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ON QUASIINVARIANTS OF S_n OF HOOK SHAPE

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Abstract

O. Chalykh, A.P. Veselov and M. Feigin introduced the notion of quasiinvariants of Coxeter groups, which is a generalization of invariants. In [2], Bandlow and Musiker showed that for the symmetric group S_n of order n, the space of quasiinvariants has a decomposition indexed by standard tableaux. They gave a description of a basis for the components indexed by standard tableaux of shape $(n-1, 1)$. In this paper, we generalize their results to a description of a basis for the components indexed by standard tableaux of arbitrary hook shape.

1. Introduction

In [3] and [5], O. Chalykh, A.P. Veselov and M. Feigin introduced the notion of *quasiinvariants* for Coxeter groups, which is a generalization of invariants. For any Coxeter group G, the quasiinvariants are determined by a multiplicity m which is a G-invariant map from the set of reflections to non-negative integers.

We denote by S_n the symmetric group of order n. In the case of S_n, the multiplicity is a constant function. Take a non-negative integer m. A polynomial $P \in \mathbb{Q}[x_1, x_2, \dots, x_n]$ is called an m-quasiinvariant if the difference

$$(1-(i, j))P(x_1, \dots, x_n)$$

is divisible by $(x_i - x_j)^{2m+1}$ for any transposition $(i, j) \in S_n$.

The notion of quasiinvariants appeared in the study of the quantum Calogero Moser system. In the case of S_n, this system is determined by the following differential operator (the generalized Calogero–Moser Hamiltonian):

$$L_m = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1\le i<j\le n} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)$$

where m is a real number.

Let G be a Coxeter group. We denote by S^G the sub ring generated by invariant polynomials for G and by I^G the ideal of the ring of quasiinvariants generated by the

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invariant polynomials of positive degree. For a generic multiplicity, there exists an isomorphism from the ring S^G to the ring of G-invariant quantum integrals of the generalized Calogero–Moser Hamiltonian (sometimes called Harish-Chandra isomorphism). We denote by $\mathcal{L}_1, \mathcal{L}_2, \dots, \mathcal{L}_n$ the operators corresponding to fundamental invariant polynomials $\sigma_1, \sigma_2, \dots, \sigma_n$. The generalized Calogero–Moser Hamiltonian is a member of this ring (see for example [5], [6]).

In the case of non-negative integer multiplicities, Chalykh and Veselov showed that there exists a homomorphism from the ring of quasiinvariants to the commutative ring of differential operators whose coefficients are rational functions (see e.g. [3]). It is shown that the restriction of such homomorphism onto S^G induces the Harish-Chandra isomorphism. In the case of non-negative integer multiplicities there are much more quantum integrals.

Let m be a non-negative integer multiplicity. In [5], Feigin and Veselov introduced the notion of m-harmonics which are defined as the solutions of the following system:

$$
\begin{aligned}
\mathcal{L}_1 \psi &= 0, \\
\mathcal{L}_2 \psi &= 0, \\
&\cdots \\
\mathcal{L}_n \psi &= 0.
\end{aligned}
$$

Feigin and Veselov also showed that the solutions of such system are polynomials. They also showed that the space of m-harmonic polynomials is a subspace of the space of m-quasiinvariants and has dimension $|G|$. In [7], G. Felder and Veselov gave a formula of the Hilbert series of the space of m-harmonic polynomials.

In [4], P. Etingof and V. Ginzburg proved the following:

(i) the ring of quasiinvariants of G is a free module over S^G, Cohen–Macaulay and Gorenstein,

(ii) there is an isomorphism from the quotient space of quasiinvariants by I^G to the dual space of m-harmonic polynomials,

(iii) the Hilbert series of the quotient space of the quasiinvariants by I^G is equal to that of m-harmonic polynomials.

Let $I_2(N)$ be the dihedral group of regular N-gon. In [5], Feigin and Veselov considered quasiinvariants of $I_2(N)$ for any constant multiplicity. Since $I_2(N)$ has rank 2, quasiinvariants can be expressed as a polynomial in z and $\bar{z}$. Feigin and Veselov gave generators over $S^{I_2(N)}$ by a direct calculation. In [6], Feigin studied quasiinvariants of $I_2(N)$ for any non-negative integer multiplicity. He gave a free basis of the module of quasiinvariants over $S^{I_2(N)}$ using the above mentioned results of Etingof and Ginzburg. An explicit description of basis of the quotient space of quasiinvariants for S_3 is contained in [5]. Another description is given in [1]. In [7], for S_n Felder and Veselov provided integral expressions for the lowest degree non-symmetric quasiinvariant polynomials (the degree $nm + 1$). However, for any integer $n \geq 4$ a basis of the quotient space of quasiinvariants of S_n is not known.

In this paper, we consider the quasiinvariants of S_n. In this case, m is a non-negative integer. We denote by $\mathbf{QI}_m$ the ring of quasiinvariants and by Λ_n the ring of symmetric polynomials. We define $\mathbf{QI}_m^*$ as the quotient space of $\mathbf{QI}_m$ by the ideal generated by the homogeneous symmetric polynomials of positive degree.

In [2], J. Bandlow and G. Musiker showed that the space $\mathbf{QI}_m$ has a decomposition into subspaces indexed by standard tableaux. Each component has a Λ_n module structure. The quotient space $\mathbf{QI}_m^*$ is also decomposed in the same way. They constructed an explicit basis of the submodules of $\mathbf{QI}_m^*$ indexed by standard tableaux of shape $(n-1, 1)$.

In this paper, we extend the result in [2]. We construct a basis of the submodules of $\mathbf{QI}_m^*$ indexed by standard tableaux of shape $(n-k+1, 1^{k-1})$ (a hook) (see Theorem 3.8). The elements of our basis are expressed as determinants of a matrix with entries similar to elements of basis introduced in [2]. We also show that our basis is a free basis of the submodule of $\mathbf{QI}_m$ indexed by a hook $(n-k+1, 1^{k-1})$ over Λ_n (Corollary 3.11).

We also show how the operator L_m acts on our basis. In [5], it is proved that the operator L_m preserves $\mathbf{QI}_m$. In [2], it is obtained explicit formulas of the action of L_m on their basis. We extend these formulas to those of our basis (Theorem 4.4).

2. Preliminaries

2.1. Symmetric group and Young diagram. We denote $\mathbb{Q}[x_1, x_2, \dots, x_n]$ by K_n and the symmetric group on $\{1, 2, \dots, n\}$ by S_n. For a finite set X, we denote the symmetric group on X by S_X.

The symmetric group S_n acts on K_n by

$$\sigma P(x_1, \dots, x_n) = P(x_{\sigma(1)}, \dots, x_{\sigma(n)}), \quad \sigma \in S_n.$$

A polynomial $P(x_1, x_2, \dots, x_n)$ is called a symmetric polynomial when for any $\sigma \in S_n$, $P(x_1, x_2, \dots, x_n)$ satisfies

$$\sigma P(x_1, \dots, x_n) = P(x_1, \dots, x_n).$$

We denote by Λ_n the subspace spanned by symmetric polynomials and by Λ_n^d the subspace of Λ_n spanned by homogeneous polynomials of degree d. We set $\Lambda_n^d = \{0\}$ if $d < 0$. The i-th elementary symmetric polynomial is denoted by e_i. For a partition $\nu = (\nu_1, \nu_2, \dots)$, we define $e_\nu = \prod_i e_{\nu_i}$. A basis of Λ_n is given by $\{e_\nu\}$.

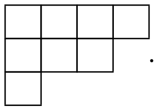
The group ring of S_n over $\mathbb{Q}$ is denoted by $\mathbb{Q}S_n$. The action of S_n on K_n is naturally extended to that of $\mathbb{Q}S_n$. For a subgroup H of S_n, we define $[H], [H]'$ in $\mathbb{Q}S_n$ by

$$[H] = \sum_{\sigma \in H} \sigma,$$

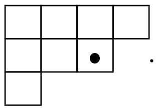
$$[H]' = \sum_{\sigma \in H} sgn(\sigma)\sigma.$$

Let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition. When λ is a partition of a positive integer n, we denote this by $\lambda \vdash n$. We define $l(\lambda) = \#\{i \mid \lambda_i \neq 0\}$ and $|\lambda| = \sum_i \lambda_i$. They are called the length and the size of λ respectively.

For a partition λ, the Young diagram of shape λ is a diagram such that its i-th row has λ_i boxes and it is arranged in left-justified rows. For example, the Young diagram of shape (4, 3, 1) is


.

We denote by (i, j) a box on the (i, j)-th position of the diagram. For instance, the box (2, 3) of the Young diagram of shape (4, 3, 1) is


.

We identify the Young diagram of shape λ with the partition λ.

Let k, n be integers such that $k \geq 2$ and $n \geq k$. We define $\eta(n,k) = (n-k+1, 1^{k-1})$. We have $l(\eta(n, k)) = k$ and $|\eta(n, k)| = n$. We call $\eta(n, k)$ (also the Young diagram of $\eta(n, k)$) the hook.

For $\lambda \vdash n$, we define the arm length $a(i, j)$ for box $(i, j) \in \lambda$ as

$$a(i, j) = \#\{(i, l) \mid j < l,\ (i, l) \in \lambda\}.$$

We also define the leg length $l(i, j)$ for box (i, j) as

$$l(i, j) = \#\{(k, j) \mid i < k,\ (k, j) \in \lambda\}.$$

We define $h(i, j) = a(i, j) + l(i, j) + 1$ called the hook length for box $(i, j) \in \lambda$.

A *tableau* of shape λ is obtained by assigning a positive integer to each box of the Young diagram λ. In this paper, we assume that entries of boxes are different each other. For a tableau D, we denote by $D_{i,j}$ the entry in the box (i, j) of D. We define

$$mem(D) = \{D_{i,j} \mid (i, j) \in \lambda\}.$$

A tableau T is called a standard tableau if T satisfies $mem(T) = \{1, 2, \dots, n\}$ and

$$T_{i,j} < T_{k,j}, \quad T_{i,j} < T_{i,l}, \quad i < k,\ j < l.$$

We denote by $ST(\lambda)$ the set of all standard tableaux of shape λ and by $ST(n)$ the set of all standard tableaux with n boxes.

For a tableau D of shape λ, we define

$$C(D) = [\{\sigma \in S_{mem(D)} \mid \sigma \text{ preserves each column of } D\}]',$$
$$R(D) = [\{\sigma \in S_{mem(D)} \mid \sigma \text{ preserves each row of } D\}],$$
$$f_\lambda = \#ST(\lambda),$$
$$\gamma_D = \frac{f_\lambda C(D)R(D)}{n!},$$
$$V_D = \prod_{(i,j)\in C_D} (x_i - x_j)$$

where $C_D = \{(i, j) \mid i < j$ and i, j are entries in a same column of $D\}$. The element $\gamma_D \in \mathbb{Q}S_{mem(D)}$ satisfies $\gamma_D^2 = \gamma_D$.

DEFINITION 2.1. Let $s_1, s_2, \dots, s_n$ be mutually distinct positive integers.
(1) We denote by $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$ the tableau of shape $\eta(n, k)$ such that the entries in the first column and in the first row are $s_1, s_2, \dots, s_k$ and $s_1, s_{k+1}, \dots, s_n$ in order, respectively.
(2) A tableau $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$ is a standard tableau of shape $\eta(n, k)$ if and only if the following holds:

$$s_1, s_2, \dots, s_n \quad \text{is a permutation of} \quad 1, 2, \dots, n,$$
$$s_1 = 1,\ s_2 \le \dots \le s_k,\ s_{k+1} \le \dots \le s_n.$$

Then we simply write $D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$ as $T(1, s_2, \dots, s_k)$.
(3) Let i be an integer such that $1 \le i \le k$ (resp. $k + 1 \le i \le n$). We set $D = D(s_1, s_2, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$. We define

$$D^{s_i} = D(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k; s_1, s_{k+1}, \dots, s_n)$$
$$(\text{resp. } D^{s_i} = D(s_1, \dots, s_k; s_1, s_{k+1}, \dots, s_{i-1}, s_{i+1}, \dots, s_n)).$$

For example, the standard tableau $T(1,3,4) = D(1,3,4; 1,2,5,6)$ of shape $(4,1,1)$ is

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ .$$

The tableau $T(1, 3, 4)^1$ is

$$\begin{array}{|c|c|c|c|}\hline 3 & 2 & 5 & 6 \\ \hline 4 \\ \cline{1-1}\end{array}$$

and $T(1, 3, 4)^2$ is

$$\begin{array}{|c|c|c|}\hline 1 & 5 & 6 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ .$$

We have the following propositions.

Proposition 2.2 ([2]). *For any* $f = \sum_{\sigma \in S_n} f_\sigma \sigma \in \mathbb{Q}S_n$, $P \in \Lambda_n$ *and* $Q \in K_n$, *we have* $f(PQ) = Pf(Q)$.

Proposition 2.3 ([2]). *Let* $i_1, i_2, \dots, i_n$ *be a permutation of* $1, 2, \dots, n$. *Then* $[S_n]$ *and* $[S_n]'$ *are expressed as follows*:

$$[S_n] = (1 + (i_1, i_n) + \cdots + (i_{n-1}, i_n)) \cdots (1 + (i_1, i_3) + (i_2, i_3))(1 + (i_1, i_2)),$$
$$[S_n]' = (1 - (i_1, i_n) - \cdots - (i_{n-1}, i_n)) \cdots (1 - (i_1, i_3) - (i_2, i_3))(1 - (i_1, i_2)).$$

2.2. The quasiinvariants of S_n. We recall the definition and the notation of m-quasiinvariants. Take a non-negative integer m. A polynomial $P \in K_n$ is called an m-quasiinvariant if the difference

$$(1 - (i, j))P(x_1, \dots, x_n)$$

is divisible by $(x_i - x_j)^{2m+1}$ for any transposition $(i, j) \in S_n$. We denote by $\mathbf{QI}_m$ the ring of quasiinvariants and by Λ_n the space of symmetric polynomials. We denote by I_m the ideal of $\mathbf{QI}_m$ generated by $e_1, \dots, e_n$. We set $\mathbf{QI}_m^* = \mathbf{QI}_m / I_m$.

We recall results in [2].

Lemma 2.4 ([2]). *The ring* $\mathbf{QI}_m$ *of quasiinvariants has following decomposition*:

$$\mathbf{QI}_m = \bigoplus_{T \in ST(n)} \gamma_T(\mathbf{QI}_m).$$

The space $\gamma_T(\mathbf{QI}_m)$ *has following description*:

$$\gamma_T(\mathbf{QI}_m) = \gamma_T(K_n) \cap V_T^{2m+1} K_n. \tag{2.1}$$

For $\lambda \vdash n$, the vector space $\bigoplus_{T \in ST(\lambda)} \gamma_T(\mathbf{QI}_m)$ is called the λ-*isotypic component of* $\mathbf{QI}_m$.

Let K be a polynomial ring. We denote by $K[i]$ the subspace spanned by homogeneous polynomials of degree i in K. The Hilbert series of K is defined as a formal power series $\sum_{i=0}^{\infty} \dim(K[i]) t^i$. We denote it by $H(K, t)$.

For $[f] \in \mathbf{QI}_m^*$, we define the degree of $[f]$ as the minimal degree in the class $[f]$. In [4] and [7], the Hilbert series of $\mathbf{QI}_m^*$ is given as follows:

Theorem 2.5 ([4], [7]).

$$H(\mathbf{QI}_m^*, t) = n!\, t^{mn(n-1)/2} \sum_{\lambda \vdash n} \prod_{(i,j) \in \lambda} \prod_{k=1}^{n} t^{w(i,j;m)} \frac{1 - t^k}{h(i, j)(1 - t^{h(i,j)})} \tag{2.2}$$

where we set $w(i, j; m) = m(l(i, j) - a(i, j)) + l(i, j)$.

In particular, for $T \in ST(\lambda)$ *the Hilbert series of* $\gamma_T(\mathbf{QI}_m^*)$ *is given as follows*:

$$H(\gamma_T(\mathbf{QI}_m^*); t) = t^{mn(n-1)/2} \prod_{(i,j)\in\lambda} \prod_{k=1}^{n} t^{w(i,j;m)} \frac{1-t^k}{1-t^{h(i,j)}}. \tag{2.3}$$

Let $s_1, s_2, \dots, s_n$ be mutually distinct positive integers. We set $D = D(s_1, s_2; s_1, s_3, \dots, s_n)$. We define the following polynomial in $\mathbb{Q}[x_{s_1}, \dots, x_{s_n}]$:

$$Q_D^{l;m} = \int_{x_{s_1}}^{x_{s_2}} t^l \prod_{i=1}^{n} (t - x_{s_i})^m \, dt. \tag{2.4}$$

Recall that we define $\eta(n, k) = (n-k+1, 1^{k-1})$. In [2], J. Bandlow and G. Musiker found an explicit basis of $\gamma_T(\mathbf{QI}_m^*)$ when $T \in ST(\eta(n, 2))$.

Theorem 2.6 ([2]). *Let* $T \in ST(\eta(n, 2))$. *The set* $\{Q_T^{0;m}, Q_T^{1;m}, \dots, Q_T^{n-2;m}\}$ *is a basis of* $\gamma_T(\mathbf{QI}_m^*)$.

REMARK 2.7. In [2], it is shown that $Q_T^{l;m}$ is divisible by $V_T = (x_1 - x_j)^{2m+1}$. We can similarly show that $Q_D^{l;m}$ is divisible by $V_D = (x_{s_1} - x_{s_2})^{2m+1}$.

Let $f \in \mathbb{Q}[x_{s_1}, x_{s_2}, \dots, x_{s_n}]$. We denote by $\deg_{x_{s_i}}(f)$ the degree of f as the polynomial in x_{s_i}. The leading term of f in x_{s_i} means the highest term of f in x_{s_i} and the leading coefficient of f in x_{s_i} means the coefficient of the leading term of f in x_{s_i}. For a homogeneous polynomial g, we define $\deg(g)$ as the degree of g.

The polynomials $Q_D^{l;m}$ have the following properties, which we will use to show Proposition 3.3.

Proposition 2.8. *Let* $s_1, s_2, \dots, s_n$ *be mutually distinct positive integers. Let* l *be a non-negative integer and take a tableau* $D = D(s_1, s_2; s_1, s_3, \dots, s_n)$ *of shape* $\eta(n, 2)$.

The polynomial $Q_D^{l;m}$ *is a homogeneous polynomial of degree* $nm + l + 1$ *and satisfies following properties.*

(1) *The polynomial* $Q_D^{l;m}$ *is symmetric in* $x_{s_3}, \dots, x_{s_n}$ *and anti-symmetric in* x_{s_1}, x_{s_2}.

(2) *We have* $\deg_{x_{s_1}}(Q_D^{l;m}) = nm + l + 1$. *The leading coefficient of* $Q_D^{l;m}$ *in* x_{s_1} *is* $(-1)^{m+1} m! / \prod_{s=0}^{m} (mn + l + 1 - s)$.

(3) *Let* $i \in \{1, \dots, n\} \setminus \{1, 2\}$. *We have* $\deg_{x_{s_i}}(Q_D^{l;m}) = m$. *The leading coefficient of* $Q_D^{l;m}$ *in* x_{s_i} *is equal to* $(-1)^m Q_{D^{s_i}}^{l;m}$.

Proof. We show the case $D = T(1, 2)$ since the proofs of other cases are similar. We set $T = T(1, 2)$.

(1) It follows from the fact that $t^l \prod_{i=1}^n (t - x_i)^m$ is symmetric in $x_1, x_2, \ldots, x_n$.

(2) We show this statement by induction on m.

When $m = 0$, the polynomial $Q_T^{l;0}$ is $(1/(l+1))(x_j^{l+1} - x_1^{l+1})$. So, the statement holds.

When $m \geq 1$, assume that the statement holds for all numbers less than m. In [2], the polynomial $Q_T^{l;m}$ is expressed as:

$$Q_T^{l;m} = \sum_{i=0}^{n} (-1)^i e_i Q_T^{n+l-i;m-1}. \tag{2.5}$$

By the induction assumption on m, we have $\deg_{x_{s_1}}(Q_T^{n+l-i;m-1}) = nm + l - i + 1$. From (2.5), we have $\deg_{x_1}(Q_T^{l;m}) = nm + l + 1$ and the leading term is in $e_0 Q_T^{n+l;m-1} - e_1 Q_T^{n+l-1;m-1}$. The leading coefficient of $Q_T^{l;m}$ in x_1 is

$$\begin{aligned} &\frac{(-1)^m (m-1)!}{\prod_{s=0}^{m-1}(mn + l + 1 - s)} - \frac{(-1)^m (m-1)!}{\prod_{s=0}^{m-1}(mn + l - s)} \\ &= \frac{(-1)^{m+1} m!}{\prod_{s=0}^{m}(mn + l + 1 - s)}. \end{aligned}$$

(3) Expanding $(t - x_i)^m$ in $Q_T^{l;m}$, we have

$$Q_T^{l;m} = \sum_{s=0}^{m} (-1)^s \binom{m}{s} Q_{T^i}^{l;m} x_i^s.$$

Thus, the statement holds. □

As a corollary of this proposition, we have $Q_D^{l;m} \neq 0$ when D is a tableau of shape $\eta(n, 2)$.

3. A basis for the isotypic component of shape $(n - k + 1, 1^{k-1})$

We give a basis for the $\eta(n, k)$-isotypic component. Let $s_1, s_2, \ldots, s_n$ be mutually distinct positive integers. Throughout this section, we set $D = D(s_1, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$ and $T = T(1, 2, \ldots, k)$.

DEFINITION 3.1. (1) Let p be a non-negative integer. For i, j such that $1 \leq i < j \leq k$, we define a polynomial $R_{D;s_i,s_j}^{p;m}$ in $\mathbb{Q}[x_{s_1}, x_{s_2}, \ldots, x_{s_n}]$ as

$$R_{D;s_i,s_j}^{p;m} = \int_{x_{s_i}}^{x_{s_j}} t^p \prod_{l=1}^{n} (t - x_{s_l})^m \, dt. \tag{3.1}$$

(2) Let k be an integer such that $k \geq 2$. Take a partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ such that $\mu_1 > \mu_2 > \cdots > \mu_{k-1} \geq 0$. We define a polynomial $Q_D^{\mu;m}$ in $\mathbb{Q}[x_{s_1}, x_{s_2}, \ldots, x_{s_n}]$ as follows:

$$Q_D^{\mu;m} = \begin{vmatrix} R^{\mu_1;m}_{D;s_1,s_2} & R^{\mu_2;m}_{D;s_1,s_2} & \cdots & R^{\mu_{k-1};m}_{D;s_1,s_2} \\ R^{\mu_1;m}_{D;s_2,s_3} & R^{\mu_2;m}_{D;s_2,s_3} & \cdots & R^{\mu_{k-1};m}_{D;s_2,s_3} \\ \vdots & \vdots & \ddots & \vdots \\ R^{\mu_1;m}_{D;s_{k-1},s_k} & R^{\mu_2;m}_{D;s_{k-1},s_k} & \cdots & R^{\mu_{k-1};m}_{D;s_{k-1},s_k} \end{vmatrix}. \tag{3.2}$$

We denote the empty sequence by $\emptyset$. When $k = 1$, μ is the empty sequence $\emptyset$. We set $Q_D^{\emptyset;m} = 1$. We simply write Q_D^m as $Q_D^{\emptyset;m}$.

REMARK 3.2. Setting $D' = D(s_1, s_2; s_1, s_3, \ldots, s_n)$, we have $R^{p;m}_{D;s_1,s_2} = Q^{p;m}_{D'}$.

The polynomial $Q_D^{\mu;m}$ has the following properties, which we will use to show our main results.

Proposition 3.3. *Let $s_1, s_2, \ldots, s_n$ be mutually distinct positive integers. We set $D = D(s_1, \ldots, s_k; s_1, s_{k+1}, \ldots, s_n)$. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$ be a partition such that $\mu_1 > \mu_2 > \cdots > \mu_{k-1} \geq 0$.*

Then, the polynomial $Q_D^{\mu;m}$ satisfies the following.

(1) *The polynomial $Q_D^{\mu;m}$ is symmetric in $x_{s_{k+1}}, x_{s_{k+2}}, \ldots, x_{s_n}$ and anti-symmetric in $x_{s_1}, x_{s_2}, \ldots, x_{s_k}$. In particular, $Q_D^{\mu;m}$ is divisible by V_D^{2m+1}.*

(2) *We have $\deg_{x_{s_1}}(Q_D^{\mu;m}) = (n+k-2)m + \mu_1 + 1$. The leading coefficient of $Q_D^{\mu;m}$ in x_{s_1} is*

$$\frac{(-1)^{(k-1)m+1} m!}{\prod_{s=0}^{m}(mn + \mu_1 + 1 - s)} Q^{(\mu_2,\ldots,\mu_{k-1});m}_{D^{s_1}}.$$

In particular, we have $\deg(Q_D^{\mu;m}) = (k-1)nm + |\mu| + k - 1$.

(3) *We have $\deg_{x_{k+1}}(Q_D^{\mu;m}) = (k-1)m$. The leading coefficient of $Q_D^{\mu;m}$ in x_{k+1} is $(-1)^{(k-1)m} Q^{\mu;m}_{D^{s_{k+1}}}$.*

(4) *The polynomial $Q_D^{\mu;m}$ is invariant under γ_D.*

Proof. We show the case $D = T$. The proofs of other cases are similar.

(1) From Proposition 2.8 (1), it follows that the polynomial $Q_T^{\mu;m}$ is symmetric in $x_{k+1}, x_{k+2}, \ldots, x_n$.

Adding the first row to the second row, we get

$$Q_T^{\mu;m} = \begin{vmatrix} R^{\mu_1;m}_{T;1,2} & R^{\mu_2;m}_{T;1,2} & \cdots & R^{\mu_{k-1};m}_{T;1,2} \\ R^{\mu_1;m}_{T;1,3} & R^{\mu_2;m}_{T;1,3} & \cdots & R^{\mu_{k-1};m}_{T;1,3} \\ \vdots & \vdots & \ddots & \vdots \\ R^{\mu_1;m}_{T;k-1,k} & R^{\mu_1;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k} \end{vmatrix}.$$

Repeating this process, we get

$$
Q_T^{\mu;m} = \begin{vmatrix} R_{T;1,2}^{\mu_1;m} & R_{T;1,2}^{\mu_2;m} & \cdots & R_{T;1,2}^{\mu_{k-1};m} \\ R_{T;1,3}^{\mu_1;m} & R_{T;1,3}^{\mu_2;m} & \cdots & R_{T;1,3}^{\mu_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;1,k}^{\mu_1;m} & R_{T;1,k}^{\mu_2;m} & \cdots & R_{T;1,k}^{\mu_{k-1};m} \end{vmatrix}. \tag{3.3}
$$

Thus, the polynomial $Q_T^{\mu;m}$ is anti-symmetric in $x_2, \dots, x_k$. We can show that $Q_T^{\mu;m}$ is anti-symmetric in $x_1, x_3, \dots, x_k$ and $x_1, x_2, x_4 \cdots, x_k$ in similar ways. Thus the first statement holds.

From Remark 2.7 and (3.3), the polynomial $Q_T^{\mu;m}$ is divisible by $\prod_{s=2}^n (x_1 - x_s)^{2m+1}$. Using this proposition (1), we see $Q_T^{\mu;m}$ is also divisible by V_T^{2m+1}.

(2) We see $Q_T^{\mu;m}$ as a polynomial in x_1. From Proposition 2.8 (2), (3), the leading term of $Q_T^{\mu;m}$ in x_{s_1} is in $R_{T;1,2}^{\mu_1;m} R_{T;2,3}^{\mu_2;m} \cdots R_{T;k-1,k}^{\mu_k;m}$. We use Proposition 2.8 (2), (3) again, and the statement holds.

(3) From Proposition 2.8 (3), the leading coefficient of $Q_T^{\mu;m}$ in x_{k+1} is

$$
\begin{vmatrix} (-1)^m R_{T^{k+1};1,2}^{\mu_1;m} & (-1)^m R_{T^{k+1};1,2}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};1,2}^{\mu_k;m} \\ (-1)^m R_{T^{k+1};2,3}^{\mu_1;m} & (-1)^m R_{T^{k+1};2,3}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};2,3}^{\mu_k;m} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^m R_{T^{k+1};k-1,k}^{\mu_1;m} & (-1)^m R_{T^{k+1};k-1,k}^{\mu_2;m} & \cdots & (-1)^m R_{T^{k+1};k-1,k}^{\mu_k;m} \end{vmatrix}. \tag{3.4}
$$

The polynomial (3.4) is equal to $(-1)^{(k-1)m} Q_{T^{k+1}}^{\mu;m}$.

(4) To prove (4), we define the following notation.

For positive integers i, j such that $i \neq j$, we define a tableau $(i, j)D$ as follows. When $i, j \notin mem(D)$, we define $(i, j)D = D$. When $i \in mem(D)$ and $j \notin mem(D)$, $(i, j)D$ is a tableau obtained by replacing the entry i in D with j. When $i, j \in mem(D)$, $(i, j)D$ is a tableau obtained by interchanging the entry i and j in D.

Using Proposition 2.3, γ_T is equal to

$$
\frac{1}{n(n-k)!\,(k-1)!}\left\{1 - \sum_{s=2}^{k}(1, s)\right\}[S_{\{2,3,\dots,k\}}]'\left\{1 + \sum_{s=k+1}^{n}(1, s)\right\}[S_{\{k+1,\dots,n\}}].
$$

From (1), we obtain

$$
\gamma_T(Q_T^{\mu;m}) = \frac{1}{n}\left\{k Q_T^{\mu;m} + \sum_{s=k+1}^{n}\{1 - (1, 2) - \cdots - (1, k)\} Q_{(1,s)T}^{\mu;m}\right\}.
$$

We consider the sum $\sum_{s=k+1}^{n}\{1-(1,2)-\cdots-(1,k)\}Q^{\mu;m}_{(1,s)T}$. We have

$$\begin{aligned}
&\sum_{s=k+1}^{n}\{1-(1,2)-(1,3)-\cdots-(1,k)\}Q^{\mu;m}_{(1,s)T}\\
&=\sum_{s=k+1}^{n}\{Q^{\mu;m}_{(1,s)T}+Q^{\mu;m}_{(2,s)T}+Q^{\mu;m}_{(3,s)T}+\cdots+Q^{\mu;m}_{(k,s)T}\}.
\end{aligned}$$

Consider the sum $Q^{\mu;m}_{(1,s)T}+Q^{\mu;m}_{(2,s)T}$. By definition, we have

$$\begin{aligned}
&Q^{\mu;m}_{(1,s)T}+Q^{\mu;m}_{(2,s)T}\\
&=\begin{vmatrix}
R^{\mu_1;m}_{T;s,2} & R^{\mu_2;m}_{T;s,2} & \cdots & R^{\mu_{k-1};m}_{T;s,2}\\
R^{\mu_1;m}_{T;2,3} & R^{\mu_2;m}_{T;2,3} & \cdots & R^{\mu_{k-1};m}_{T;2,3}\\
\vdots & \vdots & \ddots & \vdots\\
R^{\mu_1;m}_{T;k-1,k} & R^{\mu_2;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k}
\end{vmatrix}
+\begin{vmatrix}
R^{\mu_1;m}_{T;1,s} & R^{\mu_2;m}_{T;1,s} & \cdots & R^{\mu_{k-1};m}_{T;1,s}\\
R^{\mu_1;m}_{T;s,3} & R^{\mu_2;m}_{T;s,3} & \cdots & R^{\mu_{k-1};m}_{T;s,3}\\
\vdots & \vdots & \ddots & \vdots\\
R^{\mu_1;m}_{T;k-1,k} & R^{\mu_2;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k}
\end{vmatrix}.
\end{aligned}$$

Adding the first row to the second row in the second determinant, we get

$$\begin{aligned}
&Q^{\mu;m}_{(1,s)T}+Q^{\mu;m}_{(2,s)T}\\
&=\begin{vmatrix}
R^{\mu_1;m}_{T;s,2} & R^{\mu_2;m}_{T;s,2} & \cdots & R^{\mu_{k-1};m}_{T;s,2}\\
R^{\mu_1;m}_{T;s,3} & R^{\mu_2;m}_{T;s,3} & \cdots & R^{\mu_{k-1};m}_{T;s,3}\\
\vdots & \vdots & \ddots & \vdots\\
R^{\mu_1;m}_{T;k-1,k} & R^{\mu_2;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k}
\end{vmatrix}
+\begin{vmatrix}
R^{\mu_1;m}_{T;1,s} & R^{\mu_2;m}_{T;1,s} & \cdots & R^{\mu_{k-1};m}_{T;1,s}\\
R^{\mu_1;m}_{T;s,3} & R^{\mu_2;m}_{T;s,3} & \cdots & R^{\mu_{k-1};m}_{T;s,3}\\
\vdots & \vdots & \ddots & \vdots\\
R^{\mu_1;m}_{T;k-1,k} & R^{\mu_2;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k}
\end{vmatrix}.
\end{aligned}$$

Adding the two terms, we obtain

$$\begin{aligned}
&Q^{\mu;m}_{(1,s)T}+Q^{\mu;m}_{(2,s)T}\\
&=\begin{vmatrix}
R^{\mu_1;m}_{T;1,2} & R^{\mu_2;m}_{T;1,2} & \cdots & R^{\mu_{k-1};m}_{T;1,2}\\
R^{\mu_1;m}_{T;s,3} & R^{\mu_2;m}_{T;s,3} & \cdots & R^{\mu_{k-1};m}_{T;s,3}\\
\vdots & \vdots & \ddots & \vdots\\
R^{\mu_1;m}_{T;k-1,k} & R^{\mu_2;m}_{T;k-1,k} & \cdots & R^{\mu_{k-1};m}_{T;k-1,k}
\end{vmatrix}.
\end{aligned}$$

Repeating this process, we get

$$\{1-(1,2)-(1,3)-\cdots-(1,k)\}Q^{\mu;m}_{(1,s)T}=Q^{\mu;m}_{T}.$$

Thus, the statement holds. □

As a corollary of this proposition, we have $Q^{\mu;m}_{T}\in\gamma_T(\mathbf{QI}_m)$ where $T\in ST(\eta(n,k))$. We introduce the following notations.

DEFINITION 3.4. Let s, t, u be non-negative integers. When $u \geq 1$, we set the subsets $P(s;t;u)$, $P(t;u)$ and $Q(s;t;u)$ of the set of partitions as:

$$\begin{aligned} P(s;t;u) &= \{\lambda \in \mathbb{Z}^u \mid |\lambda| = s,\ t \geq \lambda_1 > \lambda_2 > \cdots > \lambda_u \geq 0\}, \\ Q(s;t;u) &= P(s;t;u) \backslash P(s;t-1;u), \\ P(t;u) &= \bigcup_{s \geq 0} P(s;t;u). \end{aligned}$$

When $u = 0$, we set

$$\begin{aligned} P(0;t;0) &= \{\emptyset\}, \\ P(t;0) &= \{\emptyset\}. \end{aligned}$$

Let l be a positive integer. We set $P(l;t;0)$ as empty set.

We define $p(s;t;u) = \# P(s;t;u)$ and $q(s;t;u) = \# Q(s;t;u)$.

REMARK 3.5. Let $\mu \in P(n-2;k-1)$ (resp. $\mu \in \bigcup_{s \geq 0} Q(s;n-2;k-1)$). We have

$$\frac{(k-1)(k-2)}{2} \leq |\mu| \leq (k-1)(n-k) + \frac{(k-1)(k-2)}{2}$$

(resp. $n - 2 + (k-2)(k-3)/2 \leq |\mu| \leq (k-1)(n-k) + (k-1)(k-2)/2$).

We have the following proposition.

Proposition 3.6. *Let k be an integer such that $k \geq 2$.*

(1) *Let l be an integer such that $0 \leq l \leq n-k-1$. Then, we have*

$$p\left(l + \frac{(k-1)(k-2)}{2}; n-3; k-1\right) = p\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right).$$

(2) *Let l be an integer such that $l \geq n-k$. Then, we have*

$$\begin{aligned} &p\left(l + \frac{(k-1)(k-2)}{2}; n-2; k-1\right) \\ &= p\left(l + \frac{(k-1)(k-2)}{2}; n-3; k-1\right) \\ &\quad + p\left(l + k - n + \frac{(k-2)(k-3)}{2}; n-3; k-2\right). \end{aligned}$$

(3) *Let l be an integer such that $0 \leq l \leq k-2$. Then, we have*

$$\begin{aligned} &p\left((k-1)(n-k) + \frac{(k-1)(k-2)}{2} - l; n-2; k-1\right) \\ &= p\left((k-2)(n-k) + \frac{(k-2)(k-3)}{2} - l; n-3; k-2\right). \end{aligned}$$

Proof. (1) By definition, we have

$$\begin{aligned} &q\left(l+\frac{(k-1)(k-2)}{2};n-2;k-1\right)\\ &= p\left(l+\frac{(k-1)(k-2)}{2};n-2;k-1\right)-p\left(l+\frac{(k-1)(k-2)}{2};n-3;k-1\right). \end{aligned}$$

Therefore we show $q(l+(k-1)(k-2)/2;n-2;k-1)=0$.

We have $l+(k-1)(k-)/2 \leq n-k-1+(k-1)(k-2)/2 < n-2+(k-2)(k-3)/2$. From Remark 3.5, we have $Q(l+(k-1)(k-2)/2;n-2;k-1)=\emptyset$. Thus, the proposition follows.

(2) To prove (2), we show

$$\begin{aligned} &q\left(l+\frac{(k-1)(k-2)}{2};n-2;k-1\right)\\ &= p\left(l+k-n+\frac{(k-2)(k-3)}{2};n-3;k-2\right). \end{aligned}$$

Let $\mu=(j,\mu_2,\dots,\mu_k)\in Q(i;j;k)$. Then, we have $(\mu_2,\dots,\mu_k)\in Q(i-j;\mu_2;k-1)$. So, we get $Q(i;j;k)=\bigcup_{s=0}^{i-1} Q(i-j;s;k-1)$. Thus, we have

$$q\left(l+\frac{(k-1)(k-2)}{2};n-2;k-1\right)=\sum_{s=0}^{n-3} q\left(l+\frac{(k-1)(k-2)}{2}-n+2;s;k-2\right).$$

We have $l+(k-1)(k-2)/2-n+2 = l+k-n+(k-2)(k-3)/2$. So, we get

$$\begin{aligned} &q\left(l+\frac{(k-1)(k-2)}{2};n-2;k-1\right)\\ &= \sum_{s=0}^{n-3} q\left(l+k-n+\frac{(k-2)(k-3)}{2};s;k-2\right). \end{aligned}$$

By definition, we obtain

$$\begin{aligned} &\sum_{s=0}^{n-3} q\left(l+k-n+\frac{(k-2)(k-3)}{2};s;k-2\right)\\ &= p\left(l+k-n+\frac{(k-2)(k-3)}{2};n-3;k-2\right). \end{aligned}$$

(3) By definition, we have

$$\begin{aligned} &p\left((k-1)(n-k)+\frac{(k-1)(k-2)}{2}-l;n-2;k-1\right)\\ &= \sum_{s=0}^{n-2} q\left((k-1)(n-k)+\frac{(k-1)(k-2)}{2}-l;s;k-1\right). \end{aligned}$$

From Remark 3.5, we have $q((k-1)(n-k)+(k-1)(k-2)/2-l;s;k-1)=0$ when $s\le n-3$. Therefore, we obtain

$$p\left((k-1)(n-k)+\frac{(k-1)(k-2)}{2}-l;n-2;k-1\right)$$
$$=q\left((k-1)(n-k)+\frac{(k-1)(k-2)}{2}-l;n-2;k-1\right).$$

From (2), we have

$$q\left((k-1)(n-k)+\frac{(k-1)(k-2)}{2}-l;n-2;k-1\right)$$
$$=p\left((k-1)(n-k)+\frac{(k-2)(k-3)}{2}-l+k-n;n-3;k-2\right)$$
$$=p\left((k-2)(n-k)+\frac{(k-2)(k-3)}{2}-l;n-3;k-2\right). \qquad \square$$

We next consider the Hilbert series of $\gamma_T(\mathbf{QI}_m^*)$. To simplify notation, we write $p_{s,n-2,k-1}=p(s+(k-1)(k-2)/2;n-2;k-1)$.

Proposition 3.6 is rewritten as:

(1) $p_{l,n-3,k-1}=p_{l,n-2,k-1}$,

(2) $p_{l,n-2,k-1}=p_{l,n-3,k-1}+p_{l+k-n,n-3,k-2}$,

(3) $p_{(k-1)(n-k)-l,n-2,k-1}=p_{(k-2)(n-k)-l,n-3,k-2}$.

Lemma 3.7. *We have*

$$H(\gamma_T(\mathbf{QI}_m^*);t)=t^{(k-1)nm+k(k-1)/2}\sum_{s=0}^{(k-1)(n-k)}p_{s,n-2,k-1}t^s. \tag{3.5}$$

Proof. From (2.3), the Hilbert series $H(\gamma_T(\mathbf{QI}_m^*);t)$ is equal to

$$t^{mn(n-1)/2}\prod_{(i,j)\in\lambda}\prod_{l=1}^{n}t^{m(l(i,j)-a(i,j))+l(i,j)}\frac{1-t^l}{1-t^{h(i,j)}}.$$

For $2\le i\le n-k+1$ and $2\le j\le k$, we have

$$a(1,1)=n-k,\quad l(1,1)=k-1,\quad h(1,1)=n,$$
$$a(1,i)=n-k+1-i,\quad l(1,i)=0,\quad h(1,i)=n-k+2-i,$$
$$a(j,1)=0,\quad l(j,1)=k-j,\quad h(j,1)=k-j+1.$$

Thus, we have

$$H(\gamma_T(\mathbf{QI}_m^*); t) = t^{(k-1)nm+k(k-1)/2} \prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)}.$$

Therefore, we must show

$$\prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)} = \sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1} t^s. \tag{3.6}$$

We show this by induction on n.

If $n = k$, then both of l.h.s. and r.h.s. are equal to 1.

When $n \geq k+1$, we assume that (3.6) holds with all numbers less than n. We have the following identity:

$$\prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)} = \prod_{s=1}^{k-1} \frac{(1-t^{n-s-1})}{(1-t^s)} + t^{n-k} \prod_{s=1}^{k-2} \frac{(1-t^{n-s-1})}{(1-t^s)}.$$

By the induction assumption, we obtain

$$\begin{aligned} &\prod_{s=1}^{k-1} \frac{(1-t^{n-s-1})}{(1-t^s)} + t^{n-k} \prod_{s=1}^{k-2} \frac{(1-t^{n-s-1})}{(1-t^s)} \\ &= \sum_{s=0}^{(k-1)(n-k-1)} p_{s,n-3,k-1} t^s + t^{n-k} \sum_{s=0}^{(k-2)(n-k)} p_{s,n-3,k-2} t^s. \end{aligned}$$

We can rewrite this as

$$\begin{aligned} &\prod_{s=1}^{k-1} \frac{(1-t^{n-s})}{(1-t^s)} \\ &= \sum_{s=n-k}^{(k-1)(n-k-1)} (p_{s-n+k,n-3,k-2} + p_{s,n-3,k-1}) t^s \\ &\quad + \sum_{s=(k-1)(n-k)-k+2}^{(k-1)(n-k)} p_{s-n+k,n-3,k-2} t^s + \sum_{s=0}^{n-k-1} p_{s,n-3,k-1} t^s. \end{aligned}$$

Using Proposition 3.6 (2), we have

$$\begin{aligned} &\sum_{s=n-k}^{(k-1)(n-k-1)} (p_{s-n+k,n-3,k-2} + p_{s,n-3,k-1}) t^s \\ &= \sum_{s=n-k}^{(k-1)(n-k-1)} p_{s,n-2,k-1} t^s. \end{aligned}$$

From Proposition 3.6 (1) and (3), the lemma holds. □

We state the main theorem in this paper.

Theorem 3.8. *The set* $\{Q_T^{\mu;m}\}_{\mu\in P(n-2;k-1)}$ *is a basis of* $\gamma_T(\mathbf{QI}_m^*)$.

To simplify notation, we set

$$\begin{aligned}
P_{s,n-2,k-1} &= P\left(s+\frac{(k-1)(k-2)}{2};n-2;k-1\right),\\
P_{n-2,k-1} &= P(n-2;k-1),\\
Q_{s,n-2,k-1} &= Q\left(s+\frac{(k-1)(k-2)}{2};n-2;k-1\right).
\end{aligned}$$

We define following notations.

Let $X=\{s_1,s_2,\dots,s_n\}$ be the set of n positive integers. We recall that S_X is the symmetric group on X and S_X acts on $\mathbb{Q}[x_{s_1},x_{s_2},\dots,x_{s_n}]$ from the left.

We define Λ_X as the subspace of $\mathbb{Q}[x_{s_1},x_{s_2},\dots,x_{s_n}]$ spanned by all polynomials which is invariant under S_X. We define Λ_X^d as the subspace of Λ_X spanned by homogeneous polynomials of degree d. We define $\Lambda_X^d=\{0\}$ if $d<0$.

Theorem 3.8 follows from the following proposition.

Proposition 3.9. *Let D be a tableau of shape $\eta(n,k)$. If*

$$\sum_{\mu\in P(n-2;k-1)} f_\mu Q_D^{\mu;m}=0 \tag{3.7}$$

where $f_\mu\in\Lambda_{mem(D)}$, then all f_μ is equal to 0.

Proof. We show this proposition by induction on the size n of tableau D.

In the case $k=1$, (3.7) is $fQ_D^m=0$ where $f\in\Lambda_{mem(D)}$. Therefore, the proposition holds when $k=1$. We assume that $k\geq 2$.

We recall that $n\geq k$. When $n=2$, we have $k=2$. Then l.h.s. of (3.7) is equal to $f_0Q_D^{0;m}$. Therefore, the lemma holds when $n=2$.

Assume that (3.7) holds when the size of the tableau D is less than n for $n\geq 3$. We show the case $D=T$ since the proofs of other cases are similar.

We recall that Λ_n is a graded ring. Therefore, we can decompose

$$f_\mu=\sum_{l\geq 0} f_{\mu,l}$$

where $f_{\mu,l} \in \Lambda_n^l$. Thus, (3.7) is written as

$$\sum_{\mu \in P(n-2;k-1)} \sum_{l \geq 0} f_{\mu,l} Q_T^{\mu;m} = 0 \tag{3.8}$$

where $f_{\mu,l} \in \Lambda_n^l$. We have $\deg(Q_T^{\mu;m}) = (k-1)nm + |\mu| + k - 1$, and we obtain $\deg(f_{\mu,l} Q_T^{\mu;m}) = (k-1)nm + |\mu| + d + k - 1$.

Thus, (3.8) is written as

$$\sum_{d \geq 0} \sum_{\mu \in P(n-2;k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0. \tag{3.9}$$

Hence, for any d we obtain

$$\sum_{\mu \in P(n-2;k-1)} f_{\mu,d-(k-1)nm-|\mu|-k+1} Q_T^{\mu;m} = 0. \tag{3.10}$$

Fix d. Recall that the set $P_{s,n-2,k-1}$ is not the empty set if $0 \leq s \leq (k-1)(n-k)$. Let s be an integer such that $0 \leq s \leq (k-1)(n-k)$ and take $\mu \in P_{s,n-2,k-1}$. Then, we have $\deg(Q_T^{\mu;m}) = (k-1)nm + k(k-1)/2 + s$. We set $d' = d - (k-1)nm - k(k-1)/2$. We express $f_{\mu,d'-s}$ as

$$\sum_{r=0}^{d'-s} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^{\mu} e_{\nu}.$$

We recall that

$$\begin{aligned}
P_{s,n-2,k-1} &= P\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right), \\
P_{n-2,k-1} &= P(n-2; k-1), \\
Q_{s,n-2,k-1} &= Q\left(s + \frac{(k-1)(k-2)}{2}; n-2; k-1\right).
\end{aligned}$$

Therefore, (3.10) is written as

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{r=0}^{d'-s} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^{\mu} e_{\nu} Q_T^{\mu;m} = 0. \tag{3.11}$$

We show $a_{r,\nu}^{\mu} = 0$ for $r \geq 0$. We show this by induction on r. To prove this, we consider the leading terms in x_{k+1}.

As a polynomial in x_{k+1}, the degree of l.h.s. of (3.11) is $(k-1)m + d'$ and the leading term is in $a_{d',(1^{d'})}^{(k-2,k-3,\ldots,0)} e_{(1^{d'})} Q_T^{(k-2,k-3,\ldots,0);m}$. Hence we have $a_{d',(1^{d'})}^{(k-2,k-3,\ldots,0)} = 0$.

Using the following lemma, we complete the proof of Proposition 3.9.

Lemma 3.10. *Let k be an integer such that $k \geq 3$. We assume that for each integer l such that $2 \leq l \leq n-1$ and each tableau of shape $\eta(n-1, l)$, the statement of* Proposition 3.9 *holds.*

Let r an integer such that $1 \leq r \leq d'-1$. If we have the following equation:

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{i=0}^{r} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=i}} a_{i,\nu}^{\mu} e_{\nu} Q_T^{\mu;m} = 0, \tag{3.12}$$

then all constants $a_{r,\nu}^{\mu}$ are equal to 0.

Proof. We set

$$I = \sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{i=0}^{r} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=i}} a_{i,\nu}^{\mu} e_{\nu} Q_T^{\mu;m}.$$

From Proposition 3.3 (3), we have $\deg_{x_{k+1}}(I) = (k-1)m + r$. The leading term of I in x_{k+1} is in

$$\sum_{s=0}^{(k-1)(n-k)} \sum_{\mu \in P_{s,n-2,k-1}} \sum_{\substack{|\nu|=d'-s \\ l(\nu)=r}} a_{r,\nu}^{\mu} e_{\nu} Q_T^{\mu;m}.$$

Recall that we have $P_{s,n-2,k-1} = Q_{s,n-2,k-1} \cup P_{s,n-3,k-1}$ and this union is disjoint. Therefore, we can rewrite this as

$$\sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s,n-2,k-1}} \sum_{\substack{|\nu^{(1)}|=d'-s \\ l(\nu^{(1)})=r}} a_{r,\nu^{(1)}}^{\mu} e_{\nu^{(1)}} Q_T^{\mu;m}$$
$$+ \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s,n-3,k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} a_{r,\nu^{(2)}}^{\mu} e_{\nu^{(2)}} Q_T^{\mu;m}.$$

We set

$$I_1 = \sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu \in Q_{s,n-2,k-1}} \sum_{\substack{|\nu^{(1)}|=d'-s \\ l(\nu^{(1)})=r}} a_{r,\nu^{(1)}}^{\mu} e_{\nu^{(1)}} Q_T^{\mu;m},$$

$$I_2 = \sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu \in P_{s,n-3,k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} a_{r,\nu^{(2)}}^{\mu} e_{\nu^{(2)}} Q_T^{\mu;m}.$$

First, we show that the constants $a^{\mu}_{r,\nu}$ in I_1 are equal to 0.

If $r > d' - n + k$, we have $|\mu| < (k-1)(k-2)/2 + n - k$. On the other hand, if $\mu \in Q_{s,n-2,k-1}$, we have $|\mu| \geq (k-1)(k-2)/2 + n - k$. Therefore if $r > d' - n + k$, the sum in I_1 is empty. We only need to consider the case when $r \leq d' - n + k$.

We define the following notations. Let $X = \{s_1, s_2, \dots, s_n\}$ be the set of n positive integers. For a partition $\nu = (\nu_1, \nu_2, \dots)$, we define

$$\begin{aligned} e_{X,i} &= \sum_{1 \leq l_1 < \cdots < l_i \leq n} x_{s_{l_1}} \cdots x_{s_{l_i}}, \\ e_{X,\nu} &= \prod_i e_{X,\nu_i}, \\ e^{(s_j)}_{X,i} &= e_i(x_{s_1}, \dots, x_{s_{j-1}}, x_{s_{j+1}}, \dots, x_{s_n}), \\ e^{(s_j)}_{X,\nu} &= \prod_{s_i} e^{(j)}_{X,\nu_i}. \end{aligned}$$

In particular, if $X = \{1, 2, \dots, n\}$, then we simply write $e^{(j)}_{X,i}$ as $e^{(j)}_i$ and $e^{(j)}_{X,\nu}$ as $e^{(j)}_\nu$.

When $r \leq d' - n + k$, the leading term of I in x_1 is in I_1. For $\mu \in Q_{s,n-2,k-1}$, there exists $\mu' = (\mu'_1, \dots, \mu'_{k-2}) \in P_{n-3,k-2}$ such that $\mu = (n-2, \mu'_1, \dots, \mu'_{k-2})$. In particular, we have $\mu' \in P_{s+k-n,n-3,k-2}$. The leading coefficient of I_1 in x_1 is

$$\sum_{s=n-k}^{(k-1)(n-k)} \sum_{\mu' \in P_{s+k-n,n-3,k-2}} \sum_{\substack{|\nu^{(1)}| = d' - s \\ l(\nu^{(1)}) = r}} b^{\mu'}_{\nu^{(1)}} e^{(1)}_{\nu^{(1)} - (1^r)} Q^{\mu';m}_{T^1}$$

where we set $b_{\mu',\nu^{(1)}} = (-1)^{(k-1)m+1} m! / \prod_{s=0}^m (mn + n - 1 - s) a^{(n-2,\mu'_1,\dots)}_{r,\nu^{(1)}}$. We can rewrite this as

$$\sum_{s=0}^{(k-2)(n-k)} \sum_{\mu' \in P_{s,n-3,k-2}} \sum_{\substack{|\nu^{(1)}| = d' - s + k - n \\ l(\nu^{(1)}) = r}} b^{\mu'}_{\nu^{(1)}} e^{(1)}_{\nu^{(1)} - (1^r)} Q^{\mu';m}_{T^1}.$$

Since $e^{(1)}_{\nu^{(1)} - (1^r)} = e_{mem(T^1),\nu^{(1)} - (1^r)}$, this is rewritten as

$$\sum_{s=0}^{(k-2)(n-k)} \sum_{\mu' \in P_{s,n-3,k-2}} \sum_{\substack{|\nu^{(1)}| = d' - s + k - n \\ l(\nu^{(1)}) = r}} b^{\mu'}_{\nu^{(1)}} e_{mem(T^1),\nu^{(1)} - (1^r)} Q^{\mu';m}_{T^1}.$$

The shape of the tableau T^1 is $(n-k+1, 1^{k-2})$. Thus T^1 has $n-1$ boxes. By the induction assumption on n, all $b^{\mu'}_{\nu^{(1)}}$ are equal to 0. Thus we have $a^{(n-2,\mu'_1,\dots)}_{r,\nu^{(1)}} = 0$. So, we get $I_1 = 0$.

We next consider I_2. The leading coefficient of I_2 in x_{k+1} is

$$\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu\in P_{s,n-3,k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} c^{\mu}_{\nu^{(2)}} e^{(k+1)}_{\nu^{(2)}-(1^r)} Q^{\mu;m}_{T^{k+1}} \tag{3.13}$$

where we set $c^{\mu}_{\nu^{(2)}} = (-1)^{(k-2)m} a^{\mu}_{r,\nu^{(2)}}$.

Since $e^{(k+1)}_{\nu^{(2)}-(1^r)} = e_{mem(T^{k+1}),\nu^{(2)}-(1^r)}$, we can rewrite (3.13) as

$$\sum_{s=0}^{(k-1)(n-k-1)} \sum_{\mu\in P_{s,n-3,k-1}} \sum_{\substack{|\nu^{(2)}|=d'-s \\ l(\nu^{(2)})=r}} c^{\mu}_{\nu^{(2)}} e_{mem(T^{k+1}),\nu^{(2)}-(1^r)} Q^{\mu;m}_{T^{k+1}}.$$

The tableau T^{k+1} has $n-1$ boxes. By the induction assumption on n, all $c^{\mu}_{\nu^{(2)}}$ are equal to 0. Thus, all $a^{\mu}_{r,\nu}$ are equal to 0.

Thus, the lemma follows. Therefore, the proposition also follows. □

From Theorem 3.8 and Proposition 3.9, we obtain the following corollary.

Corollary 3.11. *Let $T \in ST(\eta(n,k))$. The space $\gamma_T(\mathbf{QI}_m)$ is a free module over Λ_n and the set $\{Q^{\mu;m}_T\}_{\mu\in P(n-2;k-1)}$ is a free basis.*

Proof. In this proof, we simply write $Q^{\mu;m}_T$ as Q^{μ}. Using Proposition 3.9, the set $\{Q^{\mu}\}$ is linearly independent over Λ_n.

Since $H(\gamma_T(\mathbf{QI}^*_m);t) = t^{(k-1)nm+k(k-1)/2}\sum_{s=0}^{(k-1)(n-k)} p_{s,n-2,k-1}t^s$, we have

$$\gamma_T(\mathbf{QI}_m) = \bigoplus_{d\geq (k-1)nm+k(k-1)/2} \gamma_T(\mathbf{QI}_m)[d].$$

Let d be a non-negative integer such that $d \geq (k-1)nm + k(k-1)/2$. We show that the subspace of $\gamma_T(\mathbf{QI}_m)[d]$ is generated by $\{Q^{\mu}\}$ over Λ_n by induction on d.

When $d = (k-1)nm + k(k-1)/2$, the coefficient of $t^{(k-1)nm+k(k-1)/2}$ in the polynomial $H(\gamma_T(\mathbf{QI}^*_m);t)$ is equal to 1. Therefore, $\gamma_T(\mathbf{QI}_m)[d]$ is a space spanned by $Q^{(k-2,k-1,\ldots,0)}$. Thus the statement follows when $d = (k-1)nm + k(k-1)/2$.

When $d \geq (k-1)nm + k(k-1)/2 + 1$, we assume that the statement holds with all numbers less than d. We denote by V the vector space over $\mathbb{Q}$ spanned by $\{Q^{\mu}\}_{\mu\in P(n-2;k-1)}$.

Take $f \in \gamma_T(\mathbf{QI}_m)[d]$. From Theorem 3.8, we can find $g \in V[d]$ such that $[f] = [g]$ in $\gamma_T(\mathbf{QI}^*_m)$. Thus, we have $f - g \in I_m$. This is expressed as

$$f - g = \sum_{s\geq 1} A_s u_s$$

where $A_s \in \Lambda_n^s$ and $u_s \in \gamma_T(QI_m)$.

Since $\gamma_T(QI_m)$ is a graded space, we can decompose $u_s = \sum_{i\geq 0} u_{s,i}$ where $u_{s,i} \in \gamma_T(QI_m)[i]$. We have $\deg(A_s u_{s,i}) = s + i$. Thus, we have

$$f - g = \sum_{l\geq 0} \sum_{s+i=l} A_s u_{s,i}.$$

Since $f - g \in \gamma_T(\mathbf{QI}_m)[d]$, we get $\sum_{l\neq d} \sum_{s+i=l} A_s u_{s,i} = 0$. Therefore, we have

$$f - g = \sum_{s\geq 1} A_s u_{s,d-s}.$$

The polynomial A_s has the degree at least 1. So, the polynomial $u_{s,d-s}$ has the degree less than d. By the induction assumption, $u_{s,d-s}$ can be expressed as

$$u_{s,d-s} = \sum_l B_l v_l$$

where $B_l \in \Lambda_n$ and $v_l \in V$. Thus, the statement follows. $\square$

4. The operator L_m

The operator L_m is defined as

$$L_m = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - 2m \sum_{1\leq i<j\leq n} \frac{1}{x_i - x_j}\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right).$$

This operator is discussed in [4] and [5]. It is related to the quasiinvariants. In [5], Feigin and Veselov showed that the operator L_m preserves $\mathbf{QI}_m$. We consider how L_m acts on our polynomial $Q_T^{\mu;m}$. In [2], for $T(1,2)$ Bandlow and Musiker showed the following formulas for the action of L_m.

Theorem 4.1 ([2]). *Let k, m be non-negative integers.*
Then, we have $L_m(Q_{T(1,2)}^{k;m}) = k(k-1)Q_{T(1,2)}^{k-2;m}$ for $k \geq 2$ and $L_m(Q_{T(1,2)}^{k;m}) = 0$ for $k = 0, 1$.

We extend these formulas. We set $T = T(1, 2, \ldots, k)$. To write formulas simply, we define the following polynomials.

DEFINITION 4.2. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$.

We define a polynomial $Q_T^{\alpha;m}$ as follows:

$$Q_T^{\alpha;m} = \begin{vmatrix} R_{T;1,2}^{\alpha_1;m} & R_{D;1,2}^{\alpha_2;m} & \cdots & R_{T;1,2}^{\alpha_{k-1};m} \\ R_{T;2,3}^{\alpha_1;m} & R_{T;2,3}^{\alpha_2;m} & \cdots & R_{T;2,3}^{\alpha_{k-1};m} \\ \vdots & \vdots & \ddots & \vdots \\ R_{T;k-1,k}^{\alpha_1;m} & R_{T;k-1,k}^{\alpha_2;m} & \cdots & R_{T;k-1,k}^{\alpha_{k-1};m} \end{vmatrix} \tag{4.1}$$

when $\alpha_i \geq 0$, $i = 1, \ldots, k-1$. Otherwise we define $Q_T^{\alpha;m} = 0$.

REMARK 4.3. If α is a partition, $Q_T^{\alpha;m}$ is equal to a polynomial defined in Definition 3.1 (2). If $\alpha \in \mathbb{Z}_{\geq 0}^{k-1}$, $Q_T^{\alpha;m}$ is equal to $Q_T^{\mu;m}$ up to a sign where μ is a partition sorted α.

We obtain the following formulas for the action of L_m. To write the formula simply, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$ we define

$$\alpha^{(i,j)} = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_{j-1}, \alpha_j - 1, \alpha_{j+1}, \ldots, \alpha_n).$$

Theorem 4.4. *Let $\alpha = (\alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{Z}^{k-1}$ and take $T \in ST(\eta(n, k))$. Then we have*

$$\begin{aligned} L_m(Q_T^{\alpha;m}) = \sum_{i=1}^{n} \alpha_i(\alpha_i - 1) Q_T^{(\alpha_1,\ldots,\alpha_i-2,\ldots,\alpha_n);m} \\ + 2m \sum_{1\leq i<j\leq k-1} \Biggl(-\alpha_j Q_T^{\alpha^{(i,j)};m} \\ + \sum_{\substack{\alpha_i-2\geq s>t\geq 0 \\ s+t=\alpha_i+\alpha_j-2}} (s-t) Q_T^{(\alpha_1,\ldots,\alpha_{i-1},s,\alpha_{i+1},\cdots,\alpha_{j-1},t,\alpha_{j+1},\ldots,\alpha_n);m} \Biggr). \end{aligned}$$

This follows from following lemma. We define a polynomial $R_{T;1,2,3}^{s,t;m}$ as

$$R_{T;1,2,3}^{s,t;m} = \begin{vmatrix} R_{T;1,2}^{s;m} & R_{T;1,2}^{t;m} \\ R_{T;2,3}^{s;m} & R_{T;2,3}^{t;m} \end{vmatrix}.$$

Lemma 4.5. (1) *We have*

$$L_m(fg) = L_m(f)g + fL_m(g) + 2\sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i} f\right)\left(\frac{\partial}{\partial x_i} g\right).$$

(2) *Let k be a non-negative integer and m be a positive integer. Then, we have*

$$k \int_{x_i}^{x_j} t^{k-1} \prod_{s=1}^{n} (t - x_s)^m \, dt = -m \sum_{r=1}^{n} \int_{x_i}^{x_j} t^k (t - x_r)^{m-1} \prod_{s \neq r} (t - x_s)^m \, dt.$$

(3) *Let k, l be non-negative integers such that $k > l$. Then we have*

$$\begin{aligned}&\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}R^{k;m}_{T;1,2}\right)\left(\frac{\partial}{\partial x_i}R^{l;m}_{T;1,3}\right)-\left(\frac{\partial}{\partial x_i}R^{k;m}_{T;1,3}\right)\left(\frac{\partial}{\partial x_i}R^{l;m}_{T;1,2}\right)\\ &= m\left(-lR^{k-1,l-1;m}_{T;1,2,3}+\sum_{\substack{k-2\geq s>t\geq 0\\ s+t=k+l-2}}(s-t)R^{s,t;m}_{T;1,2,3}\right).\end{aligned}\tag{4.2}$$

Proof. (1) It follows from Leibniz's rule.

(2) It follows from the following identity:

$$\int_{x_i}^{x_j}\frac{\partial}{\partial t}t^k\prod_{s=1}^{n}(t-x_s)^m\,dt = 0.$$

(3) When $m = 0$, it follows from $R^{k;m}_{T;1,2} = (x_2^{k+1} - x_1^{k+1})/(k+1)$. We consider the case $m \geq 1$.

We show this formula by induction on $k - l$. We define $f(t, x) = \prod_{s=1}^{n}(t - x_s)^m$ and $f_i(t, x) = (t - x_i)^{m-1}\prod_{s\neq i}(t - x_s)^m$.

When $k - l = 1$, l.h.s. of (4.2) is equal to

$$\begin{aligned}&m^2\sum_{i=1}^{n}\int_{x_1}^{x_2}t^k f_i(t,x)\,dt\int_{x_1}^{x_3}u^{k-1}f_i(u,x)\,du\\ &-m^2\sum_{i=1}^{n}\int_{x_1}^{x_3}t^k f_i(t,x)\,dt\int_{x_1}^{x_2}u^{k-1}f_i(u,x)\,du.\end{aligned}$$

So, this is equal to

$$\begin{aligned}&m^2\sum_{i=1}^{n}\int_{x_1}^{x_2}t^{k-1}\{(t-x_i)+x_i\}f_i(t,x)\,dt\int_{x_1}^{x_3}u^{k-1}f_i(u,x)\,du\\ &-m^2\sum_{i=1}^{n}\int_{x_1}^{x_3}t^{k-1}\{(t-x_i)+x_i\}f_i(t,x)\,dt\int_{x_1}^{x_2}u^{k-1}f_i(u,x)\,du\\ &=m^2\sum_{i=1}^{n}\int_{x_1}^{x_2}t^{k-1}f(t,x)\,dt\int_{x_1}^{x_3}u^{k-1}f_i(u,x)\,du\\ &\quad-m^2\sum_{i=1}^{n}\int_{x_1}^{x_3}t^{k-1}f(t,x)\,dt\int_{x_1}^{x_2}u^{k-1}f_i(u,x)\,du.\end{aligned}$$

Using (2), we have

$$\text{l.h.s. of (4.2)} = -m(k-1)R^{k-1,k-2;m}_{T;1,2,3}.$$

We consider the case $k-l=2$. Calculating it in the same way, we have

$$\begin{aligned}\text{l.h.s. of (4.2)} = &-m(k-2)R_{T;1,2,3}^{k-1,k-3;m}\\ &+ m^2\sum_{i=1}^{n}\int_{x_1}^{x_2} t^{k-1} f_i(t,x)\,dt\int_{x_1}^{x_3} x_i u^{k-2} f_i(u,x)\,du\\ &- m^2\sum_{i=1}^{n}\int_{x_1}^{x_3} t^{k-1} f_i(t,x)\,dt\int_{x_1}^{x_2} x_i u^{k-2} f_i(u,x)\,du.\end{aligned}$$

From $x_i = u-(u-x_i)$, we get

$$\begin{aligned}\text{l.h.s. of (4.2)} = &-m(k-2)R_{T;1,2,3}^{k-1,k-3;m}\\ &+ m^2\sum_{i=1}^{n}\int_{x_1}^{x_2} t^{k-1} f_i(t,x)\,dt\int_{x_1}^{x_3} \{u-(u-x_i)\}u^{k-2} f_i(u,x)\,du\\ &- m^2\sum_{i=1}^{n}\int_{x_1}^{x_3} t^{k-1} f_i(t,x)\,dt\int_{x_1}^{x_2} \{u-(u-x_i)\}u^{k-2} f_i(u,x)\,du.\end{aligned}$$

It is equal to $-m(k-2)R_{T;1,2,3}^{k-1,k-3;m}$. Thus the statement holds when $k-l=2$.

When $k-l\geq 3$, we assume that the formula (4.2) holds with all numbers less than $k-l$. Calculating l.h.s. of (4.2) in the same way, we have

$$\begin{aligned}&\text{l.h.s. of (4.2)}\\ &= -mlR_{T;1,2,3}^{k-1,l-1;m} + m(k-1)R_{T;1,2,3}^{k-2,l;m}\\ &\quad+\sum_{i=1}^{n}\left(\frac{\partial}{\partial x_i}R_{T;1,2}^{k-1;m}\right)\left(\frac{\partial}{\partial x_i}R_{T;1,3}^{l+1;m}\right)-\left(\frac{\partial}{\partial x_i}R_{T;1,3}^{k-1;m}\right)\left(\frac{\partial}{\partial x_i}R_{T;1,2}^{l+1;m}\right).\end{aligned}$$

Hence the formula (4.2) holds by the induction assumption, and the statement has been proved. □

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