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THE DONNELLY-FEFFERMAN THEOREM ON *q*-PSEUDOCONVEX DOMAINS

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Abstract

In this paper we introduce a notion of q-subharmonicity for non-smooth functions and then using q-subharmonic exhaustion function, define a q-pseudoconvexity which is applicable to the domain with non-smooth boundary. Among others, we generalize the Donnelly-Fefferamn type theorem on q-pseudoconvex domains and as an application of this theorem, we give approximation theorem for $\bar\partial$-closed forms.

1. q-subharmonic functions and q-pseudoconvex domains

For a real valued C^2 function φ defined on $U \subset \mathbb{C}^n$, Lop-Hing Ho [5] first defined q-*subharmonicity* of φ on U and using this q-subharmonic function, he introduce the notion of *weak q-convexity* for domains with smooth boundaries. In this paper, first we investigate a natural extension of these notions to the class of upper semicontinuous functions and domains with non-smooth boundaries. After that, we deal with L^2-estimate for the $\bar\partial$-equation on this domain, which is essentially Donnelly-Fefferman theorem [3, 1, 2] in case the domain is pseudoconvex.

DEFINITION 1.1. Let φ be an upper semicontinuous function on U. Then we say that φ is *q-subharmonic* on U if for every q-complex dimension space H and for every compact set $K \subset H \cap U$, the following holds: if h is a continuous harmonic function on K and $h \leq \varphi$ on ∂K, then $h \leq \varphi$ on K.

One of the most typical examples of q-subharmonic function which is not plurisubharmonic is $-\sum_{j=1}^{q-1}|z_j|^2+(q-1)\sum_{j=q}^{n}|z_j|^2$. Also, note that an upper semicontinuous function on U is plurisubharmonic exactly when it is 1-subharmonic and q-subharmonicity implies q'-subharmonicity whenever $q \leq q'$ and an n-subharmonic function is just subharmonic function in usual sense. Before listing some properties of q-subharmonic function, we emphasize that q-plurisubharmonicity is a different notion: a C^2 smooth function u on U is called *q-plurisubharmonic* if its complex Hessian has at least $(n-q)$ non-negative eigenvalues at each point of U. Also, there is a parallel notion of q-plurisubharmonicity for upper semicontinuous functions (for example, see [4]).

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To approximate non-smooth q-subharmonic function by smooth q-subharmonic function, we define a mollifier $\rho_\epsilon(z) = \rho(z/\epsilon)/|\epsilon|^{2n}$, where ρ is a non-negative smooth radial function in $\mathbb{C}^n$ vanishing outside the unit ball and satisfying $\int_{\mathbb{C}^n} \rho\, dV = 1$. Here dV stands for the standard Lebesgue measure. We now list basic properties of q-subharmonic function.

Proposition 1.2. *Let U be an open set of $\mathbb{C}^n$ and $1 \le q \le n$. Then the following hold*:

(1) *If φ is q-subharmonic in U, then φ is subharmonic in U.*

(2) *If φ is q-subharmonic in U, then $u * \rho_\epsilon$ is smooth q-subharmonic in U_ϵ. Moreover, $u * \rho_\epsilon \searrow u$ when $\epsilon \to 0$. Here $U_\epsilon = \{z \in U : \epsilon < \operatorname{dist}(z, bU)\}$.*

(3) *In general, the set of q-subharmonic functions in U is not invariant under holomorphic maps, but in variant under unitary change of coordinates.*

(4) *If χ is a convex increasing function and φ is q-subharmonic in U, then $\chi \circ \varphi$ is φ is q-subharmonic in U.*

(5) *Let $\varphi \in C^2(U)$. Then the q-subharmonicity of φ is equivalent to*

$$\sum_{|K|=q-1}{}' \sum_{j,k} \varphi_{j\bar{k}} \alpha_{jK} \overline{\alpha}_{kK} \ge 0 \quad \textit{for all } q\textit{-forms} \quad \alpha = \sum_{|J|=q}{}' \alpha_J\, d\overline{z}^J, \tag{1.1}$$

where $\sum'$ denotes summation over strictly increasing multi-indices.

Proof. (1) is obvious. For the proof of (3), (4), and (5), see [5, 4]. Since φ is subharmonic in U, we see (2) except the q-subharmonicity of $u * \rho_\epsilon$. To see this, let H be q-dimensional complex subspace in $\mathbb{C}^n$. By (2), q-subharmonicity is invariant under the unitary change of coordinates. Hence we may assume that $H = \{(z', 0) \colon z = (z', z'') \in \mathbb{C}^n\}$, where $z' \in \mathbb{C}^q$ and $z' \in \mathbb{C}^{n-q}$. Since u is q-subharmonic, $u(z', 0)$ is subharmonic in H. Hence $u * \rho_\epsilon(\,\cdot\,, 0)$ is subharmonic in U_ϵ, i.e., $u * \rho_\epsilon(\,\cdot\,, 0)$ is q-subharmonic in U_ϵ. □

We also say that $\varphi \in C^2(U)$ is *strictly q-subharmonic* if φ satisfies (1.1) with strict inequality. With this q-subharmonicity, we define the following q-pseudoconvexity for domains so that 1-pseudoconvexity exactly coincides with pseudoconvexity in usual sense.

DEFINITION 1.3. Let D be an open set in $\mathbb{C}^n$. Then D is called *q-pseudoconvex* if there is a q-subharmonic exhaustion function for D.

Note that D is pseudoconvex if and only if it is 1-pseudoconvex, since 1-subharmonic function is just plurisubharmonic. Also, we say that D is *strictly q-pseudoconvex* if the boundary of D, bD is of C^2-class and its defining function is strictly q-subharmonic. Now we mention some elementary properties of q-pseudoconvex domains as an independent remark.

REMARK 1.4. Let D be q-pseudoconvex, $1 \le q \le n$. Then the following hold:
(1) If bD is of C^2-class, then by (1.1), D is weakly q-convex in the sense of L.-H. Ho [5].
(2) If $q \le q'$, then q-pseudoconvexity implies q'-pseudoconvexity.
(3) D has a C^∞-smooth strictly q-subharmonic exhaustion function, more precisely there are strictly q-pseudoconvex domains, D_ν's, $\nu = 1, 2, \ldots,$ satisfying

$$D = \bigcup_{\nu=1}^{\infty} D_\nu, \quad D_\nu \subset\subset D_{\nu+1} \subset\subset D. \tag{1.2}$$

Proof. For (1), we refer to [5]. From the property of q-subharmonicity, (2) is clear. We prove (3). Let φ be a q-subharmonic exhaustion function for D and $U_j = \{\varphi(z) < j\}$. Note that $U_j \nearrow D$ as $j \to \infty$. By Sard's theorem, we can find a decreasing sequence $\{\epsilon_j\}$ with $\lim_{j\to\infty} \epsilon_j = 0$ and two increasing sequences $\{a_j\}$, $\{b_j\}$ with $\lim_{j\to\infty} a_j = \infty$, $\lim_{j\to\infty} b_j = \infty$ such that for every j,
(a) $U_j \subset D_j := \{z \in D \colon u * \rho_\epsilon(z) + |z|^2/a_j < b_j\}$;
(b) $U_j \cup D_j \subset\subset D_{j+1}$;
(c) each D_j has smooth boundary. $\square$

Even though the domain is not pseudoconvex, we have the following Donnely-Fefferman type theorem [3] for the $\bar\partial$-equation on q-pseudoconvex domains.

Theorem 1.5. *Let D be a q-pseudoconvex domain in $\mathbb{C}^n$ and let φ be a given q-subharmonic function in D. Let $\psi \in C^2(D)$ be strictly plurisubharmonic and $-e^{-\psi}$ be also q-subharmonic. Let $0 < \varepsilon < 1$. Then for every $\bar\partial$-closed $(0, r)$-form g, $q \le r \le n$, there is a solution u of the equation $\bar\partial u = g$ such that*

$$\int_D |u|^2 e^{-\varphi+\varepsilon\psi}\, dV \le \frac{4}{\varepsilon(1-\varepsilon)^2} \cdot \frac{1}{r} \sum_{|K|=r-1}{}' \sum_{j,k} \int_D \psi^{j\bar k} g_{jK} \bar g_{kK} e^{-\varphi+\varepsilon\psi}\, dV, \tag{1.3}$$

whenever the right had side of (1.3) is bounded. Here $(\psi^{j\bar k}) = (\psi_{\nu\bar\mu})^{-1}$.

If ψ has the form, $\psi = -\log(-v)$, where v is a negative q-subharmonic function in D, then $-e^{-\psi}$ is q-subharmonic. This is the typical example that satisfies the assumption on ψ of Theorem 1.5. Note that $-e^{-\psi}$ is q-subharmonic means that

$$\sum_{|K|=q-1}{}' \sum_{j,k} \psi_j(z)\psi_{\bar k}(z) a_{jK} \bar a_{kK} \le \sum_{|K|=q-1}{}' \sum_{j,k} \psi_{j\bar k}(z) a_{jK} \bar a_{kK} \tag{1.4}$$

for every $(0, q)$-form $a = \sum'_{|J|=q} a_J\, d\bar z^J$ in D. In fact, (1.4) holds for any $(0, r)$, $r \ge q$ forms in D, since q-subharmonicity implies r-subharmonicity.

This kind of L^2 existence and estimate for the $\bar\partial$-equation have been thoroughly studied on strictly pseudoconvex domains by Donnely-Fefferman [3] and on general pseudoconvex domains by Berndtsson [1] and Błocki [2]. Actually, in order to prove the estimate like Błocki [2], plugging $\psi = \tilde\psi/\varepsilon$ into (1.3), we obtain the following

$$\int_D |u|^2 e^{-\varphi+\tilde\psi}\, dV \le \frac{4}{(1-\varepsilon)^2}\cdot\frac{1}{r}\sum_{|K|=r-1}{}^{\prime}\sum_{j,k}\int_D \tilde\psi^{j\bar k} g_{jK}\bar g_{kK} e^{-\varphi+\tilde\psi}\, dV,$$

whenever $\tilde\psi$ is strictly plurisubharmonic and $-e^{-\tilde\psi/\varepsilon}$ is q-subharmonic in D.

We end this section stating an approximation theorem for $\bar\partial$-closed forms as one application of Theorem 1.5. In particular, if D is 1-pseudoconvex, i.e., pseudoconvex, then this corollary corresponds to the approximation for holomorphic functions in L^2-norm.

Corollary 1.6. *Let D be a q-pseudoconvex domain in $\mathbb{C}^n$ and h a continuous q-subharmonic function in D. Assume that $K = \{z \in D\colon h(z) \le 0\} \subset\subset D$. If $\bar\partial$-closed $(0,r)$-form f, $r \ge q-1$ is smooth in a neighborhood of K, then for each $\delta > 0$, there is a $\bar\partial$-closed $(0,r)$-form g_δ whose coefficients are in $L^2(D)$ and satisfying*

$$\|f - g_\delta\|_{L^2(K)} < \delta.$$

2. Bochner identity

In this section we first prove the following Bochner identity for differential forms: for any C^2 real valued function φ and smooth $(0,r)$-form $\alpha = \sum'_{|J|=r}\alpha_J\, d\bar z^J$, we have

$$\begin{aligned}
&\sum_{|K|=r-1}{}^{\prime}\sum_{j,k}\frac{\partial^2}{\partial z_j\,\partial\bar z_k}(\alpha_{jK}\bar\alpha_{kK}e^{-\varphi})\\
&= -2\,\mathrm{Re}\langle\alpha, \overline{\bar\partial\bar\partial^*_\varphi\alpha}\rangle e^{-\varphi} + \sum_{|K|=r-1}{}^{\prime}\sum_{j,k}\varphi_{j\bar k}\alpha_{jK}\bar\alpha_{kK}e^{-\varphi}\\
&\quad + \sum_{|K|=r-1}{}^{\prime}\sum_{j,k}\Big[(\bar\partial_k\alpha_{jK})(\overline{\bar\partial_j\alpha_{kK}})e^{-\varphi} + (\delta^\varphi_j\alpha_{jK})(\overline{\delta^\varphi_k\alpha_{kK}})e^{-\varphi}\Big].
\end{aligned}\tag{2.5}$$

Here $\langle\ ,\ \rangle$ be an inner product induced by a standard Hermitian metric in $\mathbb{C}^n$ and this inner product can be naturally extended to differential forms. Also, here $\bar\partial^*_\varphi$ denotes the formal adjoint of $\bar\partial$-operator in $L^2(e^{-\varphi})$ and for C^1 function v, $\bar\partial_j v$ and $\delta^\varphi_j v$ is defined by

$$\bar\partial_j v = \frac{\partial v}{\partial\bar z_j},\quad \delta^\varphi_j v = e^\varphi\frac{\partial}{\partial z_j}(e^{-\varphi}v).$$

Note that the following two equalities hold:

$$\overline{\partial}\alpha = \sum_{|J|=r}{}' \sum_{j=1}^{n} \overline{\partial}_j \alpha_J \, d\overline{z}^J, \quad \overline{\partial}^*_\varphi \alpha = -\sum_{|K|=r-1}{}' \sum_{j=1}^{n} \delta^\varphi_j \alpha_{jK} \, d\overline{z}^K$$

for smooth $(0,r)$-form $\alpha = \sum'_{|J|=r} \alpha_J \, d\overline{z}^J$. Then (2.5) can be easily obtained by the direct calculation of the left hand side of (2.5). Let $|\alpha|^2 = \langle \alpha, \alpha \rangle$. Then, since

$$|\overline{\partial}\alpha|^2 e^{-\varphi} = \sum_{|J|=r}{}' \sum_{j=1}^{n} |\overline{\partial}_j \alpha_J|^2 e^{-\varphi} - \sum_{|K|=r-1}{}' \sum_{j,k} \langle \overline{\partial}_k \alpha_{jK}, \overline{\partial}_j \alpha_{kK} \rangle e^{-\varphi}$$

and

$$|\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} = \sum_{|K|=r-1}{}' \sum_{j,k} \langle \delta^\varphi_j \alpha_{jK}, \delta^\varphi_k \alpha_{kK} \rangle e^{-\varphi}, \tag{2.6}$$

we can rewrite (2.5) as

$$\begin{aligned} &\sum_{|K|=r-1}{}' \sum_{j,k} \frac{\partial^2}{\partial z_j \, \partial \overline{z}_k} (\alpha_{jK} \overline{\alpha}_{kK} e^{-\varphi}) \\ &= -2\,\mathrm{Re}\langle \alpha, \overline{\partial\overline{\partial}^*_\varphi \alpha} \rangle e^{-\varphi} + \sum_{|K|=r-1}{}' \sum_{j,k} \varphi_{j\overline{k}} \alpha_{jK} \overline{\alpha}_{kK} e^{-\varphi} \\ &\quad + \sum_{|J|=r}{}' \sum_{j=1}^{n} |\overline{\partial}_j \alpha_J|^2 e^{-\varphi} - |\overline{\partial}\alpha|^2 e^{-\varphi} + |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi}. \end{aligned} \tag{2.7}$$

The Bochner identity (2.7) for smooth $(0,1)$-forms can be found in [1].

Next, multiplying both sides of (2.7) by a smooth function w and integrating it over D, we obtain the following Bochner-Kodaira identity.

Lemma 2.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary and ρ its defining function of Ω. If $w, \varphi \in C^\infty(\overline{\Omega})$, $\alpha \in C^\infty_{(0,r)}(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^*)$, i.e., α is a* $(0,r)$-*form* $(1 \le r \le n)$ *which is smooth up to the boundary and satisfies the $\overline{\partial}$-Neumann boundary conditions on $b\Omega$,*

$$\sum_{j=1}^{n} \rho_j \cdot \alpha_{jK} = 0 \quad \text{on} \quad b\Omega \quad \text{for all} \quad K,$$

then we have

$$
\begin{aligned}
&2\,\mathrm{Re}\int_\Omega w\langle \overline{\partial\bar\partial^*_\varphi\alpha}, \alpha\rangle e^{-\varphi} + \int_\Omega w|\bar\partial\alpha|^2 e^{-\varphi} - \int_\Omega w|\bar\partial^*_\varphi\alpha|^2 e^{-\varphi} \\
&= \sum_{|K|=r-1}{}' \sum_{j,k}\left[\int_\Omega w\varphi_{j\bar k}\alpha_{jK}\bar\alpha_{kK}e^{-\varphi} - \int_\Omega w_{j\bar k}\alpha_{jK}\bar\alpha_{kK}e^{-\varphi}\right] \\
&\quad + \sum_{|J|=r}{}' \sum_{j=1}^n \int_\Omega w|\bar\partial_j\alpha_J|^2 e^{-\varphi} + \sum_{|K|=r-1}{}'\sum_{j,k}\int_{b\Omega} w\rho_{j\bar k}\alpha_{jK}\bar\alpha_{kK}e^{-\varphi}.
\end{aligned}
\tag{2.8}
$$

Here we omitted the standard volume form dV.

In [1], Berndtsson also proved (2.8) for for smooth (0, 1)-forms α. Though the proof of Lemma 2.1 is essentially same, for the convenience, hereunder, we give a brief verification.

Proof of Lemma 2.1. From now on, for the simplification of notation, we will omit the notation dV. Multiply the left hand side of (2.7) by w. Then we have to calculate the following integration over Ω,

$$I = \sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega w\frac{\partial^2}{\partial z_j\,\partial \bar z_k}(\alpha_{jK}\bar\alpha_{kK}e^{-\varphi}).$$

We may assume that $|\partial\rho| = 1$ on $b\Omega$. Then twice integration by parts give

$$
\begin{aligned}
I = &\sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega w_{j\bar k}\alpha_{jK}\bar\alpha_{kK}e^{-\varphi} + \sum_{|K|=r-1}{}' \sum_{j,k}\int_{b\Omega} w_j\alpha_{jK}\bar\alpha_{kK}e^{-\varphi}\rho_{\bar k}\,dS \\
&+ \sum_{|K|=r-1}{}' \sum_{j,k}\int_{b\Omega} w\bar\partial_k(\alpha_{jK}\bar\alpha_{kK}e^{-\varphi})\rho_j\,dS,
\end{aligned}
$$

where dS is the volume measure of $b\Omega$. Because of the $\bar\partial$-Neumann boundary conditions, the second integration of the right hand side of the above equality vanishes. Hence we have to evaluate the third integration of the right hand side of the above equality. Again, by the $\bar\partial$-Neumann boundary conditions, we have, on $b\Omega$

$$
\begin{aligned}
\sum_{j,k}\bar\partial_k(\alpha_{jK}\bar\alpha_{kK}e^{-\varphi})\rho_j &= \sum_{j,k}[(\bar\partial_k\alpha_{jK})\bar\alpha_{kK}e^{-\varphi} + \alpha_{jK}\bar\partial_k(\bar\alpha_{kK}e^{-\varphi})]\rho_j \\
&= \sum_{j,k}(\bar\partial_k\alpha_{jK})\bar\alpha_{kK}e^{-\varphi}\rho_j.
\end{aligned}
\tag{2.9}
$$

Since $\sum_{k=1}^n \bar\alpha_{kK}\bar\partial_k$ is tangential to $b\Omega$ and for all indices K, $\sum_{j=1}^n \rho_j\cdot\alpha_{jK} = 0$ on $b\Omega$, we have for all indices K,

$$0 = \sum_k \bar{\alpha}_{kK} \bar{\partial}_k \left(\sum_j \alpha_{jK} \rho_j \right)$$

or equivalently, on $b\Omega$

$$\sum_{j,k} \bar{\alpha}_{kK} (\bar{\partial}_k \alpha_{jK}) \rho_j = - \sum_{j,k} \alpha_{jK} \bar{\alpha}_{kK} \rho_{j\bar{k}}. \tag{2.10}$$

Plugging (2.10) into the right hand side of (2.9), we obtain

$$I = {\sum_{|K|=r-1}}' \sum_{j,k} \int_\Omega w_{j\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi} - {\sum_{|K|=r-1}}' \sum_{j,k} \int_{b\Omega} w \rho_{j\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi}\, dS. \tag{2.11}$$

We also multiply the right hand side of (2.7) by w and integrate it over Ω. Combining this with (2.11), we have (2.8) of Lemma 2.1. $\square$

3. Proof of Donnelly-Fefferman type theorem

Before proving Theorem 1.5 for general q-pseudoconvex domains and general q-subharmonic functions, we first verify our theorem for a smoothly bounded q-pseudoconvex domain Ω. Moreover, we assume that φ, ψ are smooth up to $\overline{\Omega}$, ψ is positive definite, and $-e^{-\psi}$ is q-subharmonic.

If α satisfies the $\bar{\partial}$-Neumann boundary conditions on $b\Omega$, then we have

$$2 \operatorname{Re} \int_\Omega w \langle \bar{\partial} \bar{\partial}^*_\varphi \alpha, \alpha \rangle e^{-\varphi} = 2 \int_\Omega w |\bar{\partial}^*_\varphi \alpha|^2 e^{-\varphi} - 2 \operatorname{Re} \int_\Omega \langle \bar{\partial}^*_\varphi \alpha, \partial w \,\lrcorner\, \alpha \rangle e^{-\varphi}, \tag{3.12}$$

where the interior multiplication $\partial w \,\lrcorner\, \alpha$ is defined by the following manner

$$\partial w \,\lrcorner\, \alpha = {\sum_{|K|=r-1}}' \sum_{j=1}^n \frac{\partial w}{\partial z_j} \cdot \alpha_{jK}\, d\bar{z}^K.$$

In fact, using (2.6) and integration by parts, we see (3.12). Let $w = e^{-\varepsilon\psi}$. Then using (1.4), we have

$$\begin{aligned} &- {\sum_{|K|=r-1}}' \sum_{j,k} \int_\Omega w_{j\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi} \\ &= {\sum_{|K|=r-1}}' \sum_{j,k} \left[\varepsilon \int_\Omega \psi_{j\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi-\varepsilon\psi} - \varepsilon^2 \int_\Omega \psi_j \psi_{\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi-\varepsilon\psi} \right] \\ &\geq \varepsilon(1-\varepsilon) {\sum_{|K|=r-1}}' \sum_{j,k} \int_\Omega \psi_{j\bar{k}} \alpha_{jK} \bar{\alpha}_{kK} e^{-\varphi-\varepsilon\psi}. \end{aligned} \tag{3.13}$$

Also suppose that α is $\bar\partial$-closed in $\overline{\Omega}$. Note that the second integral of the right hand side of (2.8) vanishes and the first integral and the boundary integral of the left hand side of (2.8) are non-negative. Now applying Lemma 2.1 with $w = e^{-\varepsilon\psi}$ and using (3.12), (3.13), we obtain

$$
\begin{aligned}
&\varepsilon(1-\varepsilon)\sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega \psi_{j\bar k}\alpha_{jK}\overline{\alpha}_{kK}e^{-\varphi-\varepsilon\psi} \\
&\le \int_\Omega |\bar\partial^*_\varphi\alpha|^2 e^{-\varphi-\varepsilon\psi} + 2\varepsilon\int_\Omega |\bar\partial^*_\varphi\alpha|\,|\bar\partial\psi \lrcorner \bar\alpha| e^{-\varphi-\varepsilon\psi} \\
&\le \left(1+\frac{2\varepsilon}{1-\varepsilon}\right)\int_\Omega |\bar\partial^*_\varphi\alpha|^2 e^{-\varphi-\varepsilon\psi} + \frac{\varepsilon(1-\varepsilon)}{2}\sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega \psi_{j\bar k}\alpha_{jK}\overline{\alpha}_{kK}e^{-\varphi-\varepsilon\psi}.
\end{aligned}
$$

Here we again use (1.4) for the last integral of the above estimate. Hence we have proved that

$$
\sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega \psi_{j\bar k}\alpha_{jK}\overline{\alpha}_{kK}e^{-\varphi-\varepsilon\psi} \le \frac{4}{\varepsilon(1-\varepsilon)^2}\int_\Omega |\bar\partial^*_\varphi\alpha|^2 e^{-\varphi-\varepsilon\psi} \tag{3.14}
$$

for every $\bar\partial$-closed $(0,r)$-form α which satisfies the $\bar\partial$-Neumann boundary conditions. Let g be a $\bar\partial$-closed $(0,r)$ form on Ω and assume

$$
\|g\|^2_\psi = \sum_{|K|=r-1}{}' \sum_{j,k}\int_\Omega \psi^{j\bar k} g_{jK}\overline{g}_{kK}e^{-\varphi+\varepsilon\psi} < \infty.
$$

Note that since $(\psi_{j\bar k})$ is a positive definite Hermitian matrix, the following holds:

$$
\begin{aligned}
|\langle g,\alpha\rangle|^2 &\le \sum_{|J|=r}{}' |g_J\overline{\alpha}_J|^2 = \frac{1}{r}\sum_{|K|=r-1}{}' \sum_{j=1}^{n} |g_{jK}\overline{\alpha}_{jK}|^2 \\
&\le \frac{1}{r}\left(\sum_{|K|=r-1}{}' \sum_{j,k}\psi^{j\bar k} g_{jK}\overline{g}_{kK}\right)\left(\sum_{|K|=r-1}{}' \sum_{j,k}\psi_{\nu\bar\mu} g_{\nu K}\overline{g}_{\mu K}\right).
\end{aligned}
$$

Therefore, by (3.14), we have

$$
\left|\int_\Omega \langle g,\alpha\rangle e^{-\varphi}\right|^2 \le \frac{4}{\varepsilon(1-\varepsilon)^2}\cdot\frac{1}{r}\|g\|^2_\psi \int_\Omega \left|\bar\partial^*_\varphi\alpha\right|^2 e^{-\varphi-\varepsilon\psi} \tag{3.15}
$$

for every smooth $\bar\partial$-closed $(0,r)$ form α satisfying $\bar\partial$-Neumann conditions on $b\Omega$. Now to solve the $\bar\partial$-equation for a given $\bar\partial$-closed $(0,r)$ form g, we need the following Hörmander's L^2-method.

Lemma 3.1. *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ and φ a smooth function in Ω. If g is a $\overline{\partial}$-closed $(0, r)$-form satisfying the inequality*

$$\left|\int_\Omega \langle g, \alpha\rangle e^{-\varphi}\right|^2 \leq C_1 \int_\Omega |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi}/w$$

for all $\alpha \in C^\infty_{(0,r)}(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^*)$ with $\overline{\partial}\alpha = 0$, where $1/w$ is an integrable positive function, then there is a solution u of the equation $\overline{\partial}u = g$ such that

$$\int_\Omega |u|^2 w e^{-\varphi} \leq C_1.$$

Proof. The proof is a slight modification of Hörmander's method to solve the $\overline{\partial}$-equation. This can be found in [1]. For the convenience, we give a brief proof. For $a \in \mathrm{Dom}(\overline{\partial}^*_\varphi) = \mathrm{Dom}(\overline{\partial}^*)$, define an anti-linear functional

$$L(\overline{\partial}^*_\varphi a) = \int_\Omega \langle g, a\rangle e^{-\varphi}.$$

For $a \in C^\infty_{(0,r)}(\overline{\Omega}) \cap \mathrm{Dom}(\overline{\partial}^*)$, decompose $a = \alpha + \beta$, where $\alpha \in \mathrm{Ker}\,\overline{\partial}$ and $\beta \in (\mathrm{Ker}\,\overline{\partial})^\perp \subset \mathrm{Ker}\,\overline{\partial}^*_\varphi$. By the assumption and the density, we have for any $a \in \mathrm{Dom}(\overline{\partial}^*)$,

$$L(\overline{\partial}^*_\varphi a) \leq C_1 \int_\Omega |\overline{\partial}^*_\varphi a|^2 e^{-\varphi}/w.$$

By the Hahn-Banach theorem and the Riesz representation theorem, there is a $v \in L^2(e^\varphi/w)$ such that for any $a \in \mathrm{Dom}(\overline{\partial}^*)$,

$$\int_\Omega \langle g, a\rangle e^{-\varphi} = \int_\Omega \langle v, \overline{\partial}^*_\varphi a\rangle e^{-\varphi}/w \tag{3.16}$$

and the norm of the anti-linear functional satisfies

$$\|L\| = \int_\Omega |v|^2 e^{-\varphi}/w \leq C_1. \tag{3.17}$$

Let $u = v/w$. Then by (3.16), u is a solution to $\overline{\partial}u = g$ and (3.17) gives the desired estimate for u. □

Next, we prove the following Donnelly-Fefferman type theorem for general q-pseudoconvex domain D and general q-subharmonic function φ in D (without the assumption of smoothness of φ).

Proof of Theorem 1.5. Since D is a q-*pseudoconvex* domain, we can choose strictly q-*pseudoconvex domains* D_ν with smooth boundary such that

$$D = \bigcup_{\nu=1}^{\infty} D_\nu, \quad D_\nu \subset\subset D_{\nu+1} \subset\subset D \quad \text{for all} \quad \nu.$$

Also there is a decreasing sequence $\{\varphi_\mu\}$ of smooth q-subharmonic functions which converges pointwise to φ. Now we apply the estimate (3.15) with φ_ν, $w = e^{-\varepsilon\psi}$ and $\Omega = D_\nu$ for each ν. Then we have for all smooth $\bar\partial$-closed $(0, r)$ form α satisfying $\bar\partial$-Neumann conditions on bD_ν

$$\left|\int_{D_\nu} \langle g, \alpha\rangle e^{-\varphi_\nu}\right|^2 \le C_2 \int_{D_\nu} \left|\bar\partial^*_{\varphi_\nu}\alpha\right|^2 e^{-\varphi_\nu - \varepsilon\psi}, \tag{3.18}$$

where

$$C_2 = \frac{4}{\varepsilon(1-\varepsilon)^2} \cdot \frac{1}{r} \sum_{|K|=r-1}{}' \sum_{j,k} \int_{D_\nu} \psi^{j\bar k} g_{jK} \bar g_{kK} e^{-\varphi_\nu + \varepsilon\psi}.$$

Note that $1/w = e^{\psi}$ is integrable in D_ν. Applying Lemma 3.1 with (3.18), we see that there are solutions u_ν and corresponding estimates (1.3) for all domains D_ν. Since φ_ν is decreasing, the constant C_2 is bounded by the quantity of the right hand side of (1.3) that is independent of ν. Therefore there is a limit u of some subsequence of $\{u_\nu\}$ which satisfies $\bar\partial u = g$ on D and the desired estimate (1.3). □

4. Application to approximation theorem

In this section we prove Corollary 1.6. Let h be a continuous q-subharmonic function in D. Assume that $K = \{z \in D : h(z) \le 0\} \subset\subset D$.

Proof. Let f be a given $\bar\partial$-closed $(0, s)$ form whose coefficients are in $L^2(K)$, $s \ge q-1$ and $\psi(z) = \log(1 + |z|^2)$. Note that for all $1 \le j, k \le n$,

$$\frac{\partial\psi}{\partial\bar z_k} = \frac{z_k}{1+|z|^2}, \quad \frac{\partial^2\psi}{\partial\bar z_j \partial\bar z_k} = \frac{z_k \bar z_j}{(1+|z|^2)^2} + \frac{\delta_{jk}}{1+|z|^2},$$

where δ_{jk} is the Kronecker's symbol. It follows that ψ is strictly plurisubharmonic and $-e^{-\psi}$ is plurisubharmonic, i.e., q-subharmonic. Let V be an open neighborhood of K in which f is smooth and $\bar\partial$-closed. We choose open sets U so that $K \subset U \subset\subset V \subset\subset D$ and $\chi \in C^\infty(\mathbb{C}^n)$ vanishing outside V and satisfying $\chi \equiv 1$ on U. Let $u = \bar\partial(\chi f) = \bar\partial\chi \wedge f$ and $\delta > 0$ be given. Note that u is a $\bar\partial$-closed $(0, s+1)$-form in D, $s + 1 \ge q$ whose coefficients are in $L^2(D)$. We claim that there is a v_δ such that $\bar\partial v_\delta = u$ on D and $\|v_\delta\|_{L^2(K)} < \delta$. Put $g_\delta = \chi f - v_\delta$. Note that $\chi f \equiv f$ on K. Hence we have

$$\|f - g_\delta\|_{L^2(K)} = \|v_\delta\|_{L^2(K)} < \delta.$$

We prove our claim. First, we can choose $\epsilon_0 > 0$ so that $h > \epsilon_0$ on $V \setminus U$, since h is continuous and non-negative outside K. For $\lambda > 0$ to be determined later, we apply the main theorem to $\varphi = \lambda h$, $\psi = \log(1+|z|^2)$, and $\varepsilon = 1/2$. Write $u = \sum'_{|J|=s+1} u_J \, d\bar{z}^J$. Then there is a v_λ to $\bar{\partial} v_\lambda = u$ and satisfying

$$\int_D |v_\lambda|^2 e^{-\lambda h + \log(1+|z|^2)/2} \leq \frac{32}{r} \sum_{|K|=s}' \sum_{j,k} \int_D \psi^{j\bar{k}} u_{jK} \bar{u}_{kK} e^{-\lambda h + \log(1+|z|^2)/2}.$$

Notice that for some constant C_k which is independent of λ, we have

$$C_k \int_K |v_\lambda|^2 \leq \int_D |v_\lambda|^2 e^{-\lambda h + \log(1+|z|^2)/2},$$

since $h \leq 0$ on K. On the other hand, since u has a support in $V \setminus U$, for some constant $C_{V\setminus U}$, we have

$$\frac{32}{r} \sum_{|K|=s}' \sum_{j,k} \int_D \psi^{j\bar{k}} u_{jK} \bar{u}_{kK} e^{-\lambda h + \log(1+|z|^2)/2} \leq C_{V\setminus U} e^{-\lambda \epsilon_0}.$$

Here $C_{V\setminus U}$ depends only on u, ψ and fixed ϵ_0. Hence if we choose λ large enough so that $C_{V\setminus U} e^{-\lambda\epsilon_0}/C_K < \delta$, we see that $|v_\delta|_{L^2(K)} < \delta$. □

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References

[1] B. Berndtsson: *The extension theorem of Ohsawa-Takegoshi and the theorem of Donnelly-Fefferman*, Ann. Inst. Fourier (Grenoble) **46** (1996), 1083–1094.

[2] Z. Błocki: *The Bergman metric and the pluricomplex Green function*, Trans. Amer. Math. Soc. **357** (2005), 2613–2625, electronic.

[3] H. Donnelly and C. Fefferman: *L^2-cohomology and index theorem for the Bergman metric*, Ann. of Math. (2) **118** (1983), 593–618.

[4] N.Q. Dieu: *q-plurisubharmonicity and q-pseudoconvexity in* $\mathbf{C}^n$, Publ. Mat. **50** (2006), 349–369.

[5] L.-H. Ho: *$\bar{\partial}$-problem on weakly q-convex domains*, Math. Ann. **290** (1991), 3–18.

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