N} + \mathfrak{M}\mathfrak{M}' \subseteq \mathfrak{M}'$ since $\Lambda = \mathfrak{M} + \mathfrak{M}'$. Therefore, $\mathfrak{N}' \subseteq \bigcap \mathfrak{M} = \mathfrak{N}$.

THEOREM 3.1. *Every h-order contains minimal h-orders.*

Proof. We obtain a minimal h-order Λ by Theorem 1.1. Let Γ be h-order. Since every maximal order is isomorphic, we may assume Λ and Γ are contained in a maximal order. Let $\{\mathfrak{M}_i\}$ $i=1, \cdots, r$ be the normal sequence of Γ with form $(m_1, \cdots, m_r)$, and $\Omega = \mathrm{Hom}_\Gamma^r(\mathfrak{M}_1, \mathfrak{M}_1)$. We assume that $\Omega \supseteq \Lambda$. Let $\mathfrak{N} = N(\Omega)$, and $\mathfrak{N}' = N(\Gamma)$. Since $\mathfrak{N}' \supseteqq \mathfrak{N}$, $\mathfrak{M}_1 \supseteqq \mathfrak{N}$. Now, we consider a left ideal $\mathfrak{M}_1/\mathfrak{N}'$ in $\Omega/\mathfrak{N} = \mathrm{Hom}^r_{\Gamma/\mathfrak{N}}(\mathfrak{M}_1/\mathfrak{M}_1\mathfrak{N}'$,

$\mathfrak{M}_1/\mathfrak{M}_1\mathfrak{N}'$). Since $(\mathfrak{M}_i, \mathfrak{M}_j)=1$ if $i \neq j$, there exist m in $\mathfrak{M}_1$ and y in $\mathfrak{M}_2 \cdots \mathfrak{M}_r$ such that $1=m+y$, $m^2-m=m(m-1) \in \mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_r = \mathfrak{M}_1(\mathfrak{M}_1\mathfrak{M}_2 \cdots \mathfrak{M}_r) \subseteqq \mathfrak{M}_1\mathfrak{N}'$. Therefore, $\mathfrak{M}_1/\mathfrak{M}_1\mathfrak{N}' = m\Lambda + \mathfrak{M}_1\mathfrak{N}'/\mathfrak{M}_1\mathfrak{N}' \oplus \mathfrak{N}'/\mathfrak{M}_1\mathfrak{N}'$. It is clear that $\mathfrak{M}_1(\mathfrak{N}'/\mathfrak{M}_1\mathfrak{N}')=(0)$. Hence, $\mathfrak{M}_1/\mathfrak{N}=(\Omega/\mathfrak{N})m \approx \mathfrak{l}^{m_2} \oplus \Omega/\mathfrak{L}_3 \oplus \cdots \oplus \Omega/\mathfrak{L}_r$, where the $\mathfrak{L}_i$'s are maximal ideals in Ω, and $\mathfrak{l}$ is a simple component in $\Omega/\mathfrak{L}_2$. On the other hand, since Ω contains Λ, Ω contains an h-order Γ' with form $(m_1, \cdots, m_r)$ by Corollary to Theorem 2. 3, and $\Omega=\mathrm{Hom}_{\Lambda'}^r(\mathfrak{M}_1', \mathfrak{M}_1')$, and $\Gamma'/\mathfrak{M}_1=\Delta_{m_1}$. Therefore, $\mathfrak{M}_1/\mathfrak{N} \approx \mathfrak{M}_1'/\mathfrak{N}$ by the above observation. Hence, Γ is isomorphic to Γ' which contains Λ. We can prove the theorem by induction.

COROLLARY. *Every minimal h-order is isomorphic. If two minimal h-orders are contained in an order* Γ, *then this isormorphism is given by a unit element in* Γ.

Proof. In the above, we use the fact that any h-order is isomorphic to an order containing a fixed minimal h-order, which implies the first part of the corollary. The second part is clear from the proof of the theorem.

THEOREM 3. 2. *Let* Ω *be a maximal order such that* $\Omega/N(\Omega)=\Delta_n$. *Then every length of maximal chain for h-orders is equal to n.*

Proof. It is clear from Theorems 1. 1 and 3. 1.

4. Isomorphisms of *h*-orders.

In this section, we shall discuss isomorphisms over R among h-orders. For this purpose, we shall use the following definition. Let Γ_1, Γ_2 be h-orders containing an h-order Λ. If there exists an isomorphism θ of Γ_1 to Γ_2 such that $\theta(\Lambda)=\Lambda$, we call θ "*isomorphism over* Λ", and "Γ_1, Γ_2 *are isomorphic over* Λ". Let Λ be an h-order with normal sequence $\{\mathfrak{M}_i\}$ $i=1, \cdots, r$. Then we shall call that Λ is rth order, and the rank of Λ is r. 1st order is nothing but maximal order Ω, and nth order is minimal if $\Omega/N(\Omega)=\Delta_n$.

We have introduced an equation

$$\sum_{i=1}^{r} X_i = n \tag{1}$$

in Section 2. We shall only consider nonzero integral solutions of (1). Hence, by solution we mean always such solutions. We shall define a relation among solutions $(a_1, \cdots, a_r)$ as follows: $(a_1, \cdots, a_r) \equiv (a_1', \cdots, a_r')$ if they are only different by a cyclic permutation. We shall denote the number of classes of solutions by $\varphi(n, r)$. It is clear that $\varphi(n, r)=\varphi(n, n-r)$, and that $\varphi(n, 2)=[n/2]$, and $\varphi(p, r)=\binom{p}{r}\Big/p$, where p is prime and [ ] Gauss' number.

We note that every isomorphism is given by an inner-automorphism in Σ.

Let Λ be an h-order with radical $\mathfrak{N}$. If $\mathfrak{N}$ is principal, we call Λ "*a principal h-order*". Every maximal order and minimal order are principal.

THEOREM 4.1. *Let Λ be an h-order with form $(m_1, \cdots, m_r)$. Then Λ is principal if and only if $m_1=\cdots=m_r$, (cf. [9], Theorem 1).*

Proof. If $m_1=\cdots=m_r$, Λ is principal by the fact $\Lambda=\mathrm{Hom}^r_\Lambda(\mathfrak{N}, \mathfrak{N})=\mathrm{Hom}^l_\Lambda(\mathfrak{N}, \mathfrak{N})$ and by [5], Corollary 4.5. Conversely, if $N=\alpha\Lambda=\Lambda\alpha$, then $\alpha^{-1}(\Lambda/\mathfrak{M}_i)\alpha=\Lambda/\alpha^{-1}\mathfrak{M}_{i+1}\alpha$ by Theorem 2.1, and hence, $m_i=m_{i+1}$ for all i.

PROPOSITION 4.1. *Let Λ be an h-order with radical $\mathfrak{N}$, and Γ_1, Γ_2 orders containing Λ. If Γ_1, Γ_2 are isormorphic over Λ, then this isomorphism is given by an element in $\mathfrak{N}$. In this case $C(\Gamma_2)=\mathfrak{N}^{-t}C(\Gamma_i)\mathfrak{N}^t$ for some t.*

Proof. If $\beta^{-1}\Gamma_1\beta=\Gamma_2$, and $\beta\Lambda\beta^{-1}=\Lambda$ for $\beta\in\Sigma$, then we may assume that $\beta\in\Lambda$. Since $\beta\Lambda=\Lambda\beta$ is inversible two-sided ideal in Λ, $\beta\Lambda=\mathfrak{N}^t$ for some $t\geqslant 0$. It is clear that $C(\Gamma_2)=\beta^{-1}C(\Gamma_1)\beta=\mathfrak{N}^{-t}C(\Gamma_1)\mathfrak{N}^t$.

COROLLARY. *If Λ is principal, then Γ_1 and Γ_2 are isomorphic over Λ if and only if $\mathfrak{N}(\Gamma_1)=\mathfrak{N}^{-t}C(\Gamma_1)\mathfrak{N}^t$ for some t, where $\mathfrak{N}=\mathfrak{N}(\Lambda)$.*

THEOREM 4.2. *Let Λ be a principal h-order of a form $(\overbrace{s, \cdots, s}^{m})$. Then the following statements are true:*

1) *Γ_1, Γ_2 are isomorphic if and only if Γ_1, Γ_2 are isomorphic over Λ.*

2) *The number of classes of isomorphic $m-r$th orders containing Λ is equal to $\varphi(m, r)$.*

3) *Those isomorphisms are given by inner-automorphisms of α^i for some i, where $N(\Lambda)=\alpha\Lambda=\Lambda\alpha$.*

4) *Let Λ_1, Λ_2 be h-orders. Then Λ_1 and Λ_2 are isomorphic if and only if they are of same form.*

Proof. Let Γ_1 and Γ_2 be $m-r$th orders and $\mathfrak{C}_i=C(\Omega_i)$ $i=1, 2$. Let $\mathfrak{C}_1=I(\mathfrak{M}_{i_1}, \mathfrak{M}_{i_2}, \cdots, \mathfrak{M}_{i_r})$ $\mathfrak{C}_2=I(\mathfrak{M}_{j_1}, \mathfrak{M}_{j_2}, \cdots, \mathfrak{M}_{j_r})$, $i_1<i_2<\cdots<i_r$; $j_1<j_2<\cdots<j_r$, and $\{\mathfrak{M}_i\}$ $i=1, \cdots, m$ the normal sequence of Λ. If Γ_1 and Γ_2

are isomorphic over Λ, then $\mathfrak{C}_2=\alpha^{-t}\mathfrak{C}_1\alpha^t$ for some t by the above corollary. Furthermore, $\alpha^{-t}\mathfrak{M}_{i_l}\alpha^t=\mathfrak{M}_{(i_l+t)}$, where $(i_{l_1}+t)\equiv i_l+t \bmod m$, and $0<(i+t)\leq m$. Therefore, $((i_{l+1}+t), (i_{l_1+2}+t), \cdots, (i_{l_1+s}+t), (i_{l_2+1}+t), \cdots, (i_{l_2+(r-s)}+t))\equiv(j_1, j_2, \cdots, j_r)$. We shall associate the set $(j_1, j_2, \cdots, j_r)$ to a class of solution of (1) as follows: $x_1=j_2-j_1, \cdots, x_2=j_3-j_2, \cdots, x_{r-1}=j_r-j_{r-1}$, $x_r=j_1+m-j_r$. Then $(j_1, \cdots, j_r)$, and $(i_1, \cdots, i_r)$ correspond to the same class. Coversely, for any $m-r$th h-orders Γ_1 and Γ_2 if (j_l), (i_l) correspond to the same class, then there exists some t such that $((i_l+t))=(j_l)$. Hence, $\beta^{-1}\Gamma_1\beta=\Gamma_2$. Let $(x_1, \cdots, x_r)$ be any solution of (1). Let $\mathfrak{C}=I(\mathfrak{M}_1, \mathfrak{M}_{1+x_1}, \cdots, \mathfrak{M}_{1+x_1+\cdots+x_{r-1}})$, then $\Gamma=\mathrm{Hom}_\Lambda^r(\mathfrak{C}, \mathfrak{C})$ is an h-order containing Λ and Γ corresponds to $(x_1, \cdots, x_r)$ by the above mapping, which implies 2). Next, we shall consider rth order Γ_i $(i=1, 2)$ containing Λ. If Γ_1 and Γ_2 are isomorphic, then they are of same form $(st_1, st_2, \cdots, st_r)$. If we associate $(t_1, t_2, \cdots, t_r)$ to Γ_i, then Γ_1 and Γ_2 correspond to the same class of solution of (1) replacing n by m. Conversely, for any solution (t_j) of (1), we can find an order $\Gamma(\supseteqq\Lambda)$ of a form $(st_1, \cdots, st_r)$ by Theorem 2.3. Hence, the number of classes of isomorphic rth orders is equal to or larger than $\varphi(m, r)$. On the other hand, that number does not exceed the number of classes of isomorphic rth orders over Λ, which is equal to $\varphi(m, m-r)=\varphi(m, r)$ by 2). Therefore, we have proved 1). 3) is clear by 1) and Proposition 4.1. 4) is clear from the above and Theorem 3.1.

COROLLARY 4.1. *Let Γ_1 and Γ_2 be isomorphic over Λ, then they are isomorphic over any principal h-orders Λ' contained in Λ. In this case the form of Λ has a periodicity.*[5]

Proof. The first part is clear by the theorem, and the isomorphism is given by α^t, where $\mathfrak{N}=N(\Lambda')=\alpha\Lambda'$. Hence, $\alpha^{-t}\Lambda\alpha^t=\Lambda$, which means $C_{\Lambda'}(\Lambda)=\mathfrak{N}^{-t}C_{\Lambda'}(\Lambda)\mathfrak{N}^t$.

COROLLARY 4.2. *Let Γ_1 and Γ_2 be h-orders contained in an order Ω, and which are isomorphic, then this isomorphism is given by a unit element in Ω and an element α^t, where α is a generator of radical of minimal h-order contained in Γ_1.*

It is clear by Theorem 4.2 and Corollary to Theorem 3.1.

COROLLARY 4.3. *For principal h-orders Γ_1, Γ_2, the following statements are equivalent*:

5) If a form is the following type: $(m_1, m_2, \cdots, m_1, m_2, \cdots)$, then we call the form has a periodicity.

1) Γ_1 *and* Γ_2 *are isomorphic,*

2) $\Gamma_1/N(\Gamma_1)$ *and* $\Gamma_2/N(\Gamma_2)$ *are isomorphic,*

3) Γ_1 *and* Γ_2 *are of the same rank.*

REMARK. The above corollary is not true for any h-order. For instance, let $\{\mathfrak{M}_1, \mathfrak{M}_2, \cdots, \mathfrak{M}_6\}$ be the normal sequence of a minimal h-order Λ in K_6, and $\mathfrak{C}_1 = I(\mathfrak{M}_2, \mathfrak{M}_4, \mathfrak{M}_5)$, $\mathfrak{C}_2 = I(\mathfrak{M}_1, \mathfrak{M}_4, \mathfrak{M}_5)$. Then $\Gamma_1 = \mathrm{Hom}^r_\Lambda(\mathfrak{C}_1, \mathfrak{C}_1)$ and $\Gamma_2 = \mathrm{Hom}^r_\Lambda(\mathfrak{C}_2, \mathfrak{C}_2)$ have different form (1, 2, 3) and (2, 1, 3), but $\Gamma_2/N(\Gamma_1) \approx \Gamma_2/N(\Gamma_2)$.

COROLLARY 4. 4. *Let* Γ_1 *and* Γ_2 *be h-orders containing principal h-orders* Λ_1, *and* Λ_2 *such that there exist no orders between* Γ_i *and* Λ_i. *Then the statements in Corollary* 4. 3 *are true.*

Proof. Every Γ containing Λ which satesfies the condition of the corollary is isomorphic by Theorems 2. 3 and 4. 2. Hence, the corollary is true by Corollary 4. 2.

COROLLARY 4. 5. *Let n be the length of maximal chain for h-orders. If* $n \leqq 5$, 1) *and* 2) *in Corollary* 4. 3 *are equivalent for any orders. If* $n \leqslant 3$, 1), 2), *and* 3) *in Corollary* 4. 3. *are equivalent for any orders.*

We shall recall the definition of same type in [5], Section 4. If there exists a left Γ_1 and right Γ_2 ideal $\mathfrak{A}$ in Σ for two orders Γ_1 and Γ_2 such that $\Gamma_1 = \mathrm{Hom}^r_{\Gamma_2}(\mathfrak{A}, \mathfrak{A})$, and $\Gamma_2 = \mathrm{Hom}^l_{\Gamma_1}(\mathfrak{A}, \mathfrak{A})$, we call "$\Gamma_1$ *and* Γ_2 *belong to the same type*".

LEMMA 4. 1. *Let* Λ_1 *and* Λ_2 *be h-orders which belong to the same type, and* Ω_1, Ω_2 *containing* Λ_1, Λ_2, *respectively. Then* Ω_1, Ω_2 *belong to the same type if and only if* Ω_1 *and* Ω_2 *are of same rank.*

Proof. By the assumption, we have a left Λ_1 and right Λ_2 ideal $\mathfrak{A}$ such that $\Lambda_1 = \mathrm{Hom}^r_{\Lambda_2}(\mathfrak{A}, \mathfrak{A})$, $\Lambda_2 = \mathrm{Hom}^l_{\Lambda_1}(\mathfrak{A}, \mathfrak{A})$. Then $\mathfrak{A}\mathfrak{A}^{-1} = \Lambda_1$, $\mathfrak{A}^{-1}\mathfrak{A} = \Lambda_2$, and hence, $\mathfrak{A}^{-1}\Lambda_1\mathfrak{A} = \Lambda_2$, and $\mathfrak{A}\Lambda_2\mathfrak{A}^{-1} = \Lambda_1$ by [5], Section 4. Let $\mathfrak{C} = C_{\Lambda_1}(\Omega_1)$. Then $\Omega_1 = \mathrm{Hom}^r_{\Lambda_1}(\mathfrak{C}, \mathfrak{C})$. It is clear that $\Omega_1 = \mathrm{Hom}^r_{\Lambda_1}(\mathfrak{C}, \mathfrak{C}) = \mathrm{Hom}^r_{\mathfrak{A}^{-1}\Lambda_1\mathfrak{A}}(\mathfrak{C}\mathfrak{A}, \mathfrak{C}\mathfrak{A}) = \mathrm{Hom}^r_{\Lambda_2}(\mathfrak{C}\mathfrak{A}, \mathfrak{C}\mathfrak{A})$. Let $\Omega_2' = \mathrm{Hom}^l_{\Omega_1}(\mathfrak{C}\mathfrak{A}, \mathfrak{C}\mathfrak{A})$, then $\Omega_2' \geqq \Lambda_2$. Since Ω_1, Ω_2' belong to the same type, they are of same rank. Therefore, Ω_2, Ω_2' belong to the same type by [5], Theorem 4. 2. Hence, Ω_1 and Ω_2 belong to the same type.

The following theorem is a generalization of [5], Theorem 4. 3.

THEOREM 4. 3. *Let* Γ_1, Γ_2 *be orders in* Σ. *Then* Γ_1 *and* Γ_2 *belong to the same type if and only if* Γ_1 *and* Γ_2 *are of same rank.*

Proof. Let Λ_1, Λ_2 be minimal h-orders in Γ_1, Γ_2, respectively. Then $\Lambda_1=\mathcal{E}\Lambda_2\mathcal{E}^{-1}$ by Corollary to Theorem 3.1. Hence, $\Lambda_1=\mathrm{Hom}^r_{\Lambda_2}(\mathcal{E}\Lambda_2, \mathcal{E}\Lambda_2)$, and $\Lambda_2=\mathrm{Hom}^l_{\Lambda_1}(\Lambda_1\mathcal{E}, \Lambda_1\mathcal{E})$. Thus, we obtain the theorem by Lemma 4.1.

5. Chain of h-orders.

In this section, we shall study by making use of arguments in the proof of Theorem 3.1 how we can find maximal chains of h-orders which pass a given h-order Γ. We have already known by [5], Theorem 3.3 how we can construct chains of h-orders containing Γ, which is determined by the structure of $\Gamma/N(\Gamma)$.

First, we shall study a relation between left conductor $D(\ )$ and right conductor $C(\ )$.

THEOREM 5.1. *Let $\Gamma\supseteq\Lambda$ be h-orders. Then $C(\Gamma)=\mathfrak{N}D(\Gamma)\mathfrak{N}^{-1}$, where $\mathfrak{N}=N(\Lambda)$.*

Proof. Let $\{\mathfrak{M}_i\}$ $i=1, \cdots, r$ be the normal sequence in Λ, and let $\Gamma=\mathrm{Hom}^l_\Lambda(\mathfrak{M}_2, \mathfrak{M}_2)$, then $D(\Gamma)=\mathfrak{M}_2$. There exists some $\mathfrak{M}_i$ such that $\Gamma=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_i, \mathfrak{M}_i)$, and hence, $\{I(\mathfrak{M}_i, \mathfrak{M}_j)\Gamma\}$ $i\neq j$ is the normal sequence in Γ. Since $\mathfrak{M}_2/\mathfrak{N}\mathfrak{M}_2\approx\Lambda/\mathfrak{M}_1\oplus\Lambda/\mathfrak{M}_3\oplus\cdots\oplus\Lambda/\mathfrak{M}_r\oplus\mathfrak{L}$, where $\mathfrak{L}=\mathfrak{N}/\mathfrak{N}\mathfrak{M}_2$ is a direct sum of m_2 simple components which apper in $\Lambda/\mathfrak{M}_1$, $\mathfrak{M}_2I(\mathfrak{M}_i, \mathfrak{M}_{i+1})\Gamma$ $+\mathfrak{N}\mathfrak{M}_2/\mathfrak{N}\mathfrak{M}_2=\Lambda/\mathfrak{M}_1\oplus\Lambda/\mathfrak{M}_3\oplus\cdots\overset{\overset{i}{\vee}}{\oplus}\ \overset{\overset{i+1}{\vee}}{\oplus}\cdots\oplus\Lambda/\mathfrak{M}_n\oplus\mathfrak{N}I(\mathfrak{M}_i, \mathfrak{M}_{i+1})\Gamma/\mathfrak{N}\mathfrak{M}_2$. Hence, if $i\neq 1, n$, $\Gamma/I(\mathfrak{M}_i, \mathfrak{M}_{i+1})\Gamma\approx\Delta_{m_i}$ or $\Delta_{m_{i+1}}$ by Lemma 2.1. However, $\Gamma/I(\mathfrak{M}_i, \mathfrak{M}_{i+1})\Gamma=\Delta_{m_i+m_{i+1}}$ by Lemma 2.5, which is a contradiction. If $i=n$, then $\mathfrak{L}_2(I(\mathfrak{M}_1, \mathfrak{M}_n)\Gamma)=(0)$, and hence, $\mathfrak{M}_2I(\mathfrak{M}_1, \mathfrak{M}_n)\Gamma+\mathfrak{N}\mathfrak{M}_2/\mathfrak{N}\mathfrak{M}_2=\Lambda/\mathfrak{M}_3\otimes\cdots\otimes\Lambda/\mathfrak{M}_{n-1}$, which also contradicts the fact that $I(\mathfrak{M}_1, \mathfrak{M}_n)\Gamma$ is a maximal two-sided ideal. Let $\mathfrak{C}=I(\mathfrak{M}_2, \cdots, \mathfrak{M}_t)$ and $\mathfrak{D}=I(\mathfrak{M}_1, \cdots, \mathfrak{M}_{t-1})$, then $\mathfrak{C}=\mathfrak{N}^{-1}\mathfrak{D}\mathfrak{N}$. We assume that $\Gamma=\mathrm{Hom}^l_\Lambda(\mathfrak{C}, \mathfrak{C})=\mathrm{Hom}^r_\Lambda(\mathfrak{D}, \mathfrak{D})$. Then $\Omega=\mathrm{Hom}^l_\Lambda(I(\mathfrak{C}, \mathfrak{M}_{t+1}), I(\mathfrak{C}, \mathfrak{M}_{t+1}))=\mathrm{Hom}^l_\Gamma(\Gamma I(\mathfrak{C}, \mathfrak{M}_{t+1}), \Gamma I(\mathfrak{C}, \mathfrak{M}_{t+1}))=\mathrm{Hom}^r_\Gamma$ $(\Gamma I(\mathfrak{C}, \mathfrak{M}_1), \Gamma I(\mathfrak{C}, \mathfrak{M}_1))$ by the first part. Hence, $\Omega=\mathrm{Hom}^l_\Lambda(I(\mathfrak{C}, \mathfrak{M}_{t+1}), I(\mathfrak{C}, \mathfrak{M}_{t+1}))=\mathrm{Hom}^r_\Lambda(\mathfrak{N}I(\mathfrak{C}, \mathfrak{M}_{t+1})\mathfrak{N}^{-1}, \mathfrak{N}I(\mathfrak{C}, \mathfrak{M}_{t+1})\mathfrak{N}^{-1})$. Thus, we can prove by induction that for maximal orders $\Omega_i\supseteq\Lambda$, $\mathfrak{C}_i=C(\Omega_i)=\mathfrak{N}D(\Omega_i)\mathfrak{N}^{-1}$. Let $\Gamma=\bigwedge\Omega_i=\bigwedge\mathrm{Hom}^r_\Lambda(\mathfrak{C}_i, \mathfrak{C}_i)=\bigwedge\mathrm{Hom}^l_\Lambda(\mathfrak{N}^{-1}\mathfrak{C}_i\mathfrak{N}, \mathfrak{N}^{-1}\mathfrak{C}_i\mathfrak{N})=\mathrm{Hom}^l_\Lambda(\mathfrak{N}^{-1}C(\Gamma)\mathfrak{N}, \mathfrak{N}^{-1}C(\Gamma)\mathfrak{N})$, since $\mathfrak{C}(\Gamma)=\Sigma\mathfrak{C}_i$.

THEOREM 5.2. *Let Λ be a principal h-order and Γ an order containing Λ. Then every h-order containing Λ which is isomorphic to Γ is written as $T^t(\Gamma)$, where T is the following functor: for $\Omega\supseteq\Lambda$ $T(\Omega)=\mathrm{Hom}^l_\Lambda(C(\Omega), C(\Omega))$, and $T^r(\Omega)=T(T^{r-1}(\Omega))$.*

Proof. It is clear by Theorems 4.2 and 5.1, and Proposition 4.1.

We note that for two h-orders $\Lambda\supseteq\Gamma$, $C_{\Gamma}(\Lambda)\supseteq N(\Gamma)$ by Lemma 3.2.

LEMMA 5.3. *Let Γ be an rth order with radical $\mathfrak{N}$ and $\mathfrak{L}$ a left ideal containing $\mathfrak{N}$ in Γ such that $\mathfrak{L}/\mathfrak{N}\approx\Delta_{m_1}\otimes\cdots\otimes\Delta_{m_i}\otimes\mathfrak{l}\otimes\Delta_{m_{i+2}}\cdots\otimes\Delta_{m_r}$; $\mathfrak{l}$ a proper left ideal in Δ_{m_i}. Then $\Lambda=\mathrm{Hom}^l_{\Lambda}(\mathfrak{L},\mathfrak{L})\cap\mathrm{Hom}^r_{\Lambda}(\mathfrak{L},\mathfrak{L})=\Gamma\cap\mathrm{Hom}^l_{\Lambda}(\mathfrak{L},\mathfrak{L})$ is an $r+1$th h-order and $C(\Gamma)=\mathfrak{L}$. Hence, Λ is uniquely determined by the rank and conductor. Furthermore, every $r+1$th h-order in Γ is expressed as above.*

Proof. Since $\mathfrak{L}\Gamma=\Gamma$, $\tau^l_{\Omega}(\mathfrak{L})=\Gamma$. If we put $\Gamma'=\mathrm{Hom}^l_{\Lambda}(\mathfrak{L},\mathfrak{L})$, then $\Gamma=\mathrm{Hom}^r_{\Lambda'}(\mathfrak{L},\mathfrak{L})$ by [1], Theorem A 2. By the same argument in the proof of Theorem 3.1, we can find an $r+1$th h-order Λ' such that $C_{\Lambda'}(\Gamma)/\mathfrak{N}\approx\mathfrak{L}/\mathfrak{N}$. Hence, there exists a unit element ε in Γ such that $C_{\Lambda'}(\Gamma)=\mathfrak{L}\varepsilon$. Furthermore, $\Lambda'=\Gamma\cap\mathrm{Hom}^l_{\Lambda}(C_{\Lambda'}(\Gamma),\ C_{\Lambda'}(\Gamma))=\Gamma\cap\mathrm{Hom}^l_{\Lambda}(\mathfrak{L}\varepsilon,\ \mathfrak{L}\varepsilon)=\Gamma\cap\varepsilon^{-1}\Gamma'\varepsilon=\varepsilon^{-1}(\Gamma\cap\Gamma')\varepsilon$. Therefore, $\Lambda=\Gamma\cap\Gamma'$ is an $r+1$th h-order. Since $\varepsilon^{-1}\mathfrak{L}\varepsilon=C_{\Lambda'}(\Gamma)$, $\mathfrak{L}=C_{\Lambda}(\Gamma)$. If Λ' is an $r+1$th h-order $(\subseteq\Gamma)$ such that $C_{\Lambda'}(\Gamma)=\mathfrak{L}$. Then $\Lambda=\Gamma\cap\mathrm{Hom}^l_{\Gamma}(C_{\Lambda'}(\Gamma),\ C_{\Lambda'}(\Gamma))\supseteq\Lambda'$. Hence $\Lambda=\Lambda'$. The last part is clear.

Let Λ be an h-order of form $(m_1, m_2, \cdots, m_r)$; $\Lambda/N(\Lambda)=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_r}$, and $\mathfrak{L}_{i,j}$ a left ideal in Λ such that $\mathfrak{L}_{i,j}\supseteq\mathfrak{N}$, and $\mathfrak{L}_{i,j}/\mathfrak{N}=\Delta_{m_1}\oplus\cdots\oplus\mathfrak{l}_{i,j}\oplus\cdots\oplus\Delta_{m_r}$, $\mathfrak{l}_{i,j}$ a non-zero left ideal in Δ_{m_i}. We denote $\mathrm{Hom}^l_{\Lambda}(\mathfrak{L}_{i,j},\mathfrak{L}_{i,j})$ by $\Lambda(\mathfrak{L}_{i,j})$ and $\mathfrak{l}_{i,j}$ by $l(\mathfrak{L}_{i,j})$. Let $k(\mathfrak{l}_{i,j})$ be the length of composition series of $\mathfrak{l}_{i,j}$ as a left Λ-module.

THEOREM 5.3. *Let Λ, $\Lambda(\mathfrak{L}_{i,j})$ be as above. Then $\Gamma=\Lambda\cap\bigcap_{i=1,j=1}^{t,s(i)}\Lambda(\mathfrak{L}_{i,j})$ is an h-order if and only if $\{l(\mathfrak{L}_{i,j})\}_{j=1}^{s(i)}$ is linearly ordered by inclusion for all i. Every $r+s(i)$th h-order in Λ is uniquely written as above.*

Proof. We assume that Γ is an h-order and Λ_0 is a minimal h-order in Γ. Let $S_i=\{\mathfrak{M}_{t_i},\mathfrak{M}_{t_i+1},\cdots,\mathfrak{M}_{t_i+m_i-1}\}$ be a set of maximal two-sided ideals in Λ_0 such that $C_{\Lambda_0}(\Lambda)=I(S_1,S_2,\cdots,S_r)$, (cf. Section 2). We denote $\Lambda\cap\Lambda(\mathfrak{L}_{i,j})$ by Γ_j. Since Γ_j is an $r+1$th order from Lemma 2.5 we obtain $C_{\Lambda_0}(\Gamma_j)=I(S_1,\cdots,S_{i-1},S_i^*,\cdots,S_r)$; $S_i^*=S_i-\{\mathfrak{M}_{\rho(j)}\}$. We assume $\rho(j_1)<\rho(j_2)$. Let $\bar{S}_i=S_i-\{\mathfrak{M}_{\rho(j_1)},\mathfrak{M}_{\rho(j_2)}\}$, $\mathfrak{C}=I(S_1,\cdots,S_{i-1},\bar{S}_i,S_{i+1},\cdots,S_r)$. Then $\Gamma'=\mathrm{Hom}^r_{\Lambda}(\mathfrak{C},\mathfrak{C})$ is an $r+2$th h-order and $\Gamma'=\Gamma_{j_1}\cap\Gamma_{j_2}$, $\Lambda=\Gamma_{j_1}\cup\Gamma_{j_2}$. Let $\mathfrak{N}_1=I(S_1,\cdots,S_{i-1},S_i-\{\mathfrak{M}_{\rho(j_2)}\},\cdots,S_r)\Gamma'$ and $\mathfrak{N}_2=I(S_1,\cdots,S_i-\{\mathfrak{M}_{\rho(j_1)}\},\cdots,S_r)\Gamma'$, then we obtain a normal sequence $\{\mathfrak{N}_1,\mathfrak{N}_2,\mathfrak{N}_3,\cdots\}$ in Γ' by Theorem 2.3, and $C_{\Gamma'}(\Gamma_{j_1})=\mathfrak{N}_2$, $C_{\Gamma'}(\Gamma_{j_2})=\mathfrak{N}_1$. Since $C_{\Gamma'}(\Lambda)=I(\mathfrak{N}_1,\mathfrak{N}_2)$, $C_{\Gamma'}(\Lambda)/N(\Lambda)=\Delta_{m_1}\oplus\cdots\oplus\Delta_{m_{i-1}}\oplus\mathfrak{l}^k\oplus\Delta_{m_{i+1}}\oplus\cdots$ by the usual argument in Sections 2 and 3, where $\Gamma'/\mathfrak{N}_3=\Delta_k$, and $\mathfrak{l}$ is a simple left ideal in Δ_{m_i}. On the other hand, since $\mathfrak{L}_{i,j_2}=C_{\Gamma_{j_2}}(\Lambda)=I(\mathfrak{N}_1,\mathfrak{N}_2)\Gamma_{j_2}$, and $\{I(\mathfrak{N}_1,\mathfrak{N}_2)\Gamma_{j_2},$

$I(\mathfrak{N}_1, \mathfrak{N}_3)\Gamma_{j_2}, \cdots\}$ is a normal sequence in Γ_{j_2}, we obtain $\mathfrak{L}_{i,j_2}/\mathfrak{N}(\Lambda)=\Delta_{m_1}\oplus \cdots \overset{\prime}{\oplus}\Delta_{m_{i-1}}\oplus \mathfrak{l}^k \oplus \Delta_{m_{i+1}}\oplus \cdots \approx C_{\Gamma'}(\Lambda)/N(\Lambda)$. However, $\mathfrak{L}_{i,j_2}\supseteqq C_{\Gamma'}(\Lambda)$, and hence $\mathfrak{L}_{i,j_2}=C_{\Gamma'}(\Lambda)\subsetneqq \mathfrak{L}_{i,j_1}$. Thus we have proved that $\{l(\mathfrak{L}_{i,j})\}_j$ is linearly ordered for any i. Conversely, we assume that $\{l(\mathfrak{L}_{i,j})\}_j$ is linearly ordered for all i, and $k(\mathfrak{l}_{i,1})>k(\mathfrak{l}_{i,2})>k(\mathfrak{l}_{i,s(i)})$. Let Λ_0 be a minimal order in Λ and $\{S_i\}$ be as above. If we denote $I(S_1, \cdots, S_{i-1}, S_i-M_{t_i+m_i-k(l_{i,j})}, S_{i+1}, \cdots)$ by $\mathfrak{C}_{i,j}$, then $\Gamma'_{i,j}=\mathrm{Hom}^r_{\Lambda_0}(\mathfrak{C}_{i,j}, \mathfrak{C}_{i,j})$ is an $r+1$th order in Λ and $\mathfrak{L}'_{i,j}=C_{\Gamma'_{i,j}}(\Lambda)\approx\mathfrak{L}_{i,j}$. Furthermore, we know by the above argument that $\{l(\mathfrak{L}_{i,j})\}_j$ is linearly ordered. Therefore, there exists a unit element ε in Λ such that $\mathfrak{L}_{i,j}=\mathfrak{L}'_{i,j}\varepsilon$ for all i, j. Hence $\Gamma=\Lambda_\frown\bigcap_{i,j}\Lambda(\mathfrak{L}_{i,j})=\Lambda_\frown\bigcap_{i,j}\varepsilon^{-1}\Lambda(\mathfrak{L}'_{i,j})\varepsilon$ $=\varepsilon^{-1}(\Lambda_\frown\bigcap_{i,j}\Lambda(\mathfrak{L}'_{i,j}))\varepsilon$ is an h-order containing $\varepsilon^{-1}\Lambda_0\varepsilon$. The second part is clear from the proof.

From the above proof we have

COROLLARY 5.1. *Let* $\Gamma=\Lambda_\frown\bigcap_{j,i}\Lambda(\mathfrak{L}_{i,j})$, *and* $k(i,j)=k(l(\mathfrak{L}_{i,j}))$. *If* $k_{i,j}>k_{i,j'}$, *for* $j<j'$, Γ *is of a form* $(m_1-k_{1,1}, k_{1,1}-k_{1,2}, \cdots, k_{1,s(1)}, \cdots, m_i-k_{i,1}, k_{i,1}-k_{i,2}, \cdots, k_{i,s(i)}, \cdots)$.

COROLLARY 5.2. *Let* $\{\Omega_i\}_{i=1}^n$ *be h-orders. Then* $\bigcap_i\Omega_i$ *is an h-order if and only if intersection of any two of the* Ω_i's *is an h-order.*

Proof. Since every h-order is written as an intersection of maximal orders, we may assume that the Ω_i's are maximal. If $\Omega_1\frown\Omega_i$ is an h-order, then $\Omega_i=\mathrm{Hom}^l_{\Omega_1}(\mathfrak{L}_i, \mathfrak{L}_i)$, for a left ideal $\mathfrak{L}_i$ $(\supset N(\Omega_1))$ in Ω_1. Let $\mathfrak{L}_i+\mathfrak{L}_j=\mathfrak{L}$. Then $\Omega_i\frown\Omega_j\subseteqq\mathrm{Hom}^l_{\Omega_1}(\mathfrak{L}, \mathfrak{L})$. Hence Ω_i or Ω_j is equal to $\mathrm{Hom}^l_{\Omega_1}(\mathfrak{L}, \mathfrak{L})$ by [5], Theorem 3.3. Therefore, $\mathfrak{L}=\mathfrak{L}_i$ or $\mathfrak{L}_j$ which shows that $\{\mathfrak{L}_i\}$ is linearly ordered. Hence $\bigcap\Omega_i$ is an h-order by the theorem. Converse is clear by [5], Corollary 1.4.

PROPOSITION 5.1. *Let* Λ *be an h-order and* $\mathfrak{L}$ *a left ideal containing* $N(\Lambda)$ *such that* $\mathfrak{L}\Lambda=\Lambda$. *Then* $\Gamma=\Lambda_\frown\mathrm{Hom}^l_\Lambda(\mathfrak{L}, \mathfrak{L})$ *is a unique maximal order among orders* Γ' *in* Λ *such that* $C_{\Gamma'}(\Lambda)=\mathfrak{L}$. *Hence* $\mathfrak{L}$ *is idempotent.*

Proof. Let $\mathfrak{L}=\bigcap_i\mathfrak{L}_i$; $\mathfrak{L}_i/\mathfrak{N}=\Delta_{m_1}\oplus\cdots\oplus\mathfrak{l}_i\oplus\cdots\oplus\Delta_{m_r}$. Then $\Gamma=\Lambda_\frown\bigcap_i\Lambda(\mathfrak{L}_i)$. Hence, $C_\Gamma(\Lambda)\subseteq\bigcap C_{\Lambda(\mathfrak{L}_i)}(\Lambda)=\bigcap\mathfrak{L}_i=\mathfrak{L}$. It is clear that $C_\Gamma(\Lambda)\supseteqq\mathfrak{L}$. If $C_{\Gamma'}(\Lambda)=\mathfrak{L}$ for an h-order $\Gamma'\subseteq\Lambda$. Then $\Gamma'\subseteq\Lambda_\frown\mathrm{Hom}^l_\Lambda(\mathfrak{L}, \mathfrak{L})=\Gamma$, since $C_{\Gamma'}(\Lambda)$ is a two-sided ideal in Γ'.

COROLLARY 5.3. *Let* $\Gamma=\Lambda_\frown\bigcap_{i,j}\Lambda(\mathfrak{L}_{i,j})$, *then* $C_\Gamma(\Lambda)=\bigcap_{i,j}\mathfrak{L}_{i,j}$.

Proof. Let $C_\Gamma(\Lambda)=\bigcap\mathfrak{L}_i$, where the $\mathfrak{L}_i$'s are as in the proof of

Corollary 5.2. $\Gamma' = \Lambda \cap \mathrm{Hom}^l_\Lambda(C_\Gamma(\Lambda), C_\Gamma(\Lambda)) \supseteqq \Gamma$ and $\Gamma' = \Lambda \cap \bigcap_i \Lambda(\mathfrak{L}_i)$. Since $\Lambda(\mathfrak{L}_i) \supset \Gamma$, $\mathfrak{L}_i = \mathfrak{L}_{k,j}$ for some k, j. Hence $C_\Gamma(\Lambda) = \bigcap \mathfrak{L}_{i,j}$.

PROPOSITION 5.2. *Let Λ be a principal h-order and $\mathfrak{L}$ a left ideal in Λ. Then $\mathfrak{L}$ is principal if and only if $\tau^l_\Lambda(\mathfrak{L}) = \Lambda$ and $\Lambda(\mathfrak{L})$ is principal.*

Proof. If $\mathfrak{L} = \Lambda\alpha$, then $\Lambda(\mathfrak{L}) = \alpha^{-1}\Lambda\alpha$, and hence $\Lambda(\mathfrak{L})$ is principal, and $\tau^l_\Lambda(\mathfrak{L}) = \mathfrak{L}\mathfrak{L}^{-1} = \Lambda\alpha\alpha^{-1}\Lambda = \Lambda$. If $\tau^l_\Lambda(\mathfrak{L}) = \Lambda$, $\Lambda = \mathrm{Hom}^r_{\Lambda(\mathfrak{L})}(\mathfrak{L}, \mathfrak{L})$. Furthermore if $\Lambda(\mathfrak{L})$ is principal, Λ and $\Lambda(\mathfrak{L})$ have the same form, and hence $\mathfrak{L}$ is principal by [5], Corollary 4.5.

We shall discuss further properties of one-sided ideals in the forthcoming paper [7].

PROPOSITION 5.3. *For any rth order Γ, there exist $n-r+1$ minimal h-orders Λ_i such that $\Gamma = \bigcup \Lambda_i$, where n is the length of maximal chain for h-orders in Σ.*

Proof. We prove the proposition by induction on rank r of orders. If $r = n$, then Γ is minimal. If Γ is an rth order $(r < n)$, then $\Gamma/N(\Gamma) = \Delta_{m_1} \oplus \cdots \oplus \Delta_{m_r}$, and $m_i > 1$ for some i. Therefore, there exist two distinct left ideals $\mathfrak{L}_1$ and $\mathfrak{L}_2$ in Γ by Theorem 5.3 such that $L_1 = C_{\Omega_1}(\Gamma)$, and $C_{\Omega_2}(\Gamma) = \mathfrak{L}_2$ for some $r+1$th orders Ω_1 and Ω_2. Since $\Omega_1 \neq \Omega_2$, $\Gamma = \Omega_1 \cup \Omega_2$. By induction hypothesis we obtain that $\Omega_i = \bigcup_{j=1}^{n-r} \Lambda_{i,j}$, where the $\Lambda_{i,j}$'s are minimal h-orders. Since $\Omega_1 \neq \Omega_2$, there exists $\Lambda_{2,j} \not\subseteq \Omega_1$. Hence $\Gamma = \Omega_1 \cup \Lambda_{2,j} = \bigcup_{i=1}^{n-r+1} \Lambda_i$.

6. Numbers of *h*-orders.

We shall count numbers of h-orders in an h-order.

LEMMA 6.1. *Let $\Gamma \supseteq \Lambda$ be h-orders and ε a unit in Γ. If $\varepsilon^{-1}\Lambda\varepsilon = \Lambda$ then $\varepsilon \in \Lambda$.*

Proof. Since $\varepsilon\Lambda = \Lambda\varepsilon$ is a two-sided inversible ideal with respect to Λ in Σ, $\Lambda\varepsilon = \mathfrak{N}^\rho$ by [5], Theorem 6.1, where $\mathfrak{N} = N(\Lambda)$. Let $\mathfrak{N}^t = \mathfrak{p}\Lambda$, then $\Lambda\varepsilon^t = \mathfrak{N}^{t\rho} = \mathfrak{p}^\rho\Lambda$. Hence, $\varepsilon^{-t}\mathfrak{p}^\rho$ is a unit in Λ, and hence in Γ. Therefore, $\rho = 0$, which implies $\Lambda\varepsilon = \Lambda$.

PROPOSITION 6.1. *Let Ω be an h-order. If Γ_1 and Γ_2 are isomorphic by an inner-automorpism in Ω for $\Gamma_i \subseteq \Omega$ $(i = 1, 2)$, and $\Gamma_1 \neq \Gamma_2$, then $\Gamma_1 \cap \Gamma_2$ is not an h-order.*

Proof. If $\Gamma_1 \cap \Gamma_2$ is h-order, there exists a minimal h-order Λ in

Γ_1 and Γ_2. Since Γ_1 and Γ_2 are isomorphic by an inner-automorphism in Ω, they are isomorphic over Λ by Theorem 4.2. Hence, $\varepsilon\Lambda\varepsilon^{-1}=\Lambda$. Therefore, ε is a unit in Λ, and in Γ_i, which is a contradiction to the fact $\Gamma_1 \neq \Gamma_2$.

COROLLARY 6.1. *Let Ω be a maximal order and Γ_1, Γ_2 nonmaximal distinct principal h-orders of same rank in Ω, then $\Gamma_1 \cap \Gamma_2$ is not an h-order.*

Proof. Let Λ_1 and Λ_2 be minimal h-orders contained in Γ_1 and Γ_2, respectively. Then $\Lambda_2=\varepsilon^{-1}\Lambda_1\varepsilon$; ε unit in Ω by Corollary to Theorem 3.1. However, by Theorems 2.3 and 4.1, $\Gamma_2=\varepsilon^{-1}\Gamma_1\varepsilon$.

COROLLARY 6.2. *Let Ω be an h-order, and $\{\Gamma_i\}$ the set of r th h-orders between Ω and a fixed minimal h-order Λ in Ω. Then every r th order in Ω is isomorphic by inner-automorphism in Ω to some Γ_i, and those isomorphic classes by units in Ω do not meet each other.*

It is clear by the proof of Theorem 3.1 and the proposition.

THEOREM 6.1. *The following conditions are equivalent*:

1) *The number of h-orders in a maximal order is finite,*
2) *The number of h-orders in a nonminimal h-order is finite.*
3) *$R/\mathfrak{p}$ is a finite field.*

To prove this we use the following elementary property.

LEMMA 6.2. *Let $B=\Delta_n$ be a simple ring and $L=Be_{1,1}\oplus\cdots\oplus Be_{r,r}$, then for any unit element ε in B $L\varepsilon=L$ if and only if*

$$\varepsilon = \begin{array}{c} \\ r \end{array}\!\!\begin{array}{c} \begin{array}{cc} r & \end{array} \\ \left(\begin{array}{c|c} \varepsilon_1 & 0 \\ \hline C & \varepsilon_2 \end{array}\right) \end{array}$$

ε_1, ε_2 are units in Δ_r and Δ_{n-r}, and C is an arbitrary element in $(n-r)\times r$ matrices over Δ.

Proof of Theorem 6.1. Let Γ be a nonminimal rth h-order. By Theorem 5.3 $r+1$th h-orders contained in Γ correspond uniquely to left ideals $\mathfrak{L}_i$; $\mathfrak{L}_i/N(\Gamma)=\Delta_{m_1}\oplus\cdots\oplus\mathfrak{l}_i\oplus\cdots\oplus\Delta_{m_r}$. Hence, the number of $r+1$th h-orders in Γ is equal to the number of those left ideals. The number of left ideals in $\Gamma/N(\Gamma)$ which are isomorphic to $\mathfrak{L}_i/N(\Gamma)$ is equal to $[(\Gamma/N(\Gamma))^*:1]/[E(\mathfrak{L}_i):1]$, where * means the group of units and $E(\mathfrak{L}_i)=\{\varepsilon \mid \in(\Gamma/N(\Gamma))^*, (\mathfrak{L}_i/N(\Gamma))\varepsilon\subseteqq\mathfrak{L}_i/N(\Gamma)\}$. Since $[\Delta:R/\mathfrak{p}]<\infty$, $[(\Gamma/N(\Gamma))^*:1]/[E(\mathfrak{L}_i):1]<\infty$ if and only if $[R/\mathfrak{p}:1]<\infty$ by Lemma 6.1. Thus, we obtain 2)$\Leftrightarrow$3). Since the length of maximal chain is finite, we have 1) $\Leftrightarrow$2).

If we want to count the number of h-orders in Γ, we may use the argument in the proof of Theorem 6.1. However, it is complicated a little. By virtue of Corollary 6.2, we may fix a minimal h-order in Λ. From this point, we shall study the numbers of h-orders in the special case as follows.

In Section 1, we have noted that we may restrict R to the case of a complete, discrete valuation ring. By $\wedge$ we mean completion with respect to the maximal ideal $\mathfrak{p}$ in R. Let Ω be a maximal order with radical $\mathfrak{N}$; $\Omega/\mathfrak{N}=\Delta_n$. Let $\hat{\Sigma}=T_{n'}$; T division ring, then $\hat{\Omega}=\mathfrak{O}_{n'}$, where $\mathfrak{O}$ is a unique maximal order with radical (π) in T. Since $\Omega/\mathfrak{N}\approx\hat{\Omega}/\hat{\mathfrak{N}}$, $n'=n$.

In order to decide all types of h-orders in Σ, we may consider h-orders containing a fixed minimal h-order by Theorem 3.1. By Lemma 1.2, we obtain a minimal h-order Λ, which we shall fix in this section; namely

$$\begin{aligned}\Lambda &= \{(a_{i,j})\mid\in\Sigma,\ a_{i,j}\in\mathfrak{O},\ a_{i,j}\in(\pi)\quad\text{for}\quad i>j\}\,,\\ N(\Lambda) &= \{(a_{i,j})\mid\in\Lambda,\ a_{i,i}\in(\pi)\}=\mathfrak{N}\,,\\ \mathfrak{N}^{-1} &= \{(a_{i,j})\mid\in\Sigma,\ a_{i,j}\in\mathfrak{O}\quad\text{if}\quad i\neq n,\ i\neq 1\,;\ a_{j,j}\in(\pi)\quad\text{if}\quad i+1<j\\ &\qquad\qquad\text{and}\ a_{n,1}\in(1/\pi)\mathfrak{O}\}\,.\end{aligned}$$

From now on we denote $\hat{\Sigma}$, $\hat{\Omega}$, $\hat{K}$ by Σ, Ω, R, respectively.

Let $\mathfrak{M}_i=\{(a_{i,j})\mid\in\Lambda,\ a_{ii}\in(\pi)\}$. Then the $\mathfrak{M}_i$'s are the set of maximal two-sided ideals in Λ. Since $e_{i-1,i}\pi e_{i,i}e_{i,i-1}=\pi e_{i-1,i-1}\in\mathfrak{N}^{-1}\mathfrak{M}_i\mathfrak{N}$, we know that $\mathfrak{N}^{-1}\mathfrak{M}_i\mathfrak{N}=\mathfrak{M}_{i-1}$. Hence, $\{\mathfrak{M}_n, \mathfrak{M}_{n-1}, \cdots, \mathfrak{M}_1\}$ is the normal sequence in Λ. We can easily check that $\Gamma_i=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_i, \mathfrak{M}_i)=$ the ring generated by Λ and $e_{i-1,i}$ if $i\neq 1$, and that $\Gamma_1=\mathrm{Hom}^r_\Lambda(\mathfrak{M}_1, \mathfrak{M}_1)=\{(a_{i,j})\mid\in\Sigma,\ a_{i,j}\in(\pi)$ for $i<j$, $a_{i,j}\in\mathfrak{O}$ for $i\neq n$, $j\neq 1$, and $a_{n,1}\in(1/\pi)\mathfrak{O}\}$. Hence, $\{\Gamma_2, \cdots, \Gamma_n\}$ is a complete set of $n-1$th order in Ω. For any order Γ between Ω and Λ, $C(\Gamma)=I(\mathfrak{M}_{i_1}, \cdots, \mathfrak{M}_{i_r})$ $(i_j>1)$. Then Γ is the ring generated by Λ and $\{e_{j-1,j}\}$ $j=i_1, \cdots, i_r$.

Summarizing the above, we have

THEOREM 6.2.[6] *Every h-order in Σ is isomorphic to the following type*

6) Those types are changed by the suggestion of Mr. Higikata.

$$
\begin{array}{c|c}
 & \begin{array}{cccc} m_1 & m_2 & \cdots\cdots & m_r \end{array} \\
\begin{array}{c} m_1 \\ m_2 \\ \vdots \\ m_r \end{array} &
\left(\begin{array}{c|c|c|c}
\mathfrak{O}(m_1\times m_1) & \pi\mathfrak{O}(m_1\times m_2) & \cdots\cdots & \pi\mathfrak{O}(m_1\times m_r) \\ \hline
\mathfrak{O}(m_2\times m_1) & \mathfrak{O}(m_2\times m_2) & \cdots\cdots & \pi\mathfrak{O}(m_2\times m_r) \\ \hline
\vdots & \vdots & \cdots\cdots & \vdots \\ \hline
\mathfrak{O}(m_r\times m_1) & \mathfrak{O}(m_r\times m_2) & \cdots\cdots & \mathfrak{O}(m_r\times m_r)
\end{array}\right)
\end{array}
$$

where $n=\sum m_i$, *and* $\mathfrak{O}(i\times j)$: *all* $(i\times j)$ *matrices over* $\mathfrak{O}$.

We shall return to problem of counting the number of h-orders. By virtue of Theorem 6.1, we may assume that $\mathfrak{R}/\mathfrak{p}$ is a finite field and hence, $\mathfrak{O}/\pi=GF(p^m)$.

LEMMA 6.3. *Let* Γ, Ω *be as above. Then the number of isomorphic classes of* Γ *by unit element in* Ω *is equal to* $[(\Omega/\pi\Omega)^*:(\Gamma/\pi\Omega)^*]$.

Proof. By Lemma 6.1, this number is equal to $[\Omega^*:\Gamma^*]$, and by the above remark $\pi\Gamma\subseteq N(\Gamma)$. Hence, we have $(\Omega/\pi\Omega)^*/(\Gamma/\pi\Omega)^*\approx\Omega^*/\Gamma^*$.

LEMMA 6.4. $[(\Omega/\pi\Omega)^*:(\Gamma/\pi\Omega)^*]=(p^{mn}-1)(p^{nm}-p^m)\cdots(p^{nm}-p^{m(n-1)})$ $/\prod_{i=1}^{r}(p^{m_im}-1)(p^{m_im}-p^m)\cdots(p^{m_im}-p^{m(m_i-1)})p^{ms}$, $s=\sum_{i=1}^{r}m_i(n-m_1-m_2-\cdots-m_i)$.

Proof. It is clear that $\Omega/\pi\Omega=(\mathfrak{O}/\pi)_n$ and $[(\mathfrak{O}/\pi)_n^*:1]=[GL(n,p^m):1]$ $=(p^{mn}-1)(p^{nm}-p^m)\cdots(p^{mn}-p^{m(n-1)})$ by [4], p. 77, Theorem 99. $\Gamma/\pi\Omega=$

$$\left\{\begin{pmatrix} B_{1,1} & & 0 \\ & \ddots & \\ * & & B_{r,r}\end{pmatrix}\right\},$$

and hence, $r(\in\Gamma/\pi\Omega)$ is unit if and only if the $B_{i,i}$ are unit in $(\mathfrak{O}/\pi)m_i$. Therefore, $[(\Gamma/\pi\Omega)^*:1]=\prod_{i=1}^{r}(GL(m_i,p^m):1)p^{ms}$, $s=\sum_{i=1}^{r}m_i(n-m_1-m_2-\cdots-m_i)$.

By Corollary 6.4, and Theorem 4.1, we have

THEOREM 6.3. *The number of* r *th* h*-orders in a maximal order is equal to*

$\sum_{m_1+m_2+\cdots+m_r=n}\{p^{nm}-1)(p^{nm}-p^m)\cdots(p^{nm}-p^{m(n-1)})/\prod_{i=1}^{r}(p^{m_im}-1)(p^{m_im}-p^m)\cdots$ $(p^{m_im}-p^{m(m_i-1)})p^{m(\sum_{i=1}^{r}m_i(n-m_1-\cdots-m_i)}\}$. *The number of* r *th principal* h*-orders in* r' *th principal* h*-order is equal to*

$$\{(p^{mn/r'}-1)(p^{mn/r'}-p^m)\cdots(p^{mn/r'}-p^{m(n/r'-1)})\}^{r'}/$$
$$\{(p^{mn/r}-1)(p^{mn/r}-p^m)\cdots(p^{mn/r}-p^{m(n/r-1)})\}^{r}p^{(mn^2/2)(r-r'/rr')}.$$

Especially, the number of minimal h*-orders in a maximal order is equal to*

$$\prod_{i=1}^{n-1}(1+p^{m}+\cdots+p^{m_i}).$$

We shall describe Λ as follows:

$$\Lambda=\begin{pmatrix} A_{1,1}\,\pi A_{1,2}\,\pi A_{1,3}\cdots\cdots\cdots\pi A_{1,m} \\ A_{2,1}\,A_{2,2}\,\pi A_{2,3}\ \cdots\cdots\cdots\ \pi A_{2,m} \\ \\ A_{m,1}\,A_{m,2}\ \cdots\cdots\cdots\cdots\cdots\ A_{m,m} \end{pmatrix};\ A_{i,j}\text{ is matrices of } m_i\times m_j \text{ over } \mathfrak{O}.$$

Since

$$N=\begin{pmatrix} \pi A_{1,1}\,\pi A_{1,2}\ \cdots\cdots\cdots\cdots\ \pi A_{1,m} \\ A_{2,1}\,\pi A_{2,2}\ \cdots\cdots\cdots\cdots\ \pi A_{2,m} \\ \\ A_{m,1}\,A_{m,2}\ \cdots\ A_{m,m-1}\,\pi A_{m,m} \end{pmatrix};\ N^{m}=\pi\Lambda.$$

Let t be the ramification index of a maximal order, namely $\pi^{t}=pe$, $e\in\mathfrak{O}$. Then we have a explicit result of Theorem 2.2.

PROPOSITION 6.2. *Let Λ be an rth h-order, then its ramification index is equal to tr.*

PROPOSITION 6.3. *Let Λ be an rth principal h-order, and α an element in Λ such that $\Lambda\alpha^{n/r}=N(\Lambda)$ for some n. Then $\Gamma=\Lambda\cap\alpha^{-1}\Lambda\alpha\cap\cdots\cap\alpha^{-(n/r)+1}\Lambda\alpha^{1-(n/r)}$ is an nth principal h-order, and any nth principal h-order Γ in Λ is written as above and $N(\Gamma)=\alpha\Gamma=\Gamma\alpha$,* where $r\,|\,n$.

Proof. If Γ is an nth principal h-order with $N(\Gamma)=\alpha\Gamma$ in Λ, we can easily show, by Theorems 2.1 and 2.3, that $\alpha^{n/r}\Lambda=\Lambda\alpha^{n/r}$ and $\Gamma=\Lambda\cap\alpha^{-1}\Lambda\alpha\cap\cdots\cap\alpha^{-(n/r)+1}\Lambda\alpha^{1-(n/r)}$. Since $\alpha^{n/r}\Lambda=\Lambda\alpha^{n/r}$, $\alpha^{n/r}\Lambda=N(\Lambda)^{l}$. However $\alpha^{nt}\Lambda=p\Lambda$, and hence $l=1$ by Proposition 6.2. Therefore, $\Lambda\alpha^{n/r}=N(\Lambda)$. Conversely if $\Lambda\alpha^{n/r}=N(\Lambda)$, $\Lambda\alpha^{i}$ is a left ideal in Λ containing $N(\Lambda)$ for $i\leqslant n/r$, and $\Lambda\alpha^{i}/\Lambda\alpha^{i+1}\approx\Lambda/\Lambda\alpha$ as a left Λ-module. If $\Lambda\alpha\Lambda\neq\Lambda$, $\Lambda/\Lambda\alpha\approx\mathfrak{l}_1\oplus\mathfrak{l}_2\oplus\cdots\oplus\Delta_{m_i}\oplus\cdots\oplus\mathfrak{l}_r$ for some i. Hence, since $\Lambda/\Lambda\alpha\approx\Lambda\alpha/\Lambda\alpha^{2}$, $\Lambda\alpha^{2}\supseteqq N(\Lambda)$, we have a contradiction. Since Λ is principal, $\Lambda\alpha^{(n/r)-1}/N(\Lambda)=\mathfrak{l}_1\oplus\mathfrak{l}_2\oplus\cdots\oplus\mathfrak{l}_r$, $\Lambda\alpha^{i}/N(\Lambda)=\mathfrak{l}_1^{((n/r)-i)}\oplus\mathfrak{l}_2^{((n/r)-i)}\oplus\cdots\oplus\mathfrak{l}_r^{((n/r)-i)}$. Then $\Gamma=\Lambda\cap\mathrm{Hom}_{\Lambda}^{l}(\Lambda\alpha,\Lambda\alpha)\cap\mathrm{Hom}_{\Lambda}^{l}(\Lambda\alpha^{2},\Lambda\alpha^{2})\cap\cdots\cap\mathrm{Hom}_{\Lambda}^{l}(\Lambda\alpha^{(n/r)-1},\Lambda\alpha^{(n/r)-1})=\Lambda\cap\alpha^{-1}\Lambda\alpha\cap\cdots\cap\alpha^{1-(n/r)}\Lambda\alpha^{(n/r)-1}$ is a principal nth h-order by Corollary 5.1. It is clear that $\alpha\Gamma=\Gamma\alpha$. Hence $\alpha\Gamma=N(\Gamma)^{l}\beta^{l}\Gamma$. However, $\mathfrak{p}=(\alpha^{n/r})^{rt}\varepsilon=\beta^{rlt}\varepsilon'=\mathfrak{p}^{l}\varepsilon''$, where ε, ε' and ε'' are units in Λ. Hence $l=1$.

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