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ON THE CROSS RATIO VARIETY FROM THE VIEWPOINT OF THE ROOT SYSTEM OF TYPE $3A_2$

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Abstract

We observe Naruki's cross ratio variety from the viewpoint of the root system of type $3A_2$. We construct some kind of models as moduli space of marked cubic surfaces on which the action of a Weyl subgroup of type $3A_2$ and its normalizer can be easily observed. We describe the structure of our model and its relationship to the cross ratio variety.

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Introduction

A *marked* smooth cubic surface means a smooth cubic surface in $\mathbb{P}_3$ endowed with a *marking* which is an isomorphism of the fixed combinatorial model of the 27 lines onto the actual line-configuration on the surface. The automorphism group of the model is isomorphic to the Weyl group $W(E_6)$ of type E_6, and thus the moduli space $\mathcal{M}$ of marked smooth cubic surfaces admits a natural action of $W(E_6)$ on itself. In [11], Naruki constructed a smooth equivariant compactification of $\mathcal{M}$ with respect to this action. He uses the 270 Cayley's cross ratios arising from the quadruplets of collinear tritangents of the surface, so this compactification is called the cross ratio variety and is denoted by $\mathcal{C}$ in [11]. The 270 cross ratios divide themselves into 45 systems to each of which the belonging six cross ratios are permuted by simple linear fractions, so there are essentially 45 of them. These 45 systems, attributed geometrically to the 45 tritangents, correspond exactly to the 45 root subsystems of type D_4 in the root system of type E_6 for the marked surface. By fixing one root subsystem of this kind and by using the so-called modified Cayley family of cubic surfaces, $\mathcal{C}$ is described as an equivariant modification of the Weyl-chamber-compactification of the adjoint torus $T(D_4)$ of type D_4 with respect to the action of $W(D_4)$.

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This description of $\mathcal{C}$ is naturally advantageous when observing the action on $\mathcal{C}$ of a Weyl subgroup of type D_4, or of its normalizer which is of type F_4 and is of index 45, one of the maximal subgroups in $W(E_6)$. There are known some descriptions of $\mathcal{C}$ (or of similar compactifications) focusing on the actions of some subgroups of the maximal type. For example, recently E. Colombo and B. van Geemen have given a description of this kind by using the projective space associated with the Cartan subalgebra of type D_5, on which the action of a Weyl subgroup $W(D_5)$ of $W(E_6)$ is naturally visible. This subgroup is one of the maximal subgroups of $W(E_6)$ with index 27 (cf. [4]). A model of the compactified moduli space on which the Weyl subgroup $W(A_5)$ acts naturally has essentially been known in the classical works of Coble [5, 6, 7], this can be described as a double covering of $\mathbb{P}_4$, which admits natural action of the normalizer of $W(A_5)$, one of the maximal subgroups of $W(E_6)$ with index 36 (see for example [8]). There is still missing a compactified model of the moduli space on which the maximal subgroup of index 40 acts in a natural way. This group is the normalizer of the Weyl subgroup $W(3A_2)$ associated with a root subsystem of type $3A_2 = A_2 \times A_2 \times A_2$ in E_6.

In this article, I will present one of this type of models by joining three birational varieties. Each of the three varieties is described by using (essentially two) simple relations among the six special cross ratios associated with a root subsystem of type A_2, which define a 4-dimensional variety in $\mathbb{P}_1^6$. This variety is already birational to $\mathcal{C}$, so it is a blowing-down of $\mathcal{C}$, and our model lies (as the graph) in the direct product of the three (naturally birational) varieties, each associated with one of the A_2's in $3A_2$. This is very near to $\mathcal{C}$ in the following sense: There are 27 non-singular rational curves in $\mathcal{C}$, each of which is the intersection of three A_1-divisors associated with root subsystem of type $3A_1$ lying in the fixed $3A_2$. They are disjoint and can be separately blown down to 27 singular points. Our model is exactly this blowing down.

I have received cordial guidance to the subject and many helpful advices from Professor I. Naruki. For that I would like to express here my deepest gratitude to him.

1. The fixed model of formal lines and the associated notation

We recall several basic facts about marked smooth cubic surfaces and the cross ratio variety for introducing notation.

The standard odd unimodular hyperbolic lattice H_0 of the signature (1, 6) is isomorphic to the Picard lattice of a smooth cubic surface S. We denote the standard orthogonal basis by $\{l_0; e_1, \dots, e_6\}$ and the symmetric bilinear form by $(\ ,\ )$. Then the corresponding one to the canonical class of S is given in H_0 as

$$k_0 := -3l_0 + \sum_{i=1}^{6} e_i$$

and the set (of divisor classes) of the 27 lines on S is represented in H_0 as follows:

$$\mathcal{L}_0 := \{x \in H_0 \mid (x, x) = (x, k_0) = -1\}$$

which explicitly consists of (formal) lines $e_1, \ldots, e_6$ in the basis and fifteen and six elements denoted by f_{ij} $(i \neq j)$ and g_i:

$$f_{ij} = l_0 - e_i - e_j, \quad g_i = 2l_0 - \sum_{j \neq i} e_j.$$

The set $\mathcal{L}_0$ endowed with the intersection product $(\ ,\ )$ on it is a combinatorial model of the 27 lines, which we will fix in this paper.

The orthogonal complement to the sublattice $\mathbb{Z}k_0$ in H_0 is identified with the root lattice $L(E_6)$ of type E_6 by taking a fundamental system of roots as follows:

$$\alpha_1 := e_1 - e_2, \quad \alpha_2 := l_0 - e_1 - e_2 - e_3,$$
$$\alpha_3 := e_2 - e_3, \quad \alpha_4 := e_3 - e_4, \quad \alpha_5 := e_4 - e_5, \quad \alpha_6 := e_5 - e_6$$

with its Dynkin diagram:

$$\begin{array}{ccccccccc} \alpha_1 & \text{———} & \alpha_3 & \text{———} & \alpha_4 & \text{———} & \alpha_5 & \text{———} & \alpha_6 \\ & & & & \big| & & & & \\ & & & & \alpha_2 & & & & \end{array} \tag{1.1}$$

Then the 36 positive roots are explicitly

$$\begin{aligned} r_{ij} &:= e_i - e_j & (i < j) \\ r_{ijk} &:= l_0 - e_i - e_j - e_k & (i < j < k) \\ r_0 &:= 2l_0 - \sum_{i=1}^{6} e_i \end{aligned}$$

where r_0 is the maximal root. We see that the automorphism group of $\mathcal{L}_0$ is the Weyl group $W(E_6)$, and then $W(E_6)$ acts naturally on the moduli space $\mathcal{M}$ of marked smooth cubic surfaces.

We have the 45 (formal) ones as corresponding object to the tritangents on S, which are explicitly given, under the (Schläffi) notation, as fifteen ones of the type $T_{(ij)(kl)(mn)}$ and thirty ones of the type T_{ij} $(\neq T_{ji})$:

$$T_{(ij)(kl)(mn)} = \{f_{ij}, f_{kl}, f_{mn}\}, \quad T_{ij} = \{f_{ij}, g_i, e_j\}$$

where all the indices are assumed to be different. This will also be assumed for such notations as $\Delta_{(ijk)(lmn)}$, $\Delta_{(ij)(kl)(mn)}$ appearing later.

For each tritangent T (which is a tritangent on S by a marking) and for a line on T, there are four collinear tritangents, outside T, passing through the line; which determine four points on $\mathbb{P}_1$ (as the pencil of planes through the line) and we have six cross ratios of them. Cayley showed that these only depend on T, and they are known as Cayley's cross ratios associated with T. Since the six cross ratios are transitively permuted among themselves by the simple linear fractions, one can say that there is essentially one of them, one for each T.

We have anyway 270 (= $6 \cdot 45$) cross ratios and they determine an embedding of $\mathcal{M}$ into $\mathbb{P}_1^{270}$. The closure of (the image of) $\mathcal{M}$ in $\mathbb{P}_1^{270}$ is the cross ratio variety $\mathcal{C}$. By the reason mentioned above, it is sufficient for observing $\mathcal{C}$ to choose one cross ratio for each of the tritangents and to embed $\mathcal{C}$ into $\mathbb{P}_1^{45}$; that is, $\mathcal{C}$ is embedded also through the projection of $\mathbb{P}_1^{270}$ onto $\mathbb{P}_1^{45}$ defined by the choice. (We will give and fix a choice of this kind in the next section.)

The boundary $\mathcal{C} - \mathcal{M}$ consists of 36 A_1-divisors and 40 cusp divisors (called N-divisors in [11]). These 36 and 40 divisors are associated with the 36 and 40 root subsystems of type A_1 and $3A_2$ in E_6 respectively, where $3A_2$ means the union of three root subsystems of type A_2 being orthogonal to each other. The 40 root subsystems of type $3A_2$ are explicitly given to be the ten of the type $\Delta_{(ijk)(lmn)}$ and the thirty of the type $\Delta_{(ij)(kl)(mn)}$ ($\neq \Delta_{(ij)(mn)(kl)}$):

$$\Delta_{(ijk)(lmn)} = \begin{Bmatrix} \pm r_{jk}, & \pm r_{lmn}, & \pm r_{lm} \\ \pm r_{ik}, & \pm r_0, & \pm r_{ln} \\ \pm r_{ij}, & \pm r_{ijk}, & \pm r_{mn} \end{Bmatrix}, \quad \Delta_{(ij)(kl)(mn)} = \begin{Bmatrix} \pm r_{ij}, & \pm r_{kl}, & \pm r_{mn} \\ \pm r_{jkl}, & \pm r_{lmn}, & \pm r_{ijn} \\ \pm r_{ikl}, & \pm r_{kmn}, & \pm r_{ijm} \end{Bmatrix}.$$

We note that the 40 cusp divisors, which all isomorphic to $\mathbb{P}_1^3$, are contracted to the 40 cusps of the GIT-compactification $\bar{\mathcal{M}}$, and then the 36 A_1-divisors are contracted to the 36 boundary divisors of $\bar{\mathcal{M}}$ which are all isomorphic to the Segre cubic threefold (cf. [8]).

2. Projection of the cross ratio variety and the definition of our model

To observe the cross ratio variety $\mathcal{C}$, we will introduce several projections of $\mathcal{C}$ associated with a root subsystem Δ of type $3A_2$. We choose $\Delta_{(123)(456)}$ as Δ, which seems to be natural for the standard diagram (1.1); for, that is generated by the subdiagram of type $3A_2$ in the extended diagram:

$$\begin{tikzcd}[arrows=dash]
r_{12} \arrow[r] & r_{23} \arrow[r] & r_{34} \arrow[r] \arrow[d] & r_{45} \arrow[r] & r_{56} \\
& & r_{123} \arrow[d] & & \\
& & -r_0 & &
\end{tikzcd} \tag{2.1}$$

For the three root subsystems of type A_2 in Δ, we give their labels as follows:

$$\delta_\lambda := \{\pm r_{12}, \pm r_{23}, \pm r_{13}\}, \quad \delta_\mu := \{\pm r_{123}, \pm r_{456}, \pm r_0\}, \quad \delta_\nu := \{\pm r_{45}, \pm r_{56}, \pm r_{46}\}.$$

(Here we have in mind the correspondence to the parameters λ, μ, ν in [11].)

For the action of the Weyl subgroup $W(\Delta)$, the 45 tritangents are divided into four orbits, one of which consists of 27 tritangents and each of the others of six tritangents $(45 = 27 + 6 + 6 + 6)$. The six tritangents of the latter type orbit are associated with a root subsystem δ of type A_2 in Δ, that is, they are the tritangents on which the Weyl subgroup $W(\delta)$ acts trivially; more precisely, the ones corresponding to the root subsystems of type D_4 which contain δ. In particular, the tritangents associated with $\delta_\lambda, \delta_\mu, \delta_\nu$ are explicitly given as follows:

$$\delta_\lambda \leftrightarrow \begin{cases} T_{45},\ T_{56},\ T_{64} \\ T_{54},\ T_{46},\ T_{65} \end{cases}$$

$$\delta_\mu \leftrightarrow \begin{cases} T_{(14)(26)(35)},\ T_{(16)(25)(34)},\ T_{(15)(24)(36)} \\ T_{(14)(25)(36)},\ T_{(16)(24)(35)},\ T_{(15)(26)(34)} \end{cases}$$

$$\delta_\nu \leftrightarrow \begin{cases} T_{31},\ T_{12},\ T_{23} \\ T_{32},\ T_{13},\ T_{21}. \end{cases}$$

We note that, for each δ_i $(i = \lambda, \mu, \nu)$, the upper part tritangents and the lower part, not collinear in each part, form so-called Steiner trihedral pair, and that the three trihedral pairs are the complementary ones to each other. Thus we have obtained 18 tritangents associated with Δ.

By choosing the 18 cross ratios for the tritangents associated with Δ, we obtain the projection of $\mathbb{P}_1^{45}$ onto $\mathbb{P}_1^{18}$ (after some choice of 45 cross-ratio-representatives), and by restricting this to $\mathcal{C}$, we obtain a projection of $\mathcal{C}$ into $\mathbb{P}_1^{18}$, whose image will be denoted by $\mathcal{C}_\Delta$ and this is exactly our model. Now the objective of this paper is to describe how near $\mathcal{C}_\Delta$ is to the cross ratio variety $\mathcal{C}$. (We will in fact see later that the projection $\mathcal{C} \to \mathcal{C}_\Delta$ contracts only 27 mutually disjoint non-singular rational curves in $\mathcal{C}$ separately, each onto a singular point of $\mathcal{C}_\Delta$.) Since $\mathcal{C}_\Delta$ is too complicated to describe, we have to project it further by choosing one root subsystem δ of type A_2 in Δ and by extracting the components corresponding to the six tritangents associated with δ. The image of the projection is then denoted by $\mathcal{C}_\delta$. We have thus obtained three varieties $\mathcal{C}_{\delta_\lambda}, \mathcal{C}_{\delta_\mu}, \mathcal{C}_{\delta_\nu}$, which support our model together.

To describe $\mathcal{C}_{\delta_i}$ $(i = \lambda, \mu, \nu)$ explicitly, we actually choose one cross ratio for each of the tritangents associated with δ_i and we give them labels as follows:

$$\delta_\lambda \leftrightarrow \begin{cases} u_\lambda := \dfrac{\lambda - 1}{\lambda\mu\rho - 1}, \quad v_\lambda := \dfrac{(\lambda - 1)(\lambda\mu\nu\rho^2 - 1)}{(\lambda\rho - 1)(\lambda\mu\nu\rho - 1)}, \quad w_\lambda := \dfrac{\lambda - 1}{\lambda\nu\rho - 1} \\[2ex] x_\lambda := \dfrac{\lambda - 1}{\lambda\mu\nu\rho - 1}, \quad y_\lambda := \dfrac{(\lambda - 1)(\lambda\mu\nu\rho^2 - 1)}{(\lambda\mu\rho - 1)(\lambda\nu\rho - 1)}, \quad z_\lambda := \dfrac{\lambda - 1}{\lambda\rho - 1} \end{cases}$$

$$\delta_\mu \leftrightarrow \begin{cases} u_\mu := \dfrac{\mu-1}{\mu\nu\rho-1}, & v_\mu := \dfrac{(\mu-1)(\lambda\mu\nu\rho^2-1)}{(\mu\rho-1)(\lambda\mu\nu\rho-1)}, \quad w_\mu := \dfrac{\mu-1}{\lambda\mu\rho-1} \\ x_\mu := \dfrac{\mu-1}{\lambda\mu\nu\rho-1}, & y_\mu := \dfrac{(\mu-1)(\lambda\mu\nu\rho^2-1)}{(\lambda\mu\rho-1)(\mu\nu\rho-1)}, \quad z_\mu := \dfrac{\mu-1}{\mu\rho-1} \end{cases}$$

$$\delta_\nu \leftrightarrow \begin{cases} u_\nu := \dfrac{\nu-1}{\lambda\nu\rho-1}, & v_\nu := \dfrac{(\nu-1)(\lambda\mu\nu\rho^2-1)}{(\nu\rho-1)(\lambda\mu\nu\rho-1)}, \quad w_\nu := \dfrac{\nu-1}{\mu\nu\rho-1} \\ x_\nu := \dfrac{\nu-1}{\lambda\mu\nu\rho-1}, & y_\nu := \dfrac{(\nu-1)(\lambda\mu\nu\rho^2-1)}{(\lambda\nu\rho-1)(\mu\nu\rho-1)}, \quad z_\nu := \dfrac{\nu-1}{\nu\rho-1} \end{cases}$$

where the expression of cross ratios above is the one in [11]. We have identified here the Schläfli labels of the tritangents in this paper with Cayley's ones used in [11] by putting the following correspondence of mutually skew six lines:

$$\begin{array}{ll} e_1 \leftrightarrow \overline{\mathrm{l}} \cap \overline{\mathrm{n}'} \cap \overline{\mathrm{p}} \cap \overline{\mathrm{r}'} \cap \overline{\mathrm{y}} & e_4 \leftrightarrow \mathrm{l}' \cap \mathrm{m} \cap \mathrm{p}' \cap \mathrm{q} \cap \mathrm{z} \\ e_2 \leftrightarrow \overline{\mathrm{h}} \cap \mathrm{r}' \cap \overline{\mathrm{r}} \cap \theta \cap \zeta & e_5 \leftrightarrow \overline{\mathrm{l}'} \cap \overline{\mathrm{m}} \cap \overline{\mathrm{p}'} \cap \overline{\mathrm{q}} \cap \overline{\mathrm{z}} \\ e_3 \leftrightarrow \overline{\mathrm{f}} \cap \mathrm{g} \cap \mathrm{n} \cap \mathrm{n}' \cap z & e_6 \leftrightarrow \mathrm{w} \cap \mathrm{x} \cap \overline{\mathrm{x}} \cap x \cap \xi. \end{array}$$

For the benefit of the reader, we attach the complete list of the correspondence of the names of tritangents below:

$$\begin{array}{lll} T_{(12)(34)(56)} \leftrightarrow \overline{\theta} & T_{12} \leftrightarrow \zeta & T_{21} \leftrightarrow \overline{\mathrm{r}'} \\ T_{(12)(35)(46)} \leftrightarrow \mathrm{h} & T_{13} \leftrightarrow z & T_{31} \leftrightarrow \overline{\mathrm{n}'} \\ T_{(12)(36)(45)} \leftrightarrow \mathrm{r} & T_{14} \leftrightarrow \mathrm{z} & T_{41} \leftrightarrow \overline{\mathrm{l}} \\ T_{(13)(24)(56)} \leftrightarrow \mathrm{f} & T_{15} \leftrightarrow \overline{\mathrm{z}} & T_{51} \leftrightarrow \overline{\mathrm{p}} \\ T_{(13)(25)(46)} \leftrightarrow \overline{\mathrm{g}} & T_{16} \leftrightarrow \mathrm{w} & T_{61} \leftrightarrow \overline{\mathrm{y}} \\ T_{(13)(26)(45)} \leftrightarrow \overline{\mathrm{n}} & T_{23} \leftrightarrow \mathrm{n}' & T_{32} \leftrightarrow \mathrm{r}' \\ T_{(14)(23)(56)} \leftrightarrow \mathrm{p} & T_{24} \leftrightarrow \mathrm{m} & T_{42} \leftrightarrow \overline{\mathrm{h}} \\ T_{(14)(25)(36)} \leftrightarrow \mathrm{q}' & T_{25} \leftrightarrow \overline{\mathrm{q}} & T_{52} \leftrightarrow \theta \\ T_{(14)(26)(35)} \leftrightarrow \overline{\mathrm{m}}' & T_{26} \leftrightarrow \overline{\mathrm{x}} & T_{62} \leftrightarrow \overline{\mathrm{r}} \\ T_{(15)(23)(46)} \leftrightarrow \mathrm{l} & T_{34} \leftrightarrow \mathrm{q} & T_{43} \leftrightarrow \mathrm{g} \\ T_{(15)(24)(36)} \leftrightarrow \mathrm{m}' & T_{35} \leftrightarrow \overline{\mathrm{m}} & T_{53} \leftrightarrow \overline{\mathrm{f}} \\ T_{(15)(26)(34)} \leftrightarrow \overline{\mathrm{q}'} & T_{36} \leftrightarrow \mathrm{x} & T_{63} \leftrightarrow \mathrm{n} \\ T_{(16)(23)(45)} \leftrightarrow \mathrm{y} & T_{45} \leftrightarrow \overline{\mathrm{l}'} & T_{54} \leftrightarrow \mathrm{p}' \\ T_{(16)(24)(35)} \leftrightarrow y & T_{46} \leftrightarrow x & T_{64} \leftrightarrow \mathrm{l}' \\ T_{(16)(25)(34)} \leftrightarrow \eta & T_{56} \leftrightarrow \xi & T_{65} \leftrightarrow \overline{\mathrm{p}'}. \end{array}$$

(We remark that the correspondence is a little different from the one of [12] or [3] though equivalent to it under the action of $W(E_6)$.)

We have to remark here that, under the correspondence above, reflections s_1, s_6 in [11] should be interpreted as the ones with respect to roots $\pm r_{345}, \pm r_{156}$, so s_1, s_6 are different from the actions of the reflections associated with fundamental roots α_1, α_6 (of this paper). The roots $\pm r_{12}, \pm r_{56}$ are interchanged with $\pm r_{345}, \pm r_{156}$ respectively under the action of the non-trivial central element of the Weyl subgroup of type D_4 associated with T_{16}, the central subdiagram of type D_4 in (2.1), taken for determining the parameters λ, μ, ν, ρ. Thus the reflections σ_1, σ_6 associated with α_1, α_6 act as $\gamma s_1 \gamma^{-1}, \gamma s_6 \gamma^{-1}$ where γ denotes the central action interchanging $(\lambda, \mu, \nu, \rho)$ and $(\lambda^{-1}, \mu^{-1}, \nu^{-1}, \rho^{-1})$. They are explicitly given as follows:

$$\sigma_1 : \begin{cases} \lambda \mapsto -\dfrac{\lambda(\lambda\mu\nu\rho^2 - 1)}{\lambda - 1} \\ \mu \mapsto \dfrac{\mu(\lambda\rho - 1)(\lambda\nu\rho - 1)}{(\lambda\mu\rho - 1)(\lambda\mu\nu\rho - 1)} \\ \nu \mapsto \dfrac{\nu(\lambda\rho - 1)(\lambda\mu\rho - 1)}{(\lambda\nu\rho - 1)(\lambda\mu\nu\rho - 1)} \\ \rho \mapsto \dfrac{\rho(\lambda - 1)(\lambda\mu\nu\rho - 1)}{(\lambda\rho - 1)(\lambda\mu\nu\rho^2 - 1)} \end{cases} \quad \sigma_6 : \begin{cases} \lambda \mapsto \dfrac{\lambda(\nu\rho - 1)(\mu\nu\rho - 1)}{(\lambda\nu\rho - 1)(\lambda\mu\nu\rho - 1)} \\ \mu \mapsto \dfrac{\mu(\nu\rho - 1)(\lambda\nu\rho - 1)}{(\mu\nu\rho - 1)(\lambda\mu\nu\rho - 1)} \\ \nu \mapsto -\dfrac{\nu(\lambda\mu\nu\rho^2 - 1)}{\nu - 1} \\ \rho \mapsto \dfrac{\rho(\nu - 1)(\lambda\mu\nu\rho - 1)}{(\nu\rho - 1)(\lambda\mu\nu\rho^2 - 1)}. \end{cases}$$

Now, we mention how the normalizer of the Weyl subgroup $W(\Delta)$ acts on the 18 cross ratios chosen for Δ, which naturally induces the action on our model $\mathcal{C}_\Delta$. We recall that there is a unique subgroup of $W(E_6)$ which induces the symmetric group $\mathcal{S}_3$ of λ, μ, ν (and fixes ρ) in the parameter space of [11]. This subgroup, lying originally in the normalizer of the Weyl subgroup of type D_4 above, lies also in the normalizer of $W(\Delta)$, so it is a semi-direct summand; namely, the normalizer is isomorphic to

$$W(\Delta) \rtimes \mathcal{S}_3 \cong (W(\delta_\lambda) \times W(\delta_\mu) \times W(\delta_\nu)) \rtimes \mathcal{S}_3.$$

Thus one can in principle describe the action of the normalizer.

The action of $\mathcal{S}_3$ is clearly seen, since the 18 cross ratios are given in terms of λ, μ, ν, ρ. For example, the transposition of μ and ν sends u_λ, u_μ, u_ν to w_λ, w_μ, w_ν respectively, while, for the other coordinates, only μ and ν in their labels are interchanged. We remark that the elements of order 3 in $\mathcal{S}_3$ act simply as the cyclic permutations of the indices λ, μ, ν.

For describing the action of $W(\Delta) \cong W(\delta_\lambda) \times W(\delta_\mu) \times W(\delta_\nu)$, it now suffices to describe the action of $W(\delta_\lambda)$. The reflection σ_{12} associated with $\pm r_{12}$ transforms $u_\lambda, v_\lambda, w_\lambda, x_\lambda, y_\lambda, z_\lambda$ to their inverses respectively, while the reflection σ_{23} associated with $\pm r_{23}$ transforms them to $u_\lambda/(u_\lambda - 1)$, $v_\lambda/(v_\lambda - 1)$, etc. respectively. The remaining 12 coordinates (associated with δ_μ, δ_ν) are transposed by σ_{12}, σ_{23} exactly in the same way as the corresponding tritangents to them are transposed by σ_{12}, σ_{23}; for example by σ_{12} we have $u_\mu \leftrightarrow y_\mu$, $v_\mu \leftrightarrow x_\mu$, $w_\mu \leftrightarrow z_\mu$ and $u_\nu \leftrightarrow z_\nu$, $v_\nu \leftrightarrow x_\nu$, $w_\nu \leftrightarrow y_\nu$.

Now, we will describe $\mathcal{C}_{\delta_i}$ in $\mathbb{P}_1^6$ explicitly. We begin with the proposition:

Proposition 2.1. *Between the six cross ratios $u_i, v_i, \ldots, z_i$ chosen for δ_i, we have the following three relations*:

$$u_i v_i w_i = x_i y_i z_i,$$

$$(1-u_i)(1-v_i)(1-w_i) = (1-x_i)(1-y_i)(1-z_i),$$
$$(1-u_i^{-1})(1-v_i^{-1})(1-w_i^{-1}) = (1-x_i^{-1})(1-y_i^{-1})(1-z_i^{-1}).$$

The first relation is immediately checked by the explicit expression of the cross ratios. The other two are deduced from this by using the action of $W(\delta_i)$. But, from the algebraic point of view, only two of the three relations are independent, and which two should be chosen as the basis depends on where we observe the subvariety of $\mathbb{P}_1^6$ defined by the above three equations. We should rather pass to the multi-homogeneous coordinates $((U_0 : U_\infty), \dots, (Z_0 : Z_\infty))$ for $\mathbb{P}_1^6$ by putting the identification $u_i = U_0/U_\infty, \dots, z_i = Z_0/Z_\infty$. (We have omitted the subscript i from the homogeneous coordinates for short.)

Now the above equations are rewritten as follows:

$$U_0V_0W_0X_\infty Y_\infty Z_\infty = U_\infty V_\infty W_\infty X_0Y_0Z_0,$$
$$X_\infty Y_\infty Z_\infty(U_\infty - U_0)(V_\infty - V_0)(W_\infty - W_0) = U_\infty V_\infty W_\infty(X_\infty - X_0)(Y_\infty - Y_0)(Z_\infty - Z_0),$$
$$X_0Y_0Z_0(U_\infty - U_0)(V_\infty - V_0)(W_\infty - W_0) = U_0V_0W_0(X_\infty - X_0)(Y_\infty - Y_0)(Z_\infty - Z_0).$$

An equivalent condition is obviously given by requiring

$$\operatorname{rank}\begin{pmatrix} U_\infty V_\infty W_\infty & U_0V_0W_0 & (U_\infty - U_0)(V_\infty - V_0)(W_\infty - W_0) \\ X_\infty Y_\infty Z_\infty & X_0Y_0Z_0 & (X_\infty - X_0)(Y_\infty - Y_0)(Z_\infty - Z_0) \end{pmatrix} \leq 1,$$

so the subvariety is a kind of determinantal varieties. This homogeneous reformulation of the equations is of importance since there lie singular points of the subvariety outside the patch $\mathbb{C}^6$ of $\mathbb{P}_1^6$ on which $u_i, \dots, z_i$ are given. Since there are locally two independent equations, the subvariety is 4-dimensional. Moreover, as we will see later, this is irreducible and birational to $\mathcal{C}$; so it coincides with $\mathcal{C}_{\delta_i}$.

For proving the birationality, it suffices to show that the other cross ratios are all expressed rationally by the six cross ratios $u_i, \dots, z_i$ associated with δ_i. We need the following lemma:

Lemma 2.2. *For a quintuplet of collinear tritangents, we can arrange the cross-ratio-representatives associated with them in the following form*:

$$\alpha, \quad \beta, \quad \frac{\beta}{\alpha}, \quad \frac{1-\alpha}{1-\beta}, \quad \frac{\alpha(1-\beta)}{\beta(1-\alpha)}.$$

More precisely, they are determined by the choice of cross-ratio-pair (α, β) *such that the ratio* β/α *is also a cross ratio.*

One can check this easily. We note that the three cross ratios other than α, β have been chosen such that the product of them is equal to 1.

Notice that one cross ratio in $\{u_i, v_i, w_i\}$ and another one in $\{x_i, y_i, z_i\}$ form such a pair (α, β) as is characterized in the above lemma. Since there are nine choices of such pairs, we obtain 27 $(= 9 \cdot 3)$ cross ratios from them. These cross ratios are the ones belonging to the tritangents other than the 18 ones for Δ. For example in the case $i = \lambda$, we obtain from the pair (u_λ, x_λ) three cross ratios

$$\frac{x_\lambda}{u_\lambda} = \frac{\lambda\mu\rho - 1}{\lambda\mu\nu\rho - 1}, \quad \frac{1 - u_\lambda}{1 - x_\lambda} = \frac{(\mu\rho - 1)(\lambda\mu\nu\rho - 1)}{(\mu\nu\rho - 1)(\lambda\mu\rho - 1)}, \quad \frac{u_\lambda(1 - x_\lambda)}{x_\lambda(1 - u_\lambda)} = \frac{\mu\nu\rho - 1}{\mu\rho - 1}$$

as ones belonging to tritangents $T_{(12)(36)(45)}$, $T_{(16)(23)(45)}$, $T_{(13)(26)(45)}$ respectively.

We can also express the remaining 12 $(= 6 + 6)$ cross ratios associated with the orthogonal complements to δ_i. Since we have seen that the cyclic permutation of the indices λ, μ, ν is in the normalizer of $W(\Delta)$, we discuss only the case $i = \lambda$. The result is summarized in the following:

Proposition 2.3. *By the six cross ratios $u_\lambda, v_\lambda, \ldots, z_\lambda$ chosen for δ_λ, the other* 12 *cross ratios associated with δ_μ, δ_ν are all expressed rationally as follows*:

$$\begin{aligned}
u_\mu &= \frac{x_\lambda(u_\lambda - z_\lambda)}{u_\lambda(x_\lambda - v_\lambda)}, & v_\mu &= \frac{y_\lambda(w_\lambda - x_\lambda)}{w_\lambda(y_\lambda - u_\lambda)}, & w_\mu &= \frac{z_\lambda(v_\lambda - y_\lambda)}{v_\lambda(z_\lambda - w_\lambda)} \\
x_\mu &= \frac{w_\lambda(y_\lambda - v_\lambda)}{y_\lambda(w_\lambda - z_\lambda)}, & y_\mu &= \frac{v_\lambda(z_\lambda - u_\lambda)}{z_\lambda(v_\lambda - x_\lambda)}, & z_\mu &= \frac{u_\lambda(x_\lambda - w_\lambda)}{x_\lambda(u_\lambda - y_\lambda)} \\
u_\nu &= \frac{z_\lambda(v_\lambda - y_\lambda)}{v_\lambda(z_\lambda - u_\lambda)}, & v_\nu &= \frac{y_\lambda(u_\lambda - x_\lambda)}{u_\lambda(y_\lambda - w_\lambda)}, & w_\nu &= \frac{x_\lambda(w_\lambda - z_\lambda)}{w_\lambda(x_\lambda - v_\lambda)} \\
x_\nu &= \frac{u_\lambda(y_\lambda - v_\lambda)}{y_\lambda(u_\lambda - z_\lambda)}, & y_\nu &= \frac{v_\lambda(z_\lambda - w_\lambda)}{z_\lambda(v_\lambda - x_\lambda)}, & z_\nu &= \frac{w_\lambda(x_\lambda - u_\lambda)}{x_\lambda(w_\lambda - y_\lambda)}.
\end{aligned}$$

Proof. We have already obtained the expression of the 27 cross ratios belonging to the tritangents except the 18 ones for Δ. Then, by applying Lemma 2.2 to suitable pairs of cross ratios in the 27 tritangents, one can find the expression in this proposition. For example, the first relation of Proposition 2.1 guarantees that the ratio of cross ratios z_λ/u_λ and v_λ/x_λ is the cross ratio w_λ/y_λ, so by Lemma 2.2 the ratio

$$\frac{x_\lambda(u_\lambda - z_\lambda)}{u_\lambda(x_\lambda - v_\lambda)} = \left(1 - \frac{z_\lambda}{u_\lambda}\right) \bigg/ \left(1 - \frac{v_\lambda}{x_\lambda}\right)$$

is a cross ratio, and we can immediately check that it coincide with u_μ itself by using Naruki's $(\lambda, \mu, \nu, \rho)$-expression of the cross ratios. The other expressions are obtained similarly. □

To sum up the discussions above, we have already proved the birationality of $\mathcal{C}_{\delta_i}$ and $\mathcal{C}$ $(i = \lambda, \mu, \nu)$. This implies also the birationality of $\mathcal{C}_\Delta$ and $\mathcal{C}$.

Theorem 2.4. *The varieties $\mathcal{C}_\Delta$ and $\mathcal{C}_{\delta_i}$ $(i = \lambda, \mu, \nu)$ are all birational to the cross ratio variety $\mathcal{C}$.*

We add here some remarks about cross ratios. From the six cross ratios chosen for δ_i, we have obtained the 27 cross ratios belonging to the tritangents other than the 18 ones associated with Δ. We remark here that the set of them does not depend on the index $i = \lambda, \mu, \nu$. It means that we have chosen the 45 cross-ratio-representatives. These coincide, up to the power ± 1, with those given in [11]. Also, in the construction of the 27 representatives, the product of the cross ratios associated with any triplet of collinear tritangents in the 27 ones is equal to 1.

In the above argument, the relations between the 18 cross ratios chosen for Δ are deduced through their relationship with the other 27 cross ratios. For example, as the expressions of the cross ratio belonging to $T_{(12)(34)(56)}$, we obtain the equality:

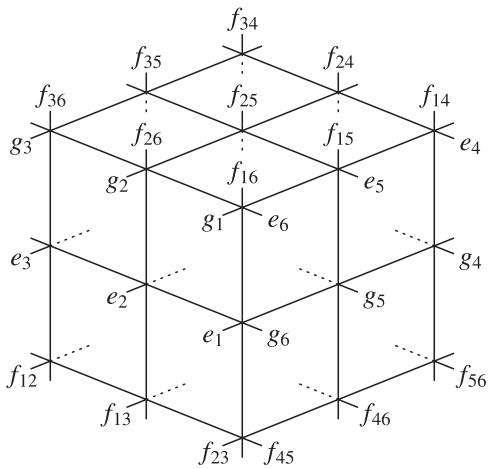
$$\frac{z_\lambda}{v_\lambda} = \frac{z_\mu}{v_\mu} = \frac{z_\nu}{v_\nu} \quad \left(= \frac{\lambda\mu\nu\rho - 1}{\lambda\mu\nu\rho^2 - 1} \right).$$

Here these expressions are all in the same form, but this is rather exceptional. In fact, we have for T_{35}

$$\frac{z_\lambda}{u_\lambda} = \frac{1 - u_\mu}{1 - y_\mu} = \frac{u_\nu(1 - x_\nu)}{x_\nu(1 - u_\nu)} \quad \left(= \frac{\lambda\mu\rho - 1}{\lambda\rho - 1} \right).$$

We can see in principle what forms of cross ratios are identified through each of the 27 tritangents. One can observe this by attaching the forms to the intersection points in the *latticed cube* below.

We close this section by explaining this latticed cube. We introduce it as the following cube constructed by the 27 lines:



All the 27 (formal) lines appear in the cube, nine by nine in the three different directions; they are indicated over the three front faces of the cube. Now one can clearly observe the 27 intersection points at each of which there meet three lines of different directions, and any two of these three lines intersect each other actually on the surface, so we can observe already 27 tritangents in the cube. We can also read the remaining 18 tritangents from the cube, that is, they divide themselves into three classes, each being observed in one of the three directions for the cube. In fact, two lines in one direction should intersect each other if and only if they are placed in neither of the other two directions of the cube. Thus we have 6 (= 3 + 3) tritangents for each direction, which form one of the three trihedral pairs associated with Δ.

3. Structure of our model and its relationship to the cross ratio variety

We begin by observing the three varieties $\mathcal{C}_{\delta_i}$ $(i = \lambda, \mu, \nu)$. They have been represented in one and the same way to be the 4-dimensional subvariety in $\mathbb{P}_1^6$, so they are isomorphic to each other. Thus we discuss mainly $\mathcal{C}_{\delta_\lambda}$ in the following.

We first observe the singular locus of $\mathcal{C}_{\delta_\lambda}$. One can easily determine it by using the multi-homogeneous equation in $\mathbb{P}_1^6$. In fact, we see that the locus consists of 36 isolated singular points and of 28 (= 1 + 27) one-dimensional irreducible components each of which is naturally isomorphic to $\mathbb{P}_1$. Each of the 36 isolated singular points is obtained by giving a pair of bijective correspondences of the sets $\{u_\lambda, v_\lambda, w_\lambda\}$, $\{x_\lambda, y_\lambda, z_\lambda\}$ to $\{0, 1, \infty\}$; for example,

$$u_\lambda = x_\lambda = 0, \quad v_\lambda = y_\lambda = 1, \quad w_\lambda = z_\lambda = \infty \tag{3.1}$$

where the given bijections are $u_\lambda \to 0$, $v_\lambda \to 1$, $w_\lambda \to \infty$ and $x_\lambda \to 0$, $y_\lambda \to 1$, $z_\lambda \to \infty$. Among the 28 one-dimensional components, there is the exceptional one which is the diagonal curve of $\mathbb{P}_1^6$:

$$u_\lambda = v_\lambda = w_\lambda = x_\lambda = y_\lambda = z_\lambda. \tag{3.2}$$

Each of the other 27 components is determined by a triplet of an element from $\{u_\lambda, v_\lambda, w_\lambda\}$, an element from $\{x_\lambda, y_\lambda, z_\lambda\}$ and an element from $\{0, 1, \infty\}$; we have for example,

$$v_\lambda = w_\lambda = y_\lambda = z_\lambda = 1, \quad u_\lambda = x_\lambda \tag{3.3}$$

where u_λ, x_λ, 1 are chosen.

Now we describe the relationship of $\mathcal{C}_{\delta_\lambda}$ to the cross ratio variety $\mathcal{C}$. Since the canonical (birational) projection of $\mathcal{C}$ onto $\mathcal{C}_{\delta_\lambda}$ can be seen in principle through the $(\lambda, \mu, \nu, \rho)$-expression of the six cross ratios chosen for δ_λ and since $\mathcal{C}$ is thus an explicit desingularization of $\mathcal{C}_{\delta_\lambda}$, we can describe the appearing exceptional set over the singular locus by using the geometry of $\mathcal{C}$ in [11].

The 27 components of the singular locus divide themselves into three classes, each containing nine of them. The nine components in one class meet in only one point on the diagonal curve, so there are three special points on this curve ($\cong \mathbb{P}_1$); they correspond to 0, 1, ∞ (under this natural isomorphism). This suggests that we should first blow up these three points (in $\mathbb{P}_1^6$) separately. Then we obtain the proper transform of $\mathcal{C}_{\delta_i}$. Now there appear the three exceptional sets to be some contractions ($\cong$ the Segre cubic) of three A_1-divisors which are special to δ_λ. They correspond exactly to the root subsystems of type A_1 contained in δ_λ (of type A_2). In this case, the three points $0, 1, \infty$ on the diagonal curve correspond to $\{\pm r_{23}\}$, $\{\pm r_{12}\}$, $\{\pm r_{13}\}$ respectively; in particular, the first two correspond to $\lambda = 1$ and $\lambda = 0$ of [11].

Next we perform the blowing up separately by taking as center each of the proper transforms of the 28 one-dimensional components, which are now disjoint from each other; we then obtain, as the exceptional sets, $\mathbb{P}_1 \times \mathbb{P}_1$-bundles over the components ($\cong \mathbb{P}_1$) which are all product bundles. In fact, they are each to each isomorphic to the corresponding cusp divisors (N-divisors in [11]) by the natural projection of $\mathcal{C}$. The cusp divisor corresponding to the diagonal curve is the one associated with Δ ($= \Delta_{(123)(456)}$) itself, which is exactly the one defined by $\rho = 0$ in [11]. The other 27 cusp divisors are associated with the 27 ($= 3 \cdot 3 \cdot 3$) root subsystems (of type $3A_2$) each of which contains a root subsystem of type $3A_1$ in Δ as the intersection with it. For example, over the component (3.3), there appears the cusp divisor which corresponds to $\Delta_{(126)(345)}$. This divisor is obtained from the exceptional set over $\nu = \rho = 1$ of [11].

The remaining 36 isolated singular points are (locally) isomorphic to each other. This singularity germ is in a sense of determinantal character, it is in fact described in the space of 2×3 matrices by requiring

$$\operatorname{rank}\begin{pmatrix} \xi & \eta & \zeta \\ \xi' & \eta' & \zeta' \end{pmatrix} \leq 1.$$

The zero matrix is clearly the unique singular point of this 4-dimensional variety; the complement of this point is the union of two open patches, the one in which (ξ, η, ζ) is non-zero and the one in which (ξ', η', ζ') is non-zero. In the first patch, $(\xi', \eta', \zeta') = c(\xi, \eta, \zeta)$, and in the second, $(\xi, \eta, \zeta) = c'(\xi', \eta', \zeta')$ for some scalars c, c'. In the intersection of patches, we have $cc' = 1$. To sum up these relations, we see that the complement is represented as a rank-3 vector bundle over $\mathbb{P}_1$ minus the zero section (since (ξ, η, ζ), (ξ', η', ζ') are not allowed to be zero). But we can obviously consider the natural mapping of this vector bundle onto the matrix space which only contracts the zero-section onto the singular point, so this is a desingularization.

In the following, every isolated singularity which is locally isomorphic to this singularity of the space of 2×3 matrices with rank at most 1 is said to be of the $(2 \times 3, 1)$-*determinantal type*, so the 36 singular points are all of this type. We can resolve this type of singularity by the way mentioned above. We say then that the singular point is modified to a non-singular rational curve.

Now we see explicitly that the cross ratio variety is obtained by resolving all the remaining isolated singular points separately in this way. We shall explain what are the 36 non-singular rational curves appearing over the singular points. Each of them is the intersection of three A_1-divisors to which the corresponding root subsystems of type A_1 are orthogonal to each other, that is, they form a subsystem of type $3A_1$. In this paper, we mean, by a $3A_1$-*curve*, any intersection of three A_1-divisors whose associated root subsystems form a subsystem of type $3A_1$.

We now explain what kinds of $3A_1$-curves are the appearing 36 ones. The root subsystems of type $3A_1$ corresponding to them have special meaning, not directly for δ_λ itself, but through its orthogonal complements δ_μ, δ_ν; this means that they are divided into two classes, each corresponding to δ_μ or δ_ν and consisting of 18 subsystems. It will suffice to explain the class corresponding to δ_ν. This is now characterized by the root subsystems of type $3A_1$ each of which has no common roots in Δ but the roots orthogonal to it are in δ_ν.

For example, over the singular point (3.1), there appears the $3A_1$-curve which corresponds to the $3A_1$: $\{\pm r_{16}, \pm r_{345}, \pm r_{136}\}$. Then the orthogonal roots to it are $\pm r_{45}$ and there is a unique root subsystem of type D_4 containing the union of these roots which is of type $4A_1$. This root subsystem is exactly the one corresponding to the tritangent T_{32}, which is associated with the cross ratio coordinate x_ν for δ_ν. We have thus obtained the root subsystem $\{\pm r_{45}\}$ in δ_ν and the root subsystem of type D_4 containing δ_ν. Now we see that we can reverse this process: Given a root subsystem of type A_1 in δ_ν and a root subsystem of type D_4 containing δ_ν, we obtain a unique root subsystem of type $3A_1$ which is orthogonal to the first and is contained in the second subsystem. This is the characterization of the above class corresponding to δ_ν, so there are obtained 18 $(= 3 \cdot 6)$ $3A_1$-curves in total as the exceptional sets over the half of isolated singular points of $\mathcal{C}_{\delta_\lambda}$. The other half is obtained from δ_μ.

Recall that we have already desingularized $\mathcal{C}_{\delta_\lambda}$ to $\mathcal{C}$. This desingularization process is essentially the same for $\mathcal{C}_{\delta_\mu}$ and $\mathcal{C}_{\delta_\nu}$. By reversing the process, we see how $\mathcal{C}_{\delta_i}$ ($i = \lambda, \mu, \nu$) is obtained as a contraction of $\mathcal{C}$. We summarize this as the following theorem:

Theorem 3.1. *By the projection of $\mathcal{C}$ onto $\mathcal{C}_{\delta_i}$, there occur exactly the following contractions*:
(1) *Each of the* 36 *isolated singular points is the contraction of a* $3A_1$*-curve whose associated root subsystem is disjoint to Δ but the roots orthogonal to it lie in $\Delta - \delta_i$,*
(2) *Each of the* 28 $(= 1 + 27)$ *one-dimensional components of the singular locus is the contraction of a cusp divisor whose associated root subsystem either coincides with Δ or has a root subsystem of type $3A_1$ as the intersection with Δ,*
(3) *Each of the three intersection points of the one-dimensional components is the contraction of an A_1-divisor whose associated root subsystem lies in δ_i.*

Now we will observe our model $\mathcal{C}_\Delta$ itself. Since this is the image of the projection of $\mathcal{C}$ to $\mathcal{C}_{\delta_\lambda} \times \mathcal{C}_{\delta_\mu} \times \mathcal{C}_{\delta_\nu}$, we see in principle how $\mathcal{C}$ is contracted onto $\mathcal{C}_\Delta$. This variety

$\mathcal{C}_\Delta$ is much nearer to $\mathcal{C}$ itself. How near $\mathcal{C}_\Delta$ is to $\mathcal{C}$ is now summarized in the following main result:

Theorem 3.2. *$\mathcal{C}_\Delta$ has only 27 isolated singular points of the $(2 \times 3, 1)$-determinantal type. Through the resolution of $\mathcal{C}_\Delta$, there appear the twenty-seven $3A_1$-curves on $\mathcal{C}$ which are in one-to-one correspondence with the root subsystems of type $3A_1$ in Δ.*

Proof. Recall that there are the eighteen $3A_1$-curves on $\mathcal{C}$ associated with δ_i ($i = \lambda, \mu, \nu$), so we have the 54 (= 18 + 18 + 18) $3A_1$-curves associated with δ_λ, δ_μ, δ_ν. As is mentioned in Theorem 3.1, the 36 (= 18 + 18) curves except the 18 curves for δ_i are contracted onto the 36 isolated singular points by the projection $\mathcal{C} \to \mathcal{C}_{\delta_i}$, but the 18 curves for δ_i themselves are all isomorphically mapped. This means that all the 54 curves are mapped isomorphically into $\mathcal{C}_\Delta$.

Now recall that every cusp divisor is (canonically) isomorphic to $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$. We see then that the second contraction of Theorem 3.1 for the projection $\mathcal{C} \to \mathcal{C}_{\delta_i}$ is regarded, when restricted to each of the 28 cusp divisors, as the projection onto one component of the product representation of the cusp divisor. We now let i run over the whole index set $\{\lambda, \mu, \nu\}$; then each of the 28 divisors remains the same, but the above projection onto $\mathbb{P}_1$ is replaced by the others, so that each cusp divisor is mapped isomorphically by $\mathcal{C} \to \mathcal{C}_\Delta$. For example, the cusp divisor corresponding to Δ is obtained by $\rho = 0$ in [11], and then we have on it

$$\begin{aligned}
u_\lambda &= v_\lambda = w_\lambda = x_\lambda = y_\lambda = z_\lambda = 1 - \lambda, \\
u_\mu &= v_\mu = w_\mu = x_\mu = y_\mu = z_\mu = 1 - \mu, \\
u_\nu &= v_\nu = w_\nu = x_\nu = y_\nu = z_\nu = 1 - \nu.
\end{aligned}$$

Thus this cusp divisor remains alive in $\mathcal{C}_\Delta$ as the direct product of three diagonal curves of $\mathcal{C}_{\delta_\lambda}, \mathcal{C}_{\delta_\mu}, \mathcal{C}_{\delta_\nu}$. Next we check, as an example, the cusp divisor defined by $\rho = \infty$ which means the one defined by $\rho' = 0$ on the patch $(\lambda', \mu', \nu', \rho') = (\lambda^{-1}, \mu^{-1}, \nu^{-1}, \rho^{-1})$ of Naruki's toroidal construction of the moduli in [11]. This cusp divisor exactly corresponds to the root subsystem $\Delta_{(16)(23)(45)}$, so it is special to Δ. Its image is defined by

$$\begin{aligned}
u_\lambda &= w_\lambda = x_\lambda = z_\lambda = 0, \quad v_\lambda = y_\lambda = 1 - \lambda', \\
u_\mu &= w_\mu = x_\mu = z_\mu = 0, \quad v_\mu = y_\mu = 1 - \mu', \\
u_\nu &= w_\nu = x_\nu = z_\nu = 0, \quad v_\nu = y_\nu = 1 - \nu'.
\end{aligned}$$

Thus the divisor is also alive isomorphically in $\mathcal{C}_\Delta$. Now, we can let the normalizer of $W(\Delta)$ act on this. To sum up, we conclude that all the 28 cusp divisors are alive isomorphically in $\mathcal{C}_\Delta$.

It still remains to be checked how the 9 (= 3 + 3 + 3) A_1-divisors corresponding to the root subsystems of type A_1 in δ_λ, δ_μ, δ_ν (i.e. in Δ) are mapped into $\mathcal{C}_\Delta$. According to Theorem 3.1, each of them are contracted onto a point for some of the three projections $\mathcal{C} \to \mathcal{C}_{\delta_i}$ ($i = \lambda, \mu, \nu$), but we see that it remains as $\mathbb{P}^3_1$ for the other projections. In fact, the A_1-divisor corresponding to $\{\pm r_{12}\}$ (in Δ), which is obtained by $\lambda = 0$ in [11], has the following:

$$u_\lambda = v_\lambda = w_\lambda = x_\lambda = y_\lambda = z_\lambda = 1,$$
$$u_\mu = y_\mu,\ v_\mu = z_\mu,\ w_\mu = x_\mu,$$
$$u_\nu = x_\nu,\ v_\nu = z_\nu,\ w_\nu = y_\nu.$$

Thus the nine A_1-divisors remain almost isomorphically in $\mathcal{C}_\Delta$. By checking up it in detail, one can see that the contraction occurs, as is mentioned in this theorem, only along the twenty-seven $3A_1$-curves on $\mathcal{C}$ corresponding to the root subsystems of type $3A_1$ in Δ. By the action of $W(\Delta)$, we can check this through one example.

The $3A_1$-curve corresponding to the $3A_1$: $\{\pm r_{12}, \pm r_0, \pm r_{56}\}$ is covered by two patches of Naruki's model, one is $\lambda = \mu = \nu = 0$ in $(\lambda, \mu, \nu, \rho)$ and the other is $\lambda' = \mu' = \nu' = 0$ in $(\lambda', \mu', \nu', \rho') = (\lambda\rho, \mu\rho, \nu\rho, \rho^{-1})$. Thus this curve is contracted to an isolated singular point of the $(2 \times 3, 1)$-determinantal type:

$$\operatorname{rank}\begin{pmatrix} \lambda & \mu & \nu \\ \lambda' & \mu' & \nu' \end{pmatrix} \leq 1.$$

Now note that all the 18 cross ratios associated with Δ have the same value 1 on the curve, so there is a local morphism of this contraction to a neighborhood of a point on $\mathcal{C}_\Delta$. By using the $(\lambda, \mu, \nu, \rho)$-expression of the 18 cross ratios, we can check directly that this morphism is an local isomorphism. $\square$

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