Journal of the Institute of Polytechnics,
Osaka City University,
Vol. 5, No. 1, Series A

Meromorphic approximations on Riemann surfaces

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(Received Feb. 1, 1954)

Let D, D' be compact domains of a Riemann surface R relative to R such that $\bar{D} \subset D'$ and D be enclosed by a finite number of closed Jordan curves. Let P be a finite point set contained in D, Q' be a selected set of the collection of compact components of D'-$\bar{D}$ relative to D', that is, any point of Q' is contained in one and only one element of the collection and conversely any element of the collection contains one and only one point of Q', and Q be a selected set of the collection of compact components of R-$\bar{D}$ relative to R. Obviously both Q' and Q are finite point sets. Then we have the following theorems:

THEOREM 1'. *There exists such a function as is meromorphic in D' and has its poles on P.*

THEOREM 2'. *Any function which is regular in a certain domain containing $\bar{D}$ is uniformly approximated on $\bar{D}$ by such a function as is meromorphic on D' and has its poles on Q'.*

THEOREM 3'. *Any functon which is meromorphic in a certain domain containing $\bar{D}$ and has its poles on P is uniformly approximated on $\bar{D}$ by such a function as is meromorphic in D' and has its poles on $P \cup Q'$.*

THEOREM 1. *There exists such a function as is meromorphic in R and has its poles on P.*

THEOREM 2. *Any function which is regular in a certain domain contiaining $\bar{D}$ is uniformly approximated on $\bar{D}$ by such a function as is meromorphic in R and has its poles on Q.*

THEOREM 3. *Any function which is meromorphic in a certain domain containing $\bar{D}$ and has its poles on P is uniformly approximated on $\bar{D}$ by such a function as is meromorphic in R and has its poles on $P \cup Q$.*

According to the method of Behenke and Stein[1], these theorems are easily derived by the following process[2]:

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1) Behnke und Stein: Entwicklung analytischer Funktionen auf Riemannschen Flächen, Math. Ann. 120 (1948), pp. 430-461.

2) Theorem 1' is trivial. Theorem 2' is a modified one of a theorem in the above paper in which D is simply connected relative to D'.

$$\left.\begin{matrix}1'\\ \\ 2'\end{matrix}\right] \longrightarrow 3' \longrightarrow 3 \begin{matrix}\nearrow 1\\ \\ \searrow 2\end{matrix}$$

The purpose of this paper is to bring Theorem 2, 3 in the following formulations.

THEOREM 2*. *Any function which is continuous on $\bar{D}$ and regular in D is uniformly approximated on $\bar{D}$ by such a function as is meromorphic in R and has its poles on Q.*

THEOREM 3*. *Any function which is continuous on $\bar{D}-P$ and meromorphic in D is uniformly approximated on $\bar{D}$ by such a function as is meromorphic in R and has its poles on $P \cup Q$.*

Since Theorem 3* follows from Theorem 1, 2*, it is sufficient to prove Theorem 2* only. To see this, it is sufficient to prove the following theorem.

THEOREM 4. *Any function which is continuous on $\bar{D}$ and regular in D is uniformly approximated on D by such a function as is regular in a certain domain containing $\bar{D}$.*

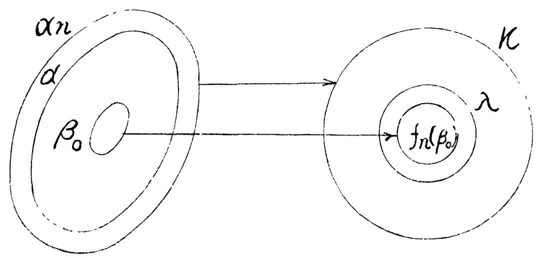
We shall begin with some preparations. A closed Jordan curve is briefly called a loop. When there exists a family of mutually homotopic loops, its order is denoted by $>$, the part enclosed by two mutually homotopic loops α, β by (α, β), and $(\alpha, \beta) \cup \alpha$, $(\alpha, \beta) \cup \beta$, $(\alpha, \beta) \cup \alpha \cup \beta$ by $[\alpha, \beta)$, $(\alpha, \beta]$, $[\alpha, \beta]$ respectively. Also the definition domain or the range of a function f is denoted by dom f or ran f, and a function which is defined on a set E by $f|E$. Throughout this paper, we assume that a function is continuous on its domain and regular in the interior of its domain.

LEMMA 1. *Let α_0, α, β, β_0 be four mutually homotopic loops arranged in this order. Then, for any positive number ε, there exist two loops α', β' and a function φ such that*

1. $\alpha_0 > \alpha' > \alpha$, $\beta > \beta' > \beta_0$;
2. dom $\varphi = [\alpha', \beta')$, ran $\varphi \subset [\alpha, \beta_0)$;
3. $|I-\varphi| < \varepsilon$ *on* $[\alpha', \beta]$.

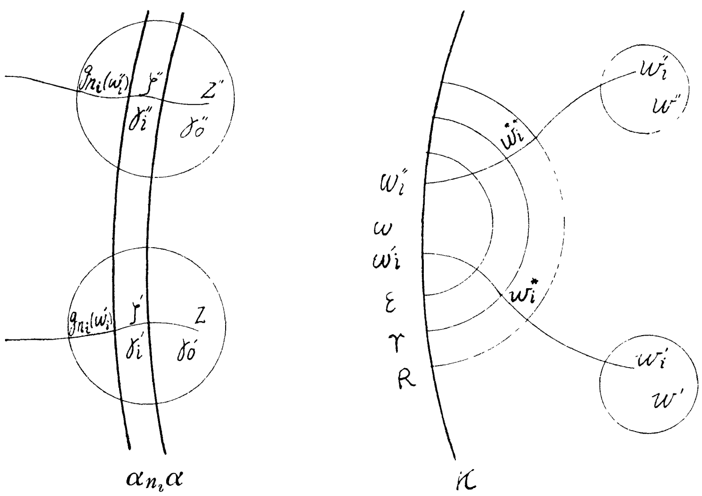
where I is the identity function.

Proof. We can assume without loss of generality that $[\alpha_0, \beta_0]$ lies on the z-plane. On the w-plane, if the Jordan domain enclosed by κ contains λ, then we define the ordering as $\kappa > \lambda$. Let $\{\alpha_n\}$ be a sequence such that $\alpha_0 > \alpha_n \downarrow \alpha$ and f_n the function which maps topologically $[\alpha_n, \beta_0]$ onto the closed ring $[\kappa, f_n(\beta_0)]$ on the w-plane, α_n onto κ, β_0 onto $f_n(\beta_0)$ and conformally (α_n, β_0) onto $(\kappa, f_n(\beta_0))$, where κ, $f_n(\beta_0)$ are concentric circles. Then the circles $f_n(\beta_0)$ are monotone-increasing and converge to a certain circle $\lambda (< \kappa)$.



Let g_n be the inverse function of f_n, then we shall show that g_n form a normal family on κ. To see this, it is sufficient to prove that g_n are equi-continuous on κ. Since f_n form a normal family on (α, β_0), if g_n were not equi-continuous on κ, there would exist an increasing sequence $\{n_i\}$ of natural numbers, a function $f \mid (\alpha, \beta_0)$, ω_i', ω_i'', $\omega(\in \kappa)$ and ς', ς'' such that

$$
\begin{aligned}
&f_{n_i} \to f \qquad \text{on } (\alpha, \beta_0)\,; \\
&\omega_i' \to \omega,\ \omega_i'' \to \omega\,; \\
&g_{n_i}(\omega_i') \to \varsigma',\ g_{n_i}(\omega_i'') \to \varsigma'',\ \varsigma' \neq \varsigma''.
\end{aligned}
$$



1. Obviously $\varsigma', \varsigma'' \in \alpha$. Describe two circles with their centers ς', ς'' and with the common radius $l = \frac{1}{3}\left|\varsigma' - \varsigma''\right|$ and let $U_l(\varsigma')$, $U_l(\varsigma'')$ be the interiors of these circles.

2. Since ς', ς'' are accessible boundary points of (α, β_0), there exist such two curves γ_0', γ_0'' as end in ς', ς'' and are contained in $[\alpha, \beta_0) \cap U_l(\varsigma')$, $[\alpha, \beta_0) \cap U_l(\varsigma'')$, respectively. Let z', z'' be the initial points of γ_0', γ_0''.

3. There exist two curves γ', γ'' such that, (i) γ', γ'' contain $g_{n_i}(\omega_i')$, $g_{n_i}(\omega_i'')$ and end in ζ', ζ'' respectively, (ii) the parts γ_i', γ_i'' of γ', γ'' rising respectively from $g_{n_i}(\omega_i')$, $g_{n_i}(\omega_i'')$ are contained in $[\alpha_{n_i}, \alpha]$.

4. Put $w' = f(z')$, $w_i' = f_{n_i}(z')$, then $w_i' \to w'$.

Put $w'' = f(z'')$, $w_i'' = f_{n_i}(z'')$, then $w_i'' \to w''$.

5. Let $U_R(\omega)$, $U(w')$, $U(w'')$ be neighbourhoods of ω, w', w'', and mutually exclusive.

Under these circumstances, for any given positive number $\varepsilon(<R)$, there exists a natural number i such that

$$
\begin{aligned}
&1.\quad \omega_i' \in U_\varepsilon(\omega),\ \omega_i'' \in U_\varepsilon(\omega)\,,\\
&2.\quad w_i' \in U(w')\,,\quad w_i'' \in U(w'')\,,\\
&3.\quad \gamma_i' \subset U_l(\varsigma')\,,\quad \gamma_i'' \subset U_l(\varsigma'')\,.
\end{aligned}
$$

Then

$$\gamma_0' + \gamma_i' \subset [\alpha_{n_i}, \beta_0) \cap U_l(\varsigma')\,,\quad \gamma_0'' + \gamma_i'' \subset [\alpha_{n_i}, \beta_0) \cap U_l(\varsigma'')$$

Hence

$$\text{dis}\ (\gamma_0' + \gamma_i', \gamma_0'' + \gamma_i'') \geqq l\,.$$

Next we consider on $[\kappa, f_{n_i}(\beta_0))$ the images of $\gamma_0' + \gamma_i'$, $\gamma_0'' + \gamma_i''$ by f_{n_i}. They form curves conbining w_i' and ω_i', w_i'' and ω_i''. For any r such that $R > r > \varepsilon$, w_i', w_i'' lie in the exterior of the circle $|w - \omega| = r$, while ω_i', ω_i'' lie in the interior of that circle. Hence $f_{n_i}(\gamma_0' + \gamma_i')$ (resp. $f_{n_i}(\gamma_0'' + \gamma_i'')$) intersects that circle. Let w_i^* (resp. w_i^{**}) be one of the intersecting points. Then

$$|g_{n_i}(w_i^{**}) - g_{n_i}(w_i^*)| \geqq l\,.$$

Hence

$$
\begin{aligned}
\text{area of } (\alpha_0, \beta_0) &> \text{area of } (\alpha_{n_i}, \beta_0)\\
&> \iint_A |g'_{n_i}(w)|^2\, du\, dv
\end{aligned}
$$

where $A = (\kappa, \lambda) \cap \{w : R > w - \omega > \varepsilon\}$, $w = u + iv$,

$$\geqq \int_\varepsilon^R dr \int_{\widehat{w_i^* w_i^{**}}} |g'_{n_i}(w)|^2 |dw|$$

$$\geqq \int_\varepsilon^R dr \frac{\widehat{w_i^* w_i^{**}}}{\pi r} \int_{\widehat{w_i^* w_i^{**}}} |g'_{n_i}(w)|^2 |dw| = \int_\varepsilon^R dr \frac{1}{\pi r} \int_{\widehat{w_i^* w_i^{**}}} |dw| \int_{\widehat{w_i^* w_i^{**}}} |g'_{n_i}(w)|^2 |dw|$$

and by Schwarz's inequality,

$$\geqq \int_\varepsilon^R dr \frac{1}{\pi r} \left(\int_{\widehat{w_i^* w_i^{**}}} |g'_{n_i}(w)|\, |dw| \right)^2$$

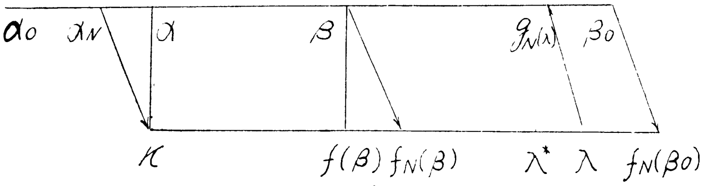
$$\geqq \int_\varepsilon^R dr \frac{1}{\pi r} \left| \int_{\widehat{w_i^* w_i^{**}}} g'_{n_i}(w)\, dw \right|^2 = \int_\varepsilon^R dr \frac{1}{\pi r} \left| g_{n_i}(w_i^{**}) - g_{n_i}(w_i^*) \right|^2$$

$$\geqq \int_{\varepsilon}^{R} dr \frac{l^2}{\pi r} = \frac{l^2}{\pi} \log \frac{R}{\varepsilon}.$$

Hence the area of $(\alpha_0, \beta_0) > \frac{l^2}{\pi} \log \frac{R}{\varepsilon}$ for all positive number ε $(< R)$, which is impossible.

Since we have seen that g_n form a normal family on κ, we shall go to the next step. Of course, g_n form a normal family on (κ, λ), so that g_n form a normal family on $[\kappa, \lambda)$, while f_n form a normal family on (α, β_0). Then there exist an increasing sequence $\{n'\}$ of natural numbers and two functions $f|(\alpha, \beta_0)$, $g|[\kappa, \lambda)$ such that

$$f_{n'} \to f \quad \text{on } (\alpha, \beta_0) .$$
$$g_{n'} \to g \quad \text{on } [\kappa, \lambda) .$$



Since $f_{n'}(\beta) > f_{n'}(\beta_0) \geqq f_1(\beta_0)$, the oscilation of $f_{n'}$ on β is not smaller than the diameter of the circle $f_1(\beta_0)$. Consulting with $f_{n'} \to f$ on β, we conculude that f is nonconstant and hence univalent and regular, so that f is an open mapping. Then ran f is an open set contained in $[\kappa, \lambda]$, and therefore ran $f \subset (\kappa, \lambda)$. As well as f, g is also an open mapping. Then $g((\kappa, \lambda))$ is an open set contained in $[\alpha, \beta_0]$, and therefore $g((\kappa, \lambda)) \subset (\alpha, \beta_0)$. Also $g(\kappa) \subset \alpha$. We have then

$$\text{ran } g = g([\kappa, \lambda]) \subset [\alpha, \beta_0) .$$

From ran $f \subset (\kappa, \lambda)$, it follows $f(\beta) \subset (\kappa, \lambda)$. Take λ^* such that $f(\beta) > \lambda^* > \lambda$, then

$$f(\beta) \subset (\kappa, \lambda^*) .$$

Consulting with $f_{n'} \to f$ on β and $g_{n'} \to g$ on $[\kappa, \lambda)$, for any positive number ε, there exists a suitably large natural number $N (= n')$ such that

$$f_N(\beta) \subset (\kappa, \lambda^*) ,$$
$$|g_N - g| < \varepsilon \text{ on } [\kappa, \lambda^*] .$$

From the former, it follows

$$f_N([\alpha_N, \beta]) = [\kappa, f_N(\beta)] \subset [\kappa, \lambda^*) ,$$

and from the latter and the above fact,

$$|g_N \circ f_N - g \circ f_N| < \varepsilon \quad \text{on} \quad [\alpha_N, \beta] ,$$

that is,

$$|I - g \circ f_N| < \varepsilon \text{ on } [\alpha_N, \beta] ,$$

and $\alpha_0 > \alpha_N > \alpha$, $\beta > g_N(\lambda) > \beta_0$ from $f_N(\beta) > \lambda^* > \lambda > f_N(\beta_0)$, dom $(g \circ f_N) = g_N$ (dom g) $= [\alpha_N, g_N(\lambda))$, ran $(g \circ f_N) =$ ran $g \subset [\alpha, \beta_0)$.

Putting $\alpha' = \alpha_N$, $\beta' = g_N(\lambda)$, $\varphi = g \circ f_N$, we get the statement.

LEMMA 2. *Let α_0, α, β, β_0 be four mutually homotopic loops arranged in this order. Let f be a function such that* dom $f = [\alpha, \beta_0]$. *Then, for any positive number ε, there exist two loops α', β' and a function g such that*

1. $\alpha_0 > \alpha' > \alpha$, $\beta > \beta' > \beta_0$;
2. dom $g = [\alpha', \beta')$;
3. $|f - g| < \varepsilon$ on $[\alpha, \beta]$.

Proof. Since f is uniformly continuous on $[\alpha, \beta_0]$, for any positive number ε, there exists a positive number δ such that

if $z, z' \in [\alpha, \beta_0]$, $|z - z'| < \delta$, then $|f(z) - f(z')| < \varepsilon$.

By Lemma 1, for this δ, there exist two loops α', β' and a function φ such that

1. $\alpha_0 > \alpha' > \alpha$, $\beta > \beta' > \beta_0$;
2. dom $\varphi = [\alpha', \beta')$, ran $\varphi \subset [\alpha, \beta_0)$;
3. $|I - \varphi| < \delta$ on $[\alpha', \beta]$.

From these conditions we have

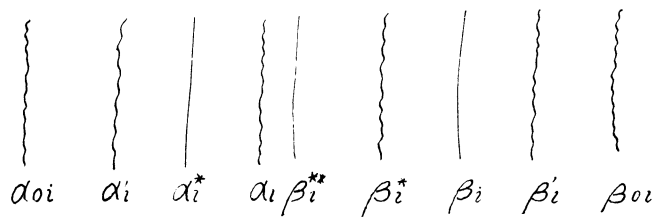
$$|f - f \circ \varphi| < \varepsilon \text{ on } [\alpha, \beta],$$

$$\mathrm{dom}(f \circ \varphi) = \mathrm{dom}\ \varphi = [\alpha,' \beta').$$

Putting $g = f \circ \varphi$, we get the statement.

All preparatians have been achieved; now we proceed to prove the theorem.

Proof of Theorem 4. Let $\{\alpha_i\}$ be a finite number of loops enclosing D. In a planer neighbourhood of α_i, we take α_{0i}, β_i^*, β_i, β_{0i} such that $\alpha_{0i} > \alpha_i > \beta_i^* > \beta_i > \beta_{0i}$, where α_{0i} lies in the exterior of D, β_i^*, β_i, β_{0i} lie in the interior of D and β_i is rectifiable.



By lemma 2, for any positive number ε, there exist two curves α_i', β_i', and a function g_i such that

1. $\alpha_{0i} > \alpha_i' > \alpha_i$, $\beta_i > \beta_i' > \beta_{0i}$;
2. dom $g_i = [\alpha_i', \beta_i')$;
3. $|f - g_i| < \varepsilon$ on $[\alpha_i, \beta_i]$.

Moreover we take a rectifiable loop α_i^* such that $\alpha_i' > \alpha_i^* > \alpha_i$, whence α_i^* depends on ε.

Let D' be the domain enclosed by $\{\alpha_i'\}$, then there exists a many valued function $\omega_p(\pi) | \pi \in D'$ depending on the parameter $p(\in D')$ such that, for any univalent regular function ς defined in $G'(\subset D')$,

$$H(\pi, p) = H_p(\pi) = \frac{d\omega_p(\pi)}{d\varsigma(\pi)}$$

is meromorphic for (π, p) in $G' \times D'$ and $H_p(\pi)$ has its singular part $\dfrac{1}{\varsigma(\pi) - \varsigma(p)}$ at $p(\in G')$.[3]

Take any rectifiable loop β_i^{**} such that $\alpha_i > \beta_i^{**} > \beta^*$. Then for all $p \in (\beta_j^{**}, \beta_j^*)$,

$$\frac{1}{2\pi i}\int_{\beta_i^{**}} f(\pi) d\omega_p(\pi) = \frac{1}{2\pi i}\int_{\beta_i} f(\pi) d\omega_p(\pi) + \delta_{ij} f(p)$$

$$\text{where} \quad \delta_{ij} = \begin{cases} 1 (i = j) \\ 0 (i \neq j) \end{cases},$$

$$= \frac{1}{2\pi i}\int_{\beta_i} f(\pi) H_i(\pi, p) d\varsigma_i(\pi) + \delta_{ij} f(p)$$

where ς_i is a univalent regular function defined in a planar neighbourhood of α_i and

$$H_i(\pi, p) = \frac{d\omega_p(\pi)}{d\varsigma_i(\pi)}.$$

Similarly

$$\frac{1}{2\pi i}\int_{\alpha_i^*} g_i(\pi) d\omega_p(\pi) = \frac{1}{2\pi i}\int_{\beta_i} g_i(\pi) H_i(\pi, p) d\varsigma_i(\pi) + \delta_{ij} g_i(p).$$

Putting

$$\max_{(\pi, p) \in \beta_j \times [\alpha_i, \beta_i^*]} |H_i(\pi, p)| = M_{ij},$$

M_{ij} does not depend on ε, and

$$\left|\frac{1}{2\pi i}\int_{\beta_i^{**}} f(\pi) d\omega_p(\pi) - \frac{1}{2\pi i}\int_{\alpha_i^*} g_i(\pi) d\omega_p(\pi)\right| \leq \left(\frac{M_{ij}}{2\pi}\int_{\beta_i} |d\varsigma_i(\pi)| + \delta_{ij}\right)\varepsilon.$$

Putting

$$\max_j \left(\frac{M_{ij}}{2\pi}\int_{\beta_i} |d\varsigma_i(\pi)| + \delta_{ij}\right) = M_i,$$

for all $p \in \bigcup_j (\beta_j^{**}, \beta_j^*)$,

$$\left|\frac{1}{2\pi i}\int_{\beta_i^{**}} f(\pi) d\omega_p(\pi) - \frac{1}{2\pi i}\int_{\alpha_i^*} g_i(\pi) d\omega_p(\pi)\right| \leq M_i \varepsilon.$$

Putting

$$\sum_i M_i = M,$$

3) Behnke und Stein, ibid.

for all $p \in \bigcup_j (\beta_j^{**}, \beta_j^*)$,

$$\left| f(p) - \frac{1}{2\pi i} \sum_i \int_{\alpha_i^*} g_i(\pi) d\omega_p(\pi) \right| \leqq M\varepsilon .$$

Since β_j^{**} was arbitrarily chosen under the condition $\alpha_j > \beta_j^{**} > \beta_j^*$, the above formula is satisfied for all $p \in \bigcup (\alpha_j, \beta_j^*)$. Putting

$$g(p) = \frac{1}{2\pi i} \sum_i \int_{\alpha_i^*} g_i(\pi) d\omega_p(\pi) ,$$

$g(p)$ is defined on the domain enclosed by $\{\alpha_i^*\}$, that is, a certain domain containing $\overline{D}$. Since $\bigcup_j (\alpha_j, \beta_j^*)$ is a boundary strip of D, we have by the maximum principle,

$$|f-g| \leqq M\varepsilon \quad \text{on } D .$$

Since M does not depend on ε, g is the function which we have desired.