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2-BRIDGE KNOT BOUNDARY SLOPES: DIAMETER AND GENUS

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Abstract

We prove that for 2-bridge knots, the diameter, D, of the set of boundary slopes is twice the crossing number, c. This constitutes partial verification of a conjecture that, for all knots in S^3, $D \leq 2c$. In addition, we characterize the 2-bridge knots with four or fewer boundary slopes and show that they each have a boundary slope of genus two or less.

Introduction

Ichihara [3] told us of a conjecture for knots in S^3. Let $D(K)$ denote the diameter of the set of boundary slopes of a knot K and $c(K)$ be the crossing number.

Conjecture 1. *For K a knot in S^3, $D(K) \leq 2c(K)$.*

(To be precise, Ichihara proposed the conjecture only for Montesinos knots and he and Mizushima [4] have recently given a proof of that case.)

Since 0, being the slope of a Seifert surface, is always included in the set of boundary slopes, we have, as an immediate consequence, a conjecture due to Ishikawa and Shimokawa [5]:

Conjecture 2. *Let b be a finite boundary slope for K a knot in S^3. Then $|b| \leq 2c(K)$.*

For example, it is easy to verify these conjectures for torus knots. For the unknot, $D(K) = 0 = 2c(K)$. For a non-trivial torus knot K of type (p, q) we can assume p, q relatively prime with $2 \leq q < p$. The boundary slopes are 0 and pq [8, 11] while the crossing number is $c(K) = pq - p$ [9]. Thus, $D(K) = pq \leq pq + p(q - 2) = 2c(K)$.

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Moreover, we have equality for the torus 2-bridge knots which are of the form $(p, 2)$ with p odd.

We will show that this equality obtains for all 2-bridge knots:

Theorem 1. *For K a 2-bridge knot, $D(K) = 2c(K)$.*

Corollary 1. *Let b be a boundary slope for a 2-bridge knot K. Then $|b| \leq 2c(K)$.*

This bound is sharp for the $(p, 2)$ torus knots and there are many examples showing that it is also sharp for hyperbolic 2-bridge knots. Such examples are given by 2-bridge knots that are "checkerboard;" an alternating knot K is called checkerboard if it possesses a reduced alternating diagram such that one of the checkerboard surfaces is an essential Seifert surface for K. In this case, the boundary slope b of the other checkerboard surface satisfies the equality in Corollary 1.

In Section 1 we review Hatcher and Thurston's [2] method for computing the boundary slopes of a 2-bridge knot K. Using Conway notation, we can associate to K a rational number p/q where $0 \leq p/q < 1$; we will use $K(p/q)$ to denote this knot. The crossing number is given by summing the terms in a simple continued fraction for p/q (see [1]) and the boundary slopes are given by continued fraction expressions of p/q with partial quotients at least two in absolute value. By identifying the maximum and minimum boundary slopes, we can verify that their difference is twice the crossing number.

In Section 2 we present two substitution rules for continued fractions. These substitution rules will allow us to produce all possible boundary slope continued fractions for a given rational number:

Theorem 2. *The boundary slope continued fractions of $K(p/q)$ are among the continued fractions obtained by applying substitutions at non-adjacent positions in the simple continued fraction of p/q.*

The proof of Theorem 2 is presented in Section 3 along with the following corollary.

Corollary 2. *If $p/q = [0, a_0, a_1, \ldots, a_n]$ is a simple continued fraction, then $K(p/q)$ has at most F_{n+2} boundary slopes where F_n is the n-th Fibonacci number.*

In Section 4 we show how to compare the boundary slopes obtained from different substitution patterns. This allows us to identify the patterns corresponding to the maximum and minimum boundary slopes and thereby to prove Theorem 1.

In Section 5 we characterize the 2-bridge knots that have no more than four boundary slopes.

Theorem 3. *Let $K = K(p/q)$ be a 2-bridge knot.*

- *If K has only two distinct boundary slopes, then K is a torus knot and $p = 1$ or $p = q - 1$.*
- *If K has precisely three boundary slopes, then $p \mid (q \pm 1)$ or $(q - p) \mid (q \pm 1)$.*
- *If K has precisely four boundary slopes, then one of the following holds: $p \mid (q+1)$, $(q - p) \mid (q + 1)$, $(p \pm 1) \mid q$, or $(q - p \pm 1) \mid q$.*

Note that the torus knots are also the only 2-bridge knots with a genus 0 boundary slope (see [2, Theorem 2 (a)]). Thus, the set of 2-bridge knots admitting a genus 0 boundary slope exactly coincides with those having two boundary slopes.

The situation for genus 1 and 2 boundary slopes is similar. Using [2] (see also [10]) the genus of a k-sheeted surface carried by a continued fraction $[0, b_0, b_1, \ldots, b_n]$ is $g = (2 + k(n - 1))/2$ which is 1 only if $n = 1$. In other words, the 2-bridge knots having a genus 1 boundary slope are exactly the trefoil knot along with the hyperbolic knots for which $p \mid (q \pm 1)$ or $(q - p) \mid (q \pm 1)$. Thus, if a 2-bridge knot has exactly three boundary slopes, then it has a genus 1 boundary slope. Similarly, if $K(p/q)$ has exactly four boundary slopes, then it has a boundary slope of genus 2 or less.

The converses of these statements are not quite true. For example, a knot with a genus 1 boundary slope may have four boundary slopes (and not just three), e.g., $K(4/11)$ has boundary slopes -4, 0, 2, 8, the last being of genus 1. Still, this suggests the following:

QUESTION. If the knot K has a boundary slope of small genus, does it follow that K has few boundary slopes? Conversely, do few boundary slopes imply a slope of small genus?

That is, does the pattern we observe for 2-bridge knots persist beyond genus 2? What if we consider more general classes of knots?

The first half of this paper is an abbreviated version of [7] and we refer the reader to that paper for additional details.

1. Boundary slopes of 2-bridge knots

In this section we introduce terminology and review Hatcher and Thurston's [2] method for computing the boundary slopes of a 2-bridge knot. Let $K(p/q)$ denote the 2-bridge knot associated to the fraction p/q. Recall that $K(p/q)$ is equivalent to $K(p'/q)$ iff $p' \equiv p^{\pm 1} \bmod q$. Therefore, we can assume that $0 \leq p/q < 1$. As $p/q = 0$ corresponds to the unknot, we will often further assume that $0 < p/q < 1$.

A continued fraction expansion of p/q is a fraction of the form

$$\frac{p}{q} = c + \cfrac{1}{b_0 + \cfrac{1}{b_1 + \cfrac{1}{\cdots + \cfrac{1}{b_n}}}} = [c, b_0, b_1, \ldots, b_n],$$

where $c \in \mathbb{Z}$ and each b_i, for $0 \le i \le n$, is a nonzero integer. The b_i are called *partial quotients* or *terms*. The *simple continued fraction* of p/q is the unique one having all terms positive and $b_n > 1$. A *boundary slope continued fraction* is one for which $|b_i| \ge 2$ $(0 \le i \le n)$. Among the boundary slope continued fractions there is a unique one, the longitude continued fraction, having all b_i even.

Following [2], in order to calculate the boundary slope associated with a boundary slope continued fraction, compare the partial quotients to the pattern $[+ - + - \cdots]$. The number of terms matching this pattern we call n^+, and the number of terms not matching this pattern (i.e., the total number of terms minus n^+) we call n^- (since these terms match the pattern $[- + - + \cdots]$). In this way, we associate to each continued fraction two non-negative integers n^+ and n^-. The boundary slope is then given by comparing the difference $n^+ - n^-$ with that corresponding to the longitude: $n_0^+ - n_0^-$; the boundary slope associated with the continued fraction is $2((n^+ - n^-) - (n_0^+ - n_0^-))$. Applying this calculation to every continued fraction with terms at least two in absolute value gives the set of boundary slopes $B(K) = B$. B is a finite set of even integers. The diameter $D(K) = D$ is the difference between the biggest and smallest elements of B.

2. Continued fraction substitution rules

In this section, we present two substitution rules that will be used to derive equal continued fractions. (The straightforward proofs by induction are omitted; see [7] for details.) As we will illustrate at the end of the section, these substitutions can be used to derive all the boundary slope continued fractions of $K(p/q)$ from the simple continued fraction of p/q.

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$. We will use the notation $(b_0, \ldots, b_m)^c$ to mean that the pattern "$b_0, \ldots, b_m$" is repeated c times, with c being any nonnegative integer, e.g., $[0, (-2, 2)^2] = [0, -2, 2, -2, 2]$ and $[0, (-2, 2)^0, 2] = [0, 2]$.

SUBSTITUTION 1. Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}$. If $n = 2$ then let $b_2 \in \mathbb{Z} \setminus \{0, -1\}$. If $n \ge 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \le i \le n$. With these assumptions, $[b_0, 2b_1, b_2, b_3, \ldots, b_n] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1, b_3, b_4, \ldots, b_n]$. In particular, $[b_0, 2b_1, b_2] = [b_0 + 1, (-2, 2)^{b_1 - 1}, -2, b_2 + 1]$.

SUBSTITUTION 2. Let $n \in \mathbb{N}$. Let $b_0 \in \mathbb{Z}$ and $b_1 \in \mathbb{N}_0$. If $n = 2$ then let $b_2 \in \mathbb{Z} \setminus \{0, -1\}$. If $n \ge 3$ then let $b_i \in \mathbb{Z}^*$ for all $2 \le i \le n$. With these assumptions,

$[b_0, 2b_1+1, b_2, b_3, \ldots, b_n] = [b_0+1, (-2, 2)^{b_1}, -b_2-1, -b_3, -b_4, \ldots, -b_n]$. In particular, $[b_0, 2b_1+1, b_2] = [b_0+1, (-2, 2)^{b_1}, -b_2-1]$.

2.1. An example of the application of the substitutions. Let us illustrate how the above results can be used to generate a list of all boundary slope continued fractions starting from the simple continued fraction. As an example, suppose we start with $[0, 2a, 2b+1, 2c]$, where $a, c \in \mathbb{N}$ and $b \in \mathbb{N}_0$. By applying Substitution 1, we can immediately derive another continued fraction: $[1, (-2, 2)^{a-1}, -2, 2b+2, 2c]$. We will refer to this as *applying* Substitution 1 *at position* 0 as it is the a_0 term, $2a$, that has been replaced by the sequence $-2, 2, \ldots, -2$.

Applying the same substitution at position 2, we get $[1, (-2, 2)^{a-1}, -2, 2b+3, (-2, 2)^{c-1}, -2]$. We could continue on this path, but it is easy to see that any further substitutions will result in a ± 1 term. Therefore, we return to the original sequence and use Substitution 2 (at position 1) to obtain $[0, 2a+1, (-2, 2)^b, -2c-1]$. Finally, applying Substitution 1 at position 2, we have $[0, 2a, 2b+2, (-2, 2)^{c-1}, -2]$.

Thus, there are five boundary slope continued fractions that can be derived from the simple continued fraction $[0, 2a, 2b+1, 2c]$: three obtained by substitutions at positions 0, 1, and 2; one by substitutions at 0 and 2; and the original continued fraction itself (with no substitutions). These are precisely the fractions obtained by applying substitutions at non-adjacent positions.

Note that when a substitution is applied at position i, the element a_i is replaced by $(a_i - 1) \pm 2$'s and the adjacent terms a_{i-1} and a_{i+1} both have their magnitude increased by one. We will return to these observations when proving Theorem 1.

3. Proof of Theorem 2

In this section we will prove Theorem 2, that the boundary slope continued fractions are among the fractions obtained by applying substitutions at non-adjacent positions in the original simple continued fraction. Our strategy is to first review Langford's argument [6] that the boundary slopes are determined by the leaves of a binary tree. We then show, by induction, that applying substitutions at non-adjacent positions accounts for all the leaves of the tree.

3.1. The boundary slope binary tree. Before we can prove Theorem 2, we must first state a lemma. The straightforward proof by induction may be found in Langford [6] which is also the source for the following definition: the *k-th subexpansion* of $[c, a_0, \ldots, a_n]$ is the continued fraction $[0, a_k, \ldots, a_n]$ where $0 \le k \le n$.

Lemma 1. *Let $[c, a_0, \ldots, a_n]$ be a boundary slope continued fraction, that is, $|a_i| \ge 2$ $(0 \le i \le n)$. Then every subexpansion r of $[c, a_0, \ldots, a_n]$ satisfies $|r| < 1$.*

As Langford [6] has shown, a complete list of boundary slope continued fractions for $K(p/q)$, where $0 < p/q < 1$, can be calculated by means of a binary tree. We

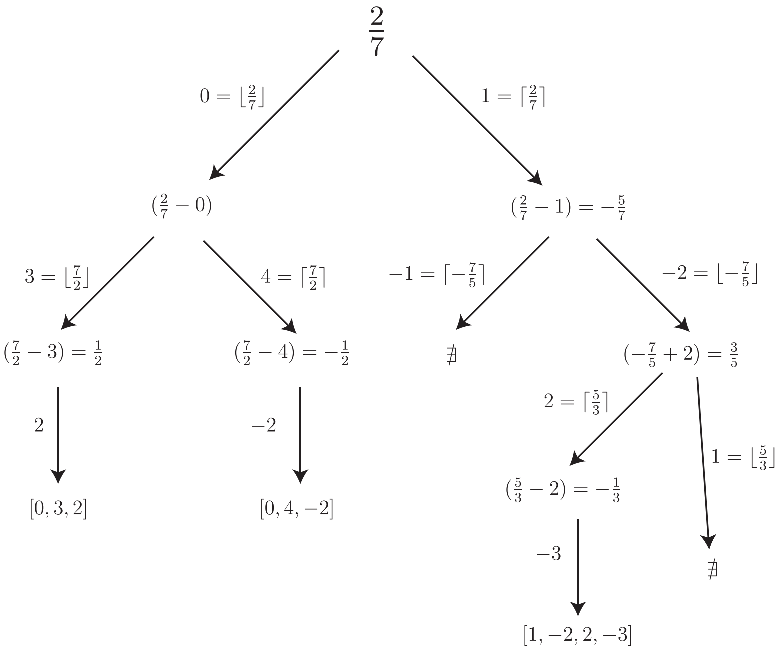

Fig. 1. The boundary slope binary tree for $p/q = 2/7$ (the 5_2 knot).

will now outline the creation of this binary tree which follows from Lemma 1.

The root vertex is labeled with the fraction p/q and the two edges coming from the root are labeled $0 = \lfloor p/q \rfloor$ and $1 = \lceil p/q \rceil$. At every other vertex in the tree, we arrive with the first k terms in a continued fraction for p/q and a rational number r representing the $(k-1)$st subexpansion. The k terms are found as labels of the edges of the tree starting from the root and continuing to the vertex in question. We label the vertex with r. Since, by Lemma 1, any k-th subexpansion is less than one in absolute value, we know that the next term in the continued fraction, a_{k-1}, is within 1 of $1/r$: $|a_{k-1} - 1/r| < 1$. However, a_{k-1} is an integer. Therefore, a_{k-1} is either the floor $\lfloor 1/r \rfloor$ or the ceiling $\lceil 1/r \rceil$ of $1/r$. If $1/r$ is not an integer, there will be two edges coming out of the vertex, one labeled with $\lfloor 1/r \rfloor$, and the other labeled with $\lceil 1/r \rceil$. Since $|r| < 1$, neither of these arrows is 0. If either is ± 1, we terminate that edge with a leaf labeled "$\nexists$" to indicate that this path does not lead to a boundary slope continued fraction. (When we refer to the leaves of the binary tree below, we will be excluding these "dead" leaves.) If $1/r$ is an integer, then, there is only one edge coming out of the vertex. Label the edge with $1/r$ and label the leaf vertex at the end of this edge with the continued fraction expansion for p/q given by the labels of the edges from the root to the leaf.

For example, Fig. 1 shows the binary tree for the fraction $2/7$ (which corresponds to the 5_2 knot).

Thus, by Lemma 1, the algorithm used to construct the tree will provide all the boundary slope continued fractions of p/q as leaf vertices.

3.2. Binary tree from substitutions. Now, let's prove the theorem by showing that the leaves of Langford's binary tree (and therefore the set of boundary slopes) correspond to applying substitutions at non-adjacent positions in the simple continued fraction.

Theorem 2. *The boundary slope continued fractions of $K(p/q)$ are among the continued fractions obtained by applying substitutions at non-adjacent positions in the simple continued fraction of p/q.*

Proof. We proceed by induction on the length n of the simple continued fraction $[0, a_0, a_1, \ldots, a_n]$.

CASE 1 ($n = 0$): Here, $p/q = 1/a_0$. We wish to show that the boundary slope continued fractions are among the two continued fractions given by substituting or not at position 0. There are several cases depending on the sign and parity of a_0. Here, we'll look at the case where $a_0 = 2a$ is an even, positive integer. We refer the reader to [7] for treatment of additional subcases.

The binary tree is shown in Fig. 2. There are two boundary slope continued fractions, and they are the fractions $[0, a_0]$ and $[1, (-2, 2)^{a-1}, -2]$ given by substituting or not at position 0.

CASE 2 ($n = 1$): Our goal is to show that the boundary slope continued fractions are among the fractions given by substituting at position 0, at position 1, and by not substituting at all. The result of substitution at position 0 will depend on whether a_0 is even or odd:

$$[0, 2a, a_1] \xrightarrow{\text{Sub. 1}} [1, (-2, 2)^{(a-1)}, -2, a_1 + 1],$$
$$[0, 2a + 1, a_1] \xrightarrow{\text{Sub. 2}} [1, (-2, 2)^{a}, -a_1 - 1].$$

Similarly, substitution at position 1 depends on the parity of a_1:

$$[0, a_0, 2b] \xrightarrow{\text{Sub. 1}} [0, a_0 + 1, (-2, 2)^{(b-1)}, -2],$$
$$[0, a_0, 2b + 1] \xrightarrow{\text{Sub. 2}} [0, a_0 + 1, (-2, 2)^{b}].$$

As Fig. 3 shows, these two boundary slopes, along with the original continued fraction $[0, a_0, a_1]$ (no substitutions) are precisely those that arise in the binary tree. Note that if, for example, a_0 or a_1 is 1, then the $[0, a_0, a_1]$ leaf is not in fact a boundary slope continued fraction. The point is that all leaves of the binary tree are included in the set of continued fractions obtained by substitutions at non-adjacent positions. So, every boundary slope continued fraction appears in this set.

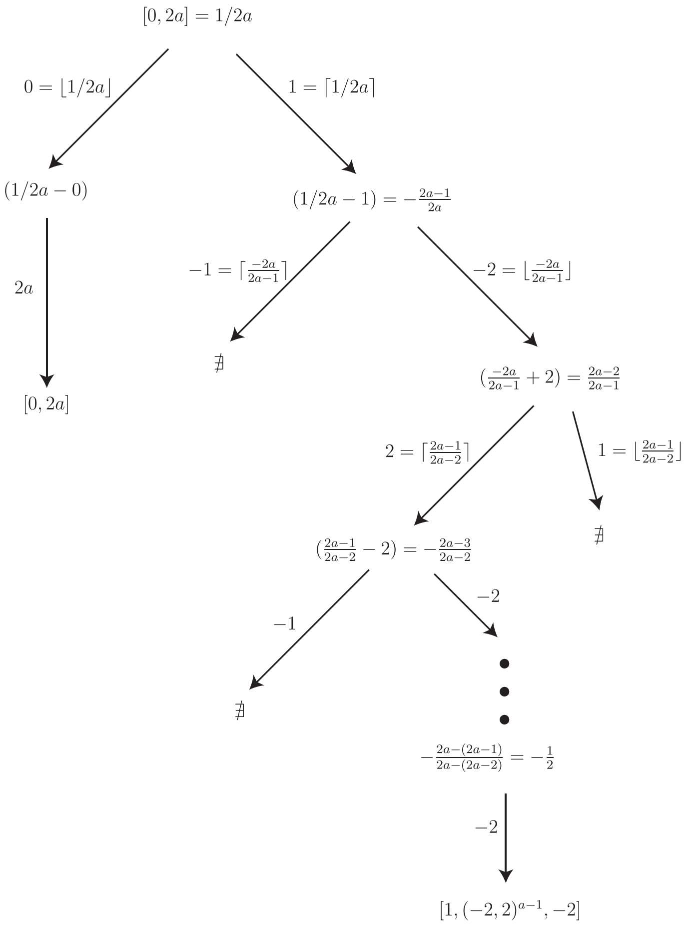

Fig. 2. The binary tree for [0, 2a].

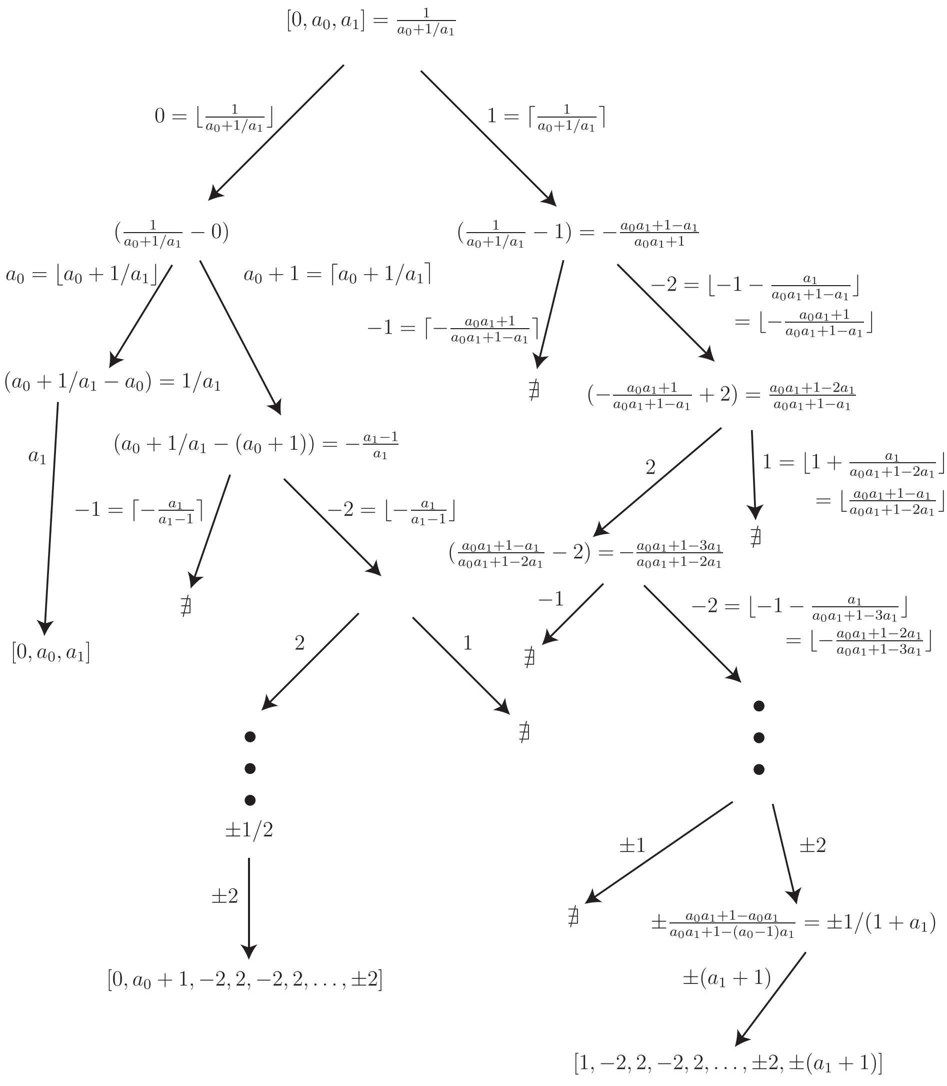

Fig. 3. The binary tree for $[0, a_0, a_1]$.

CASE 3 ($n = 2$): This case will illustrate how the induction works. There are five continued fractions given by substitutions at non-adjacent positions (compare with the example of Section 2.1): three obtained by substitutions at positions 0, 1, and 2; one by substitutions at 0 and 2; and the original continued fraction itself (with no substitutions). Let us denote these choices of substitutions by a sequence of three 0's and 1's where a 1 in the i-th place denotes a substitution at that i-th position. Thus, the five continued fractions will be denoted 100, 010, 001, 101, and 000.

We can think of the binary tree (Fig. 4) as a union of two subtrees. The one at left corresponds to making no substitution at position 0. This subtree ends in the three boundary slopes which have: no substitutions (000); substitution at position 1 (010); and substitution at position 2 (001), i.e., the sequences that begin in 0. This subtree is essentially the same as that for the continued fraction $[0, a_1, a_2]$ (compare Fig. 3) as we can obtain these three sequences by adding a 0 at the front of the three boundary slopes sequences 00, 10, and 01 of that case. The other subtree corresponds to making a substitution at position 0 and no substitution at position 1. This subtree contains the remaining two boundary slopes: substitution at position 0 (100); and substitution at positions 0 and 2 (101), i.e., sequences that begin in 10. This subtree is similar to that for $[0, a_2]$ (compare Fig. 2) as it remains only to decide whether or not to substitute in the second position. Again, some of these five sequences may not result in a boundary slope continued fraction, for example, if one of the a_i is 1. However, every leaf of the tree will be included in the set of continued fractions obtained by substituting at non-adjacent positions.

CASE 4 ($n \geq 3$): As in Case 3, we can decompose the binary tree (Fig. 5) into two subtrees. One corresponds to sequences that begin with 0, the other to sequences beginning with 10. The first will be, essentially, the tree that arises from the simple continued fraction $[0, a_1, a_2, \ldots, a_n]$. By induction, the leaves of this subtree correspond to non-adjacent substitutions in this simple continued fraction. By its placement in the $[0, a_0, a_1, \ldots, a_n]$ tree, this ensures that the leaves of this part of the tree will correspond to continued fractions obtained by substitution sequences into $[0, a_0, a_1, \ldots, a_n]$ that begin with 0.

The other subtree is isomorphic to the tree that arises from the simple continued fraction $[0, a_2, a_3, \ldots, a_n]$. By induction, the leaves of the subtree correspond to substitutions into this continued fraction. By its placement in the tree for $[0, a_0, a_1, \ldots, a_n]$, the leaves here can be obtained by non-adjacent substitutions into that continued fraction that begin with 10.

Thus, every leaf of the binary tree and, therefore, every boundary slope continued fraction can be obtained by non-adjacent substitutions into the simple continued fraction. □

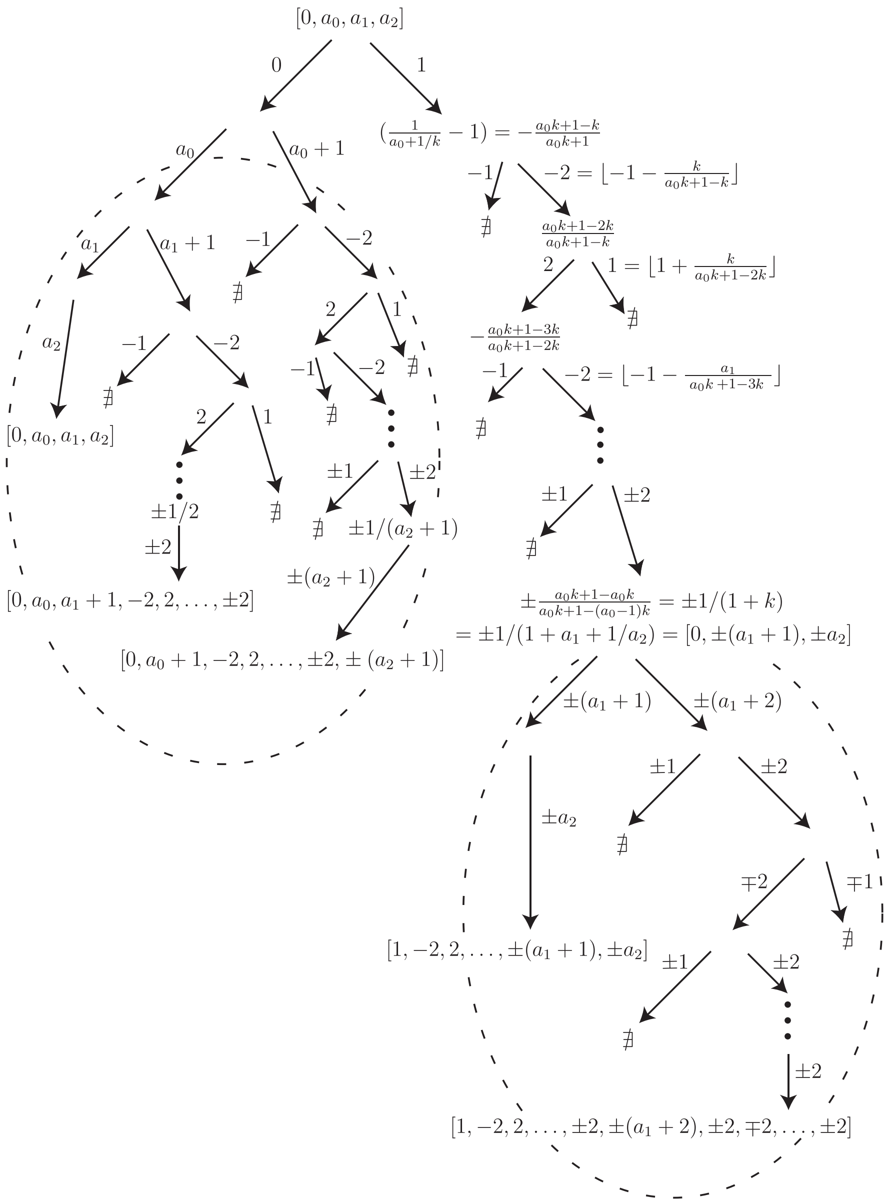

Fig. 4. The $[0, a_0, a_1, a_2]$ tree is a union of two subtrees.

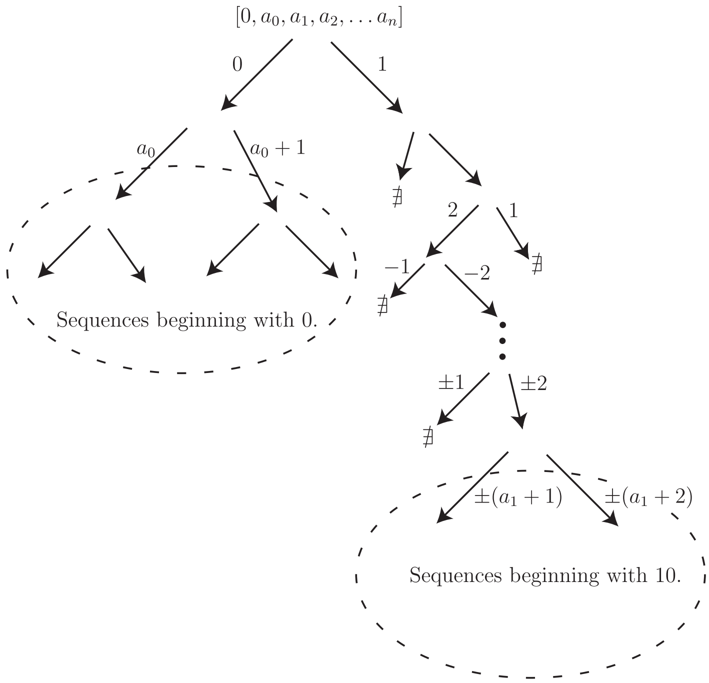

Fig. 5. The general case also results in two subtrees.

Corollary 2. *If $p/q = [0, a_0, a_1, \ldots, a_n]$ is a simple continued fraction, then $K(p/q)$ has at most F_{n+2} boundary slopes where F_n is the n-th Fibonacci number.*

Proof. We have shown that the boundary slope continued fractions lie among those given by substitution at non-adjacent positions which in turn are in bijection with sequences of $n + 1$ 0's or 1's containing no pair of consecutive 1's. Thus the number of boundary slopes is at most P_n, where P_n is the number of 0, 1 sequences of length $n + 1$ with no consecutive 1's. We will show that $P_n = F_{n+2}$ by induction.

There are two base cases. If $n = 0$, there are two sequences: 0 and 1. So, $P_0 = 2 = F_2$. For $n = 1$, there are three sequences: 00, 10, and 01. So, $P_1 = 3 = F_3$.

For the inductive step, sequences of length $n + 1$ are obtained by either adding a 0 to the beginning of a n sequence or 10 to the beginning of a $n - 1$ sequence. Thus $P_n = P_{n-1} + P_{n-2} = F_{n+1} + F_n = F_{n+2}$. □

In general, F_{n+2} is an overestimate since the continued fractions obtained by substitutions will not necessarily have terms at least two in absolute value. In particular, if the simple continued fraction includes any 1's, then the continued fraction obtained by making no substitutions $(000 \cdots 0)$ will not be a boundary slope continued fraction. Moreover, different boundary slope continued fractions could result in the same boundary slope. For example, this will occur when, in the simple continued fraction, we have two equal terms separated by an even distance: $a_i = a_{i+2k}$.

4. Proof of Theorem 1

In this section we prove Theorem 1. We will argue that the maximum and minimum boundary slopes are given by the substitution patterns $010101\cdots$ and $101010\cdots$ respectively. This will allow us to compare the diameter to the crossing number.

Denote by $\partial[s_0 s_1 \cdots s_n]$ the boundary slope obtained by applying the substitution pattern $s_0 s_1 \cdots s_n$ to some simple continued fraction $[c, a_0, a_1, \ldots, a_n]$. That is, s_0, $s_1, \ldots, s_n$ is a sequence of 0's and 1's with no adjacent 1's. Let $\delta[s_0 s_1 \cdots s_n]$ be the $n^+ - n^-$ portion of this boundary slope. Clearly, if S and S' are substitution patterns, then $\partial[S] < \partial[S'] \Leftrightarrow \delta[S] < \delta[S']$ and $\partial[S] = \partial[S'] \Leftrightarrow \delta[S] = \delta[S']$.

A key observation is that, substitution at an "even" position will decrease the value of the boundary slope. This is because, no matter which substitution is made at position $2i$, the positive number a_{2i} (which counted towards n^+) will be replaced by a sequence of $(a_{2i} - 1) \pm 2$'s that count towards n^-. Similarly, substituting at an "odd" position will increase the value:

Lemma 2.

$$\partial[1] < \partial[0], \tag{1}$$

$$\partial[10 s_2 s_3 \cdots s_n] < \partial[00 s_2 s_3 \cdots s_n] < \partial[01 s_2 s_3 \cdots s_n], \tag{2}$$

$$\partial[001 s_3 s_4 \cdots s_n] < \partial[000 s_3 s_4 \cdots s_n] < \partial[010 s_3 s_4 \cdots s_n]. \tag{3}$$

Moreover,

$$\partial[t_1 \cdots t_{2n} 10 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n} 00 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n} 01 s_1 \cdots s_n]$$

and

$$\partial[t_1 \cdots t_{2n+1} 01 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n+1} 00 s_1 \cdots s_n] < \partial[t_1 \cdots t_{2n+1} 10 s_1 \cdots s_n].$$

Proof. Equation (1): In this case, $p/q = [0, a_0]$ and we are comparing the boundary slope $\partial[1]$ obtained by a substitution at position 0 with that $\partial[0]$ obtained by no substitutions.

Let $\delta[0] = n^+ - n^-$. Then

$$\begin{aligned}\delta[1] &= (n^+ - 1) - (n^- + a_0 - 1)\\ &= n^+ - n^- - a_0\\ &< n^+ - n^-\\ &= \delta[0].\end{aligned}$$

Equation (2): Let $\delta[00s_2s_3\cdots s_n] = n^+ - n^-$. Then

$$\begin{aligned}
\delta[10s_2s_3\cdots s_n] &= n^+ - n^- - a_0 \\
&< n^+ - n^- = \delta[00s_2s_3\cdots s_n] \\
&< n^+ - n^- + a_1 \\
&= (n^+ + a_1 - 1) - (n^- - 1) \\
&= \delta[01s_2s_3\cdots s_n].
\end{aligned}$$

Equation (3): Let $\delta[000s_3s_4\cdots s_n] = n^+ - n^-$. Then

$$\begin{aligned}
\delta[001s_3s_4\cdots s_n] &= n^+ - n^- - a_2 \\
&< n^+ - n^- = \delta[000s_3s_4\cdots s_n] \\
&< n^+ - n^- + a_1 \\
&= (n^+ + a_1 - 1) - (n^- - 1) \\
&= \delta[010s_3s_4\cdots s_n].
\end{aligned}$$

The remaining two equations follow since adding the same sequence of substitutions at the beginning of the continued fraction will have a similar effect on all three of the boundary slopes. □

Let $p/q = [0, a_0, \ldots, a_n]$ be the simple continued fraction for the knot $K = K(p/q)$ where $0 < p/q < 1$. It follows from the lemma that the minimum boundary slope is $\partial[101010\cdots]$ while the maximum is $\partial[010101\cdots]$.

Note that these two are indeed boundary slopes; that is, each term in the resulting continued fraction is at least two in absolute value. For example, under the substitution $101010\cdots$ the even position terms a_{2i} of the original simple continued fraction will be replaced by a sequence of $(a_{2i} - 1) \pm 2$'s while the terms in the odd positions will be augmented in absolute value by at least one. Moreover, this substitution pattern will result in a continued fraction for which all terms satisfy the pattern $[- + - + \cdots]$. So, if we let n_1^+ and n_1^- be the numbers used in calculating this boundary slope, we have $n_1^+ = 0$ while n_1^- simply counts the number of terms in the resulting continued fraction. Again, as each a_{2i} is replaced by $(a_{2i} - 1)$ terms and there are $\lceil n/2 \rceil$ terms resulting from the a_{2i+1}'s, we have

$$\begin{aligned}
n_1^- &= \left\lceil \frac{n}{2} \right\rceil + \sum_{i=0}^{\lfloor n/2 \rfloor} (a_{2i} - 1) \\
&= \left\lceil \frac{n}{2} \right\rceil - \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + \sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} \qquad \text{if } n \text{ is odd}
\end{aligned}$$

$$= -1 + \sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i} \qquad \text{if } n \text{ is even.}$$

Similarly, for $\partial[010101 \cdots]$, $n_2^- = 0$ and

$$n_2^+ = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{2i+1} \qquad \text{if } n \text{ is odd}$$
$$= 1 + \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{2i+1} \qquad \text{if } n \text{ is even.}$$

We can now prove that twice the crossing number of a 2-bridge knot K is equal to the diameter of the boundary slopes.

Theorem 1. *For K a 2-bridge knot, $D(K) = 2c(K)$.*

Proof. Let K be a 2-bridge knot with associated fraction p/q. We may assume $0 \le p/q < 1$. If $p/q = 0$, then K is the unknot and the theorem is valid in this case. So, we will assume $0 < p/q < 1$.

If $[0, a_0, \dots, a_n] = p/q$ is the simple continued fraction for K, then $c(K) = \sum_{i=0}^n a_i$ (see [1]).

The diameter of $B(K)$ is also easy to calculate. If we use the n_1^- and n_2^+ found above, we get $D(K) = 2n_2^+ - 2(n_0^+ - n_0^-) - (-2n_1^- - 2(n_0^+ - n_0^-)) = 2n_2^+ + 2n_1^-$. At this point, n_1^- and n_2^+ may vary depending on whether n is even or odd. However, the differences cancel each other out in either instance, leaving us with

$$D(K) = 2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} a_{2i+1} + 2 \sum_{i=0}^{\lfloor n/2 \rfloor} a_{2i}$$
$$= 2 \sum_{i=0}^{n} a_i .$$

This concludes the proof that $2c(K) = D(K)$. □

5. Knots with at most four boundary slopes

In this section we characterize 2-bridge knots with four or fewer boundary slopes. We will prove Theorem 3 in two steps by first examining knots with at most three boundary slopes and then those with four boundary slopes.

Theorem 4. *Let $K = K(p/q)$ be a 2-bridge knot. If K has only two distinct boundary slopes, then K is a torus knot. If K has precisely three boundary slopes, then $p \mid (q \pm 1)$ or $(q - p) \mid (q \pm 1)$.*

We will break the proof up into several lemmas which taken together imply the theorem.

In the following, let $K = K(p/q)$ be a 2-bridge knot where $0 < p/q < 1$, p and q are relatively prime, and p/q has simple continued fraction $[0, a_0, a_1, \ldots, a_n]$ with $a_n > 1$. We also assume that q is odd (otherwise p/q represents a 2-bridge link and not a knot); although this places constraints on the parity of the a_i, we will not mention these constraints explicitly. Unless otherwise stated, "K has n distinct boundary slopes" should be taken to mean "K has precisely n distinct boundary slopes".

Lemma 3. *K has two distinct slopes if and only if K is a torus knot.*

Proof. We proceed with several cases depending on n, the length of the simple continued fraction of p/q. Note that a 2-bridge torus knot will have fraction p/q of the form $1/q$ or $(q-1)/q$.

CASE 1 ($n = 0$): $p/q = [0, a_0] = 1/a_0$. So, K is a torus knot and, by Lemma 2, has two distinct boundary slopes $\partial[0]$ and $\partial[1]$.

CASE 2 ($n = 1$): $p/q = [0, a_0, a_1] = a_1/(a_0a_1 + 1)$. This represents a torus knot only when $a_0 = 1$ (since $a_1 > 1$, by assumption). When $a_0 = 1$, we get two boundary slopes, $\partial[01]$ and $\partial[10]$; $\partial[00]$ is not a boundary slope (since $|a_0| < 2$). Also, this is a torus knot, since $a_1/(a_1 + 1)$ is of the form $(q-1)/q$.

CASE 3 ($n \geq 2$): As there are at least 3 terms, p/q is not of the form $1/q$ or $(q-1)/q$ and this is not a torus knot. Also, there are at least three distinct boundary slopes (by Lemma 2): $\partial[1010101\cdots] < \partial[10010101\cdots] < \partial[01010101\cdots]$. □

Lemma 4. *If $n = 1$ then $p \mid (q-1)$.*

Proof. $p/q = [0, a_0, a_1] = a_1/(a_0a_1 + 1)$. Note that $a_1 \mid a_0a_1$ and $a_0a_1 = q - 1$. □

Lemma 5. *If $n = 2$, then K has three distinct boundary slopes if and only if either $a_0 = 1$, or else $a_1 = 1$ and $a_0 = a_2$.*

Proof. By Theorem 2, there are at most five boundary slopes: $\partial[000]$, $\partial[001]$, $\partial[010]$, $\partial[100]$, and $\partial[101]$. Recall that $\partial[S]$ is a boundary slope only if the substitution pattern S results in a continued fraction with each term at least two in absolute value.

($\Rightarrow$) Assume that $a_0 > 1$ and that either $a_1 > 1$ or $a_0 \neq a_2$. By Lemma 2, we have at least three distinct boundary slopes: $\partial[101]$, $\partial[100]$, and $\partial[010]$. We will show that a fourth boundary slope also exists.

CASE 1 ($a_0 \neq a_2$): In this case, $\partial[001]$ will be a boundary slope different from $\partial[100]$. Indeed, using the proof of Lemma 2, $\partial[001] = \partial[000] - a_2$ while $\partial[100] = \partial[000] - a_0$. $\partial[001]$ is also different from $\partial[101]$ and $\partial[010]$ (by Lemma 2).

CASE 2 ($a_0 = a_2$ and $a_1 > 1$): Since $a_1, a_2, a_3 > 1$, $\partial[000]$ is a boundary slope. Further, by Lemma 2, it is different from $\partial[100]$, and it is also different from $\partial[101]$ and $\partial[010]$.

($\Leftarrow$)

CASE 1 ($a_0 = 1$): $\partial[000]$ and $\partial[001]$ are not boundary slopes, so K has three distinct boundary slopes.

CASE 2 ($a_1 = 1$ and $a_0 = a_2$): $\partial[000]$, once again, is not a boundary slope. Also, $\partial[100] = \partial[001]$ since $a_0 = a_2$. So K has three distinct boundary slopes. $\square$

Lemma 6. *If $n = 2$ and K has three distinct boundary slopes, then $(q-p)\,|\,(q-1)$ or $p \mid (q+1)$.*

Proof. Since K has precisely three boundary slopes, Lemma 5 tells us that either $a_0 = 1$, or else $a_1 = 1$ and $a_0 = a_2$.

CASE 1 ($a_0 = 1$): $[0, 1, a_1, a_2] = (a_1a_2 + 1)/(a_1a_2 + a_2 + 1)$. Then $q - p = a_2$, so $(q-p) \mid (q-1)$.

CASE 2 ($a_1 = 1$ and $a_0 = a_2$): $[0, a_0, 1, a_0] = (a_0 + 1)/(a_0^2 + 2a_0) = (a_0 + 1)/((a_0 + 1)^2 - 1)$, so, in this case, $p \mid (q+1)$. $\square$

Lemma 7. *If $n = 3$ and K has three distinct boundary slopes, then $(q-p)\,|\,(q+1)$.*

Proof. First, we determine the form of the simple continued fraction given that there are precisely three boundary slopes. Note that, by Lemma 2, there exist at least four boundary slope continued fractions, obtained from substitution patterns 0101, 0100, 1001, 1010. Also note that $\partial[1000]$ must not be a boundary slope, since, if it were, it would be different from $\partial[1010]$, $\partial[0101]$, and $\partial[1001]$, giving us a fourth boundary slope. Similarly, $\partial[0010]$ cannot exist since it would be different from $\partial[1010]$, $\partial[0101]$, and $\partial[0100]$, also giving us a fourth boundary slope. Therefore, since $\partial[1000]$ isn't a boundary slope, $a_2 = 1$. Similarly, since $\partial[0010]$ isn't a boundary slope, $a_0 = 1$. From this, we can also conclude that $\partial[0000]$ and $\partial[0001]$ are not boundary slopes.

Now, we have four boundary slopes: $\partial[1010]$, $\partial[0101]$, $\partial[1001]$, and $\partial[0100]$. In order to have only three distinct boundary slopes, we need two of these to be equal. By Lemma 2, the only possibility is, $\partial[1001] = \partial[0100]$. Using the proof of Lemma 2, we have $\partial[0100] = \partial[0000] + a_1$ and $\partial[1001] = \partial[0000] + a_3 - a_0$. Thus, $a_3 = a_1 + a_0$ and, since $a_0 = 1$, we have $a_3 = a_1 + 1$.

So, the simple continued fraction must be of the form $[0, 1, a, 1, a+1] = (a^2 + 3a + 1)/(a^2 + 4a + 3) = (a^2 + 3a + 1)/((a+2)^2 - 1)$. Then, $q - p = a + 2$, and so $(q-p) \mid (q+1)$. $\square$

Lemma 8. *If $n \geq 4$, then K has at least four distinct boundary slopes.*

Proof. When $n = 4$, by Lemma 2, the four boundary slopes $\partial[10101]$, $\partial[01010]$, $\partial[10010]$, and $\partial[10100]$ are all distinct. If $n > 4$, by appending $101010\cdots$ or $010101\cdots$ to the patterns for $n = 4$, we will have still have at least four distinct boundary slopes. □

Theorem 4 is now proved.

Next, we investigate knots with four boundary slopes.

Theorem 5. *Let $K = K(p/q)$ be a 2-bridge knot. If K has precisely four boundary slopes, then one of the following holds: $p \mid (q+1)$, $(q-p) \mid (q+1)$, $(p \pm 1) \mid q$, or $(q - p \pm 1) \mid q$.*

Proof. As the argument is quite similar to that of Theorem 4, we will only give an outline. A knot with four boundary slopes must have $2 \leq n \leq 5$.

If $n = 2$, there are two ways to obtain exactly four boundary slopes. Simple continued fractions of the form $[0, a, b, a]$ with a, b both odd and greater than one result in a fraction p/q with $(p+1) \mid q$. (Note that the condition on the parity of a and b ensures that p/q has odd denominator and, therefore, represents a knot rather than a link.) Those of the form $[0, a, 1, b]$ with $a, b > 1$, $a \neq b$, and a, b not both even yield $p \mid (q+1)$.

If $n = 3$, there are four types of simple continued fractions that result in exactly four boundary slopes. If $p/q = [0, a+1, a, 1, a]$ with $a > 1$ even, then $(p+1) \mid q$. When $p/q = [0, 1, a, 1, b]$ with $b > 1$ and either a even or b odd, we have $(q-p) \mid (q+1)$. The case that $p/q = [0, 1, a, b, a+1]$ with a even and $b \geq 3$ odd, results in $(q - p + 1) \mid q$. Finally, if $p/q = [0, a, 1, a, a+1]$ with a even, then $(p-1) \mid q$.

When $n = 4$, there is one way for a two bridge knot to have exactly four boundary slopes, namely $[0, 1, a, a, 1, a]$ with $a > 1$ even. In this case, $(q - p + 1) \mid q$. Finally, the only two bridge knot with a simple continued fraction of length $n = 5$ having exactly four boundary slopes is $13/21 = [0, 1, 1, 1, 1, 1, 2]$. □

This completes the proof of Theorem 3.

References

[1] C. Ernst and D.W. Sumners: *The growth of the number of prime knots*, Math. Proc. Cambridge Philos. Soc. **102** (1987), 303–315.

[2] A. Hatcher and W. Thurston: *Incompressible surfaces in 2-bridge knot complements*, Invent. Math. **79** (1985), 225–246.

[3] K. Ichihara: Private Communication.

[4] K. Ichihara and S. Mizushima: *Crossing number and diameter of boundary slope set of Montesinos knot*, preprint (2005), `math.GT/0510370`.

[5] M. Ishikawa and K. Shimokawa: *Boundary slopes and crossing numbers of knots*, preprint (2002).

[6] R. Langford: *The computer-assisted visualization of rational knots and torus knots*, M.S. Thesis, CSU, Chico, (2006), available at `http://www.csuchico.edu/math/mattman`.

[7] T. Mattman, G. Maybrun and K. Robinson: *Boundary slope diameter and crossing number of 2-bridge knots*, preprint (2005), `math.GT/0510496`.

[8] L. Moser: *Elementary surgery along a torus knot*, Pacific J. Math. **38** (1971), 737–745.

[9] K. Murasugi: *On the braid index of alternating links*, Trans. Amer. Math. Soc. **326** (1991), 237–260.

[10] D. Tanguay: *Chirugies finies et noeuds rationnels*, Ph.D. Thesis, Univ. Quebec à Montréal, (1995).

[11] C.M. Tsau: *Incompressible surfaces in the knot manifolds of torus knots*, Topology **33** (1994), 197–201.

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