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ON MOTION OF AN ELASTIC WIRE AND SINGULAR PERTURBATION OF A 1-DIMENSIONAL PLATE EQUATION

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1. Introduction and preliminaries

Consider a springy circle wire in the euclidean space $\mathbf{R}^3$. We characterize such a wire as a closed curve $\gamma = \gamma(x)$ with unit line element and fixed length. For such a curve, its elastic energy is given by

$$E(\gamma) = \int_0^L |\gamma_{xx}|^2\,dx.$$

Solutions of the corresponding Euler-Lagrange equation are called *elastic curves*. Closed elastic curves in the euclidean space are classified in [7]. We discuss on motion of a circle wire governed by the elastic energy.

We will see that the equation becomes an initial value problem for $\gamma = \gamma(x,t)$:

$$\text{(EW)}\qquad \begin{cases} \gamma_{tt} + \partial_x^4\gamma + \mu\gamma_t = \partial_x\{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\ -w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3\gamma|^2 + |\gamma_{tx}|^2, \\ \gamma(x,0) = \gamma_0(x), \quad \gamma_t(x,0) = \gamma_1(x), \quad (\gamma_{0x},\gamma_{1x}) = 0. \end{cases}$$

Here, μ is a constant which represents the resistance, and the ODE for w corresponds to the constrained condition $(\gamma_x, \gamma_{tx}) \equiv 0$ (i.e., $|\gamma_x| \equiv 1$.) When the resistance μ is very large, we can analyze the behavior of the solution replacing the time parameter t to $\tau = \mu^{-1}t$. Then, (EW) becomes

$$(\text{EW}^\tau)\qquad \begin{cases} \mu^{-2}\gamma_{\tau\tau} + \partial_x^4\gamma + \gamma_\tau = \partial_x\{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\ -w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3\gamma|^2 + \mu^{-2}|\gamma_{\tau x}|^2, \\ \gamma(x,0) = \gamma_0(x), \quad \gamma_\tau(x,0) = \mu\gamma_1(x), \quad (\gamma_{0x},\gamma_{1x}) = 0. \end{cases}$$

And, when $\mu \to \infty$, we get, omitting initial data $\gamma_\tau(x,0)$,

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$$
\text{(EP)} \qquad \begin{cases} \gamma_\tau + \partial_x^4\gamma = \partial_x\{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\ -w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3\gamma|^2, \\ \gamma(x,0) = \gamma_0(x). \end{cases}
$$

The equation (EP), treated in [4] and [5], has a unique all time solution for any initial data, and the solution converges to an elastic curve. In this paper, we will prove:

1) *The equation* (EW) *has a unique short time solution for any initial data.* (Corollary 3.13.)
2) *If μ is large, then the solution of* (EW$^\tau$) *exists for long time, and converges to a solution of* (EP) *when $\mu \to \infty$.* (Corollary 4.10.)

Note that in 2), the derivative $\gamma_\tau(x,0) = \mu\gamma_1(x)$ diverges when $\mu \to \infty$.

If (EW) contained no 3rd derivatives $\partial_x^3\gamma$ and was not coupled with ODEs, i.e., if our equation was $\gamma_{tt}+\partial_x^4\gamma+\mu\gamma_t = F(\gamma, \gamma_x, \gamma_{xx}, \gamma_t)$, it is standard to show the short time existence of solutions. (See [9] Section 11.7.) Being coupled is not main difficulty to solve the equation. We can overcome it by careful estimation similar to [4]. However, the difficulty due to the presence of 3rd derivatives is essential. We will overcome the difficulty using the new unknown variable $\xi := \gamma_x \in S^2$. As we will see in Lemma 2.2, the equation for ξ does not contain 3rd derivatives $\nabla_x^2\xi_x$. Owing to the lack of the term, we will be able to solve (EW$^\xi$) by a usual method: perturb to a parabolic equation and show the solution of the parabolic equation converges to a solution of the original equation. This will be done in Section 3.

Remark 1.1. In this paper, we only treat curves in the 3-dimensional euclidean space $\mathbf{R}^3$. But, the result holds also on the case of any dimensional euclidean space, with no modification of proofs.

By similarity, we may assume that the length of the initial curve γ_0 is 1. From now on, a closed curve means a map from $S^1 \equiv \mathbf{R}/\mathbf{Z}$ into the euclidean space $\mathbf{R}^3$ or the unit sphere S^2. The inner product of vectors is denoted by $(*, *)$, and the norm is denoted by $|*|$. We also use the covariant derivation ∇ on S^2. For a tangential vector field $X(x)$ along a curve $\gamma(x)$ on S^2, the covariant derivative is defined by $\nabla_x X := (X'(x))^T$. The covariant differentiation is non-commutative, because the curvature tensor R of S^2 is non-zero. For example, if $X(x,t)$ is a tangential vector field along a family $\gamma(x,t)$ of curves on S^2, we have

$$
\nabla_x\nabla_t X - \nabla_t\nabla_x X = R(\gamma_x, \gamma_t)X = (\gamma_t, X)\gamma_x - (\gamma_x, X)\gamma_t.
$$

For functions on S^1 and vector fields along a closed curve, we use L_2-inner product $\langle *, * \rangle$ and L_2-norm $\| * \|$. Sobolev H^n-norm is denoted by $\| * \|_n$. For a tensor field along the closed curve on S^2, $\| * \|_n$ is defined using covariant derivation. That is, $\|\zeta\|_n^2 = \sum_{i=0}^n \|\nabla_x^i\zeta\|^2$. We also use C^n norm $\| * \|_{(n)}$. In particular, $\| * \|_{(0)} = \max|*|$.

2. The equations

To derive the equation of motion, we use Hamilton's principle. For a moving curve $\gamma = \gamma(t, x)$, the velocity energy is given by $\|\gamma_t\|^2$ and the elastic energy is given by $\|\gamma_{xx}\|^2$. (By rescaling, we omit coefficients.) Therefore, the real motion is a stationary point of the integral

$$L(\gamma) := \int_{t_1}^{t_2} \|\gamma_t\|^2 - \|\gamma_{xx}\|^2\, dt.$$

That is, the integral

$$L' := \int_{t_1}^{t_2} \langle \gamma_t, \delta_t \rangle - \langle \gamma_{xx}, \delta_{xx} \rangle\, dt$$

should vanish for all $\delta = \delta(t, x)$ satisfying $\delta(t_1, x) = \delta(t_2, x) = 0$ and the constrained condition $\langle \gamma_x, \delta_x \rangle \equiv 0$.

From integration by parts, we see

$$L' = \int_{t_1}^{t_2} -\langle \gamma_{tt} + \partial_x^4 \gamma, \delta \rangle\, dt.$$

On the other hand, the orthogonal complement of the space $V := \{\delta \mid \langle \gamma_x, \delta_x \rangle \equiv 0\}$ at each time t is $V^\perp = \{(u\gamma_x)_x \mid u = u(x)\}$. Therefore, γ is stationary if and only if $\gamma_t \in V$ and $\gamma_{tt} + \partial_x^4 \gamma = (u\gamma_x)_x$ for some function $u = u(t, x)$.

REMARK 2.1. Many papers (e.g., [2], [3]) apply Hamilton's principle using $|\gamma_{xt}|^2 + |\gamma_t|^2$ as the kinetic energy, and gets a wave equation. The wave equation is completely different from (EW). A linear version of our equation can be found, for example, in [1] p. 246.

This difference can be explained as follows. We characterize a planer thick wire of length L, of radius R and of unit weight per length as a map $u = u(x, y) : [0, L] \times [-R, R] \to \mathbf{R}^2$ such that $u(x, y) = \gamma(x) + yJ\gamma_x(x)$, where γ is a curve of unit line element and J is the $\pi/2$ rotation. When u moves, i.e. when we consider a family $u = u(x, y, t)$ of such curves, the velocity energy becomes

$$\frac{1}{2R} \int_0^L dx \int_{-R}^R |u_t(x, y)|^2 dy = \|\gamma_t\|^2 + \frac{1}{3} R^2 \|\gamma_{xt}\|^2.$$

Hence, our wire is infinitely thin, while previous papers treat thick wires.

In this paper, we treat slightly more general equation, equation with resistance μ. That is,

$$\gamma_{tt} + \mu\gamma_t + \partial_x^4 \gamma = (u\gamma_x)_x,$$

coupled with an ODE for u, which is derived from the constrained condition: $|\gamma_x| \equiv 1$. From

$$0 = \partial_t^2 |\gamma_x|^2 = 2(\gamma_{ttx}, \gamma_x) + 2|\gamma_{tx}|^2,$$

the unknown u satisfies

$$\left(-\partial_x^5 \gamma + \partial_x^2(u\gamma_x) - \mu\gamma_{tx}, \gamma_x\right) = -|\gamma_{tx}|^2.$$

Using $|\gamma_x|^2 \equiv 1$, we can rewrite this to

$$-u_{xx} + |\gamma_{xx}|^2 u = 2\partial_x^2 |\gamma_{xx}|^2 - |\partial_x^3 \gamma|^2 + |\gamma_{tx}|^2,$$

and, putting $w := u + 2|\gamma_{xx}|^2$, we get (EW).

Since the principal part of (EW) is the operator of the plate equation:

$$u_{tt} + \partial_x^4 u,$$

we perturb it to a parabolic operator:

$$\begin{aligned} u_{tt} &- 2\varepsilon u_{txx} + (1 + \varepsilon^2)\partial_x^4 u \\ &= \left(\partial_t - (\varepsilon + \sqrt{-1})\partial_x^2\right)\left(\partial_t - (\varepsilon - \sqrt{-1})\partial_x^2\right)u \end{aligned}$$

with $\varepsilon > 0$. It is possible to show that a perturbed equation of (EW)

$$\begin{cases} \gamma_{tt} - 2\varepsilon\gamma_{txx} + (1 + \varepsilon^2)\partial_x^4 \gamma + \mu\gamma_t = \partial_x\{(w - 2|\gamma_{xx}|^2)\gamma_x\}, \\ -w_{xx} + |\gamma_{xx}|^2 w = 2|\gamma_{xx}|^4 - |\partial_x^3 \gamma|^2 + |\gamma_{tx}|^2, \\ \gamma(x, 0) = \gamma_0(x), \quad \gamma_t(x, 0) = \gamma_1(x), \quad (\gamma_{0x}, \gamma_{1x}) = 0 \end{cases}$$

has a short-time solution. However, we cannot get uniform estimate when $\varepsilon \to 0$, because $\partial_x\{(w - 2|\gamma_{xx}|^2)\gamma_x\}$ contains the third derivative of γ. To overcome this difficulty, we convert (EW) to an equation on S^2, and "remove" the third derivative.

We introduce a new unknown function ξ by $\xi = \gamma_x$. The function ξ is a family of closed curves on S^2.

Lemma 2.2. *The equation* (EW) *is equivalent to equation*

$$(\mathrm{EW}^\xi) \qquad \begin{cases} \nabla_t \xi_t + \nabla_x^3 \xi_x + \mu\xi_t = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \dfrac{3}{2}\partial_x |\xi_x|^2\, \xi_x, \\ -w_{xx} + |\xi_x|^2 w = |\xi_t|^2 - |\nabla_x \xi_x|^2 + |\xi_x|^4, \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x), \quad \displaystyle\int_0^1 \xi_0\, dx = \int_0^1 \xi_1\, dx = 0, \end{cases}$$

and (EP) *is equivalent to equation*

$$
(\mathrm{EP}^{\xi}) \qquad \begin{cases} \xi_T + \nabla_x^3 \xi_x = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2}\partial_x |\xi_x|^2 \, \xi_x, \\ -w_{xx} + |\xi_x|^2 w = -|\nabla_x \xi_x|^2 + |\xi_x|^4, \\ \xi(x,0) = \xi_0(0), \quad \int_0^1 \xi_0 \, dx = 0. \end{cases}
$$

Proof. It is straightforward to check the following decomposition:

$$
\begin{aligned}
\xi_{xx} &= \nabla_x \xi_x - |\xi_x|^2 \xi, \quad \xi_{tt} = \nabla_t \xi_t - |\xi_t|^2 \xi, \\
\partial_x^3 \xi &= \nabla_x^2 \xi_x - |\xi_x|^2 \xi_x - \frac{3}{2}\partial_x |\xi_x|^2 \xi, \\
\partial_x^4 \xi &= \nabla_x^3 \xi_x - |\xi_x|^2 \nabla_x \xi_x - \frac{5}{2}\partial_x |\xi_x|^2 \xi_x + \{|\nabla_x \xi_x|^2 + |\xi_x|^4 - 2\partial_x^2 |\xi_x|^2\}\xi.
\end{aligned}
$$

Using these formulas, we see that the x-derivatives of (EW) imply (EW^{ξ}). Conversely, (EW^{ξ}) implies the equation

$$
\xi_{tt} + \partial_x^4 \xi + \mu \xi_t = \partial_x^2 \{(w - 2|\xi_x|^2)\xi\}.
$$

Under the assumption: $\int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0$, we see that the closedness condition: $\int_0^1 \xi \, dx \equiv 0$ is satisfied. Let γ be the solution of an ODE:

$$
\begin{aligned}
&\gamma_{tt} + \mu \gamma_t = -\partial_x^3 \xi + \partial_x \{(w - 2|\xi_x|^2)\xi\}, \\
&\gamma(x,0) = \gamma_0(x), \quad \gamma_t(x,0) = \gamma_1(x).
\end{aligned}
$$

Then

$$
\gamma_{xtt} + \mu \gamma_{xt} = -\partial_x^4 \xi + \partial_x^2 \{(w - 2|\xi_x|^2)\xi\} = \xi_{tt} + \mu \xi_t
$$

and $(\gamma_x - \xi)_{tt} + \mu(\gamma_x - \xi)_t \equiv 0$. Hence $\gamma_x \equiv \xi$ and γ is a solution of (EW).

A similar calculation gives the equivalence of (EP) and (EP^{ξ}). □

3. Short time existence

In this section, we fix $\mu \in \mathbf{R}$.

To perturb (EW^{ξ}), we introduce a function $\rho(x, y)$. Since ξ_0 is the derivative of a closed curve γ_0 in the euclidean space, each component of ξ_0 takes 0 at some x. Therefore, by Wirtinger's inequality, we have $\|\xi_{0x}\|^2 \geq \pi^2 \|\xi_0\|^2 \geq \pi^2$. (It is known in fact that $\|\xi_{0x}\|^2 \geq 4\pi^2$.) Let $\delta(r)$ be a C^∞ function on $\mathbf{R}$ such that $\delta(r) = 1$ on $|r| \leq \pi^2/8$, $\delta(r) = 0$ on $\pi^2/4 \leq |r|$ and $0 \leq \delta(r) \leq 1$ on $\pi^2/8 \leq |r| \leq \pi^2/4$. We put

$$
\rho(x, y) = \pi^2 + \delta(y^2 - |\xi_{0x}(x)|^2)(y^2 - \pi^2).
$$

Fix an interval I such that $|\xi_{0x}(x)|^2 \geq \pi^2/2$ for any $x \in I$. If $x \in I$ and $\left|y^2 - |\xi_{0x}(x)|^2\right| \leq \pi^2/4$, then $\rho(x, y) \geq \min\{\pi^2, y^2\} \geq \pi^2/4$. And if $\left|y^2 - |\xi_{0x}(x)|^2\right| \geq \pi^2/4$, then $\rho(x, y) = \pi^2$. Therefore, for any function $u(x)$,

$$\int_0^1 \rho(x, u(x))\,dx \geq \frac{\pi^2}{4}\int_I dx.$$

REMARK 3.1. Below, we use the function ρ only to ensure $\rho \geq 0$ everywhere and $\int_0^1 \rho(x, u(x))\,dx$ is bounded from below by a positive constant. Note that $\rho(x, y) := y$ satisfies this requirement if $\xi = \gamma_x$ for some closed curve γ in the euclidean space.

Proposition 3.2. *Let $\xi_0(x)$ be a C^∞ closed curve on S^2 with $\|\xi_{0x}\| \geq \pi$ and $\xi_1(x)$ a C^∞ tangent vector field along ξ_0. Let ρ be the function defined as above. Then, equation*

$$(\mathrm{EW}^{\xi\varepsilon}) \qquad \begin{cases} \nabla_t\xi_t - 2\varepsilon\nabla_x^2\xi_t + (1+\varepsilon^2)\nabla_x^3\xi_x + \mu\xi_t \\ \qquad = (w - |\xi_x|^2)\nabla_x\xi_x + 2w_x\xi_x - \dfrac{3}{2}\partial_x|\xi_x|^2\,\xi_x, \\ -w_{xx} + \rho(x, |\xi_x|^2)w = |\xi_t|^2 - |\nabla_x\xi_x|^2 + |\xi_x|^4, \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x) \end{cases}$$

has a C^∞ solution on some interval $0 \leq t < T$.

Proof. We can prove unique short-time existence of $(\mathrm{EW}^{\xi\varepsilon})$ by a similar method with that used in [4]. Here, we mention only two steps. One is an estimation of the ODE for w. Lemma 3.3 with the function ρ ensures estimation of w by ξ. Another, Lemma 3.4, is a crucial point to use the contraction principle. □

Lemma 3.3 ([4] Lemma 4.1, Lemma 4.2). *Let a and b be L_1-functions on S^1 such that $a \geq 0$ and $\|a\|_{L_1} > 0$. Then, the ODE for a function w on S^1*

$$-w'' + aw = b$$

has a unique solution w, and the solution w is estimated as

$$\max|w| \leq 2\{1 + \|a\|_{L_1}^{-1}\} \cdot \|b\|_{L_1},$$
$$\max|w'| \leq 2\{1 + \|a\|_{L_1}\} \cdot \|b\|_{L_1}.$$

Moreover, there exists universal constants $C > 0$ and $N > 0$ depending on n such that

$$\|w\|_{n+2} \leq C\left(1 + \|a\|_n^N\right)\|b\|_n,$$
$$\|w\|_{(n+2)} \leq C\left(1 + \|a\|_{(n)}^N\right)\|b\|_{(n)}.$$

Lemma 3.4. *We consider a linear PDE for u*

$$\begin{cases} u_{tt} - 2\varepsilon u_{txx} + (1+\varepsilon^2)\partial_x^4 u = f, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \end{cases}$$

If $f \in C^{2\alpha}$, $u_0 \in C_x^{4+2\alpha}$ and $u_1 \in C_x^{2+2\alpha}$, then there is a unique solution $u \in C^{4+2\alpha}$. Moreover, we have an estimation:

$$\|u\|_{C^{4+2\alpha}} \le C\{\|f\|_{C^{2\alpha}} + \|u_0\|_{C_x^{4+2\alpha}} + \|u_1\|_{C_x^{2+2\alpha}}\},$$

where $\| * \|_{C_x^{n+2\alpha}}$ means the Hölder norm for x-direction, and $\| * \|_{C^{n+2\alpha}}$ means the weighted Hölder norm (t-derivatives are counted twice of x-derivatives.)

Proof. We decompose the equation to a parabolic equation as

$$u_t - (\varepsilon + \sqrt{-1})u_{xx} = v, \quad v_t - (\varepsilon - \sqrt{-1})v_{xx} = f.$$

Using the fundamental solution

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi}\sqrt{\varepsilon \pm \sqrt{-1}}\sqrt{t}} \exp\Big(-\frac{x^2}{4(\varepsilon \pm \sqrt{-1})t}\Big)$$

of the parabolic operator $\partial_t - (\varepsilon \pm \sqrt{-1})\partial_x^2$, we can estimate as

$$\begin{aligned} \|u\|_{C^{4+2\alpha}} &\le C\{\|v\|_{C^{2+2\alpha}} + \|u_0\|_{C_x^{4+2\alpha}}\} \\ &\le C\{\|f\|_{C^{2\alpha}} + \|v_0\|_{C_x^{2+2\alpha}} + \|u_0\|_{C_x^{4+2\alpha}}\} \\ &\le C\{\|f\|_{C^{2\alpha}} + \|u_1\|_{C_x^{2+2\alpha}} + \|u_0\|_{C_x^{4+2\alpha}}\}. \end{aligned}$$

□

When we take the limit $\varepsilon \to 0$ in (EW$^{\xi\varepsilon}$), we should note that the term $\nabla_x^3 \xi_x$ is quasi-linear, and contains the third derivative of ξ. In fact, in local coordinate system,

$$\nabla_x^3 \xi_x = \{\partial_x^4 \xi^p + 4\Gamma_q{}^p{}_r(\xi)\xi_x^q \partial_x^3 \xi^r\}\frac{\partial}{\partial x^p} + [\text{lower order terms}].$$

However, when we integrate it by parts, we can treat it as though it contained no third derivatives.

Lemma 3.5. *For any $K > 0$, there are $T > 0$ and $M > 0$ with the following property:*

Let ξ be a solution of (EW$^{\xi\varepsilon}$) *with $\varepsilon \in [0,1]$ on an interval $[0, t_1) \subset [0, T)$. If its initial value satisfies $\|\xi_1\|^2 + \|\xi_{0x}\|_1^2 \le K$, then $\|\xi_t\|^2 + \|\xi_x\|_1^2 \le M$ holds on $0 \le t < t_1$.*

Proof. Put

$$f = \big(w - \rho(x, |\xi_x|^2)\big)\nabla_x \xi_x + 2w_x \xi_x - \frac{3}{2}\partial_x |\xi_x|^2 \xi_x.$$

We can estimate

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\|\xi_t\|^2 + (1+\varepsilon^2)\|\nabla_x \xi_x\|^2\right\} \\
&= \langle \xi_t, \nabla_t \xi_t\rangle + (1+\varepsilon^2)\langle \nabla_x \xi_x, \nabla_t \nabla_x \xi_x\rangle \\
&= \langle \xi_t, \nabla_t \xi_t + (1+\varepsilon^2)\nabla_x^3 \xi_x\rangle + (1+\varepsilon^2)\langle R(\xi_t, \xi_x)\xi_x, \nabla_x \xi_x\rangle \\
&\le \langle \xi_t, 2\varepsilon \nabla_x^2 \xi_t + f\rangle - \mu\|\xi_t\|^2 + C \max|\xi_x|^2 \|\xi_t\| \|\nabla_x \xi_x\| \\
&\le -2\varepsilon\|\nabla_x \xi_t\|^2 + \langle \xi_t, f\rangle - \mu\|\xi_t\|^2 + C\|\xi_x\|_1^2 \|\xi_t\| \|\nabla_x \xi_x\| \\
&\le (1-\mu)\|\xi_t\|^2 + \|f\|^2 + C\|\xi_x\|_1^2(\|\xi_t\|^2 + \|\nabla_x \xi_x\|^2),
\end{aligned}$$

and,

$$\frac{1}{2}\frac{d}{dt}\|\xi_x\|^2 = \langle \xi_x, \nabla_t \xi_x\rangle = -\langle \nabla_x \xi_x, \xi_t\rangle \le \|\nabla_x \xi_x\|^2 + \|\xi_t\|^2.$$

Here, by Lemma 3.3, $\|f\| \le C(1 + \|\xi_t\|^2 + \|\xi_x\|_1^2)^{N_1}$. Therefore, putting $X(t) := 1 + \|\xi_t\|^2 + (1+\varepsilon^2)\|\xi_x\|_1^2$, we get

$$X'(t) \le C_1 X(t)^{N_2},$$

and, $X(t)$ is bounded from above by a solution of the ODE: $y'(t) = C_1 y(t)^{N_2}$. □

REMARK 3.6. If we use original equation of γ, which contains $\partial_x^3 \gamma$ in the right hand side, the term $\langle \gamma_t, \partial_x^3 \gamma\rangle$ appears in the estimation. Since we need the term $-2\varepsilon\|\gamma_{tx}\|^2$ to cancel $\langle \gamma_t, \partial_x^3 \gamma\rangle$, we cannot get uniform estimate with respect to ε, and the following proof will fail.

Lemma 3.7. *For any $K > 0$ and $n \ge 0$, there is $M > 0$ with the following property*:

Let ξ be a solution of ($\mathrm{EW}^{\xi\varepsilon}$) *with $\varepsilon \in [0, 1]$ on $[0, T)$. If its initial value satisfies $\|\xi_1\|_n$, $\|\xi_{0x}\|_{n+1} \le K$, and if it satisfies $\|\xi_t\|$, $\|\xi_x\|_1^2 \le K$ on $0 \le t < T$, then $\|\xi_t\|_n$, $\|\xi_x\|_{n+1}^2 \le M$ holds on $0 \le t < T$.*

Proof. The claim holds for $n = 0$ by taking $M = K$. We prove the claim by induction. Suppose that the claim holds for n. In particular, we know bounds of $\|\xi_x\|_{(n)}$,

$\|\xi_t\|_{(n-1)}$, $\|w\|_{n+2}$ and $\|w\|_{(n+1)}$. Therefore, we have

$$\|\nabla_t\nabla_x^{n+1}\xi_t - \nabla_x^{n+1}\nabla_t\xi_t\| = \left\|\sum_{i=0}^{n}\nabla_x^i(R(\xi_t,\xi_x)\nabla_x^{n-i}\xi_t)\right\|$$

$$\leq C\sum_{i+j\leq n}\left\||\nabla_x^i\xi_t||\nabla_x^j\xi_t|\right\| \leq C\sum_{i+j\leq n}\|\xi_t\|_i\,\|\xi_t\|_{j+1} \leq C\|\xi_t\|_{n+1},$$

$$\|\nabla_t\nabla_x^{n+2}\xi_x - \nabla_x^{n+3}\xi_t\| = \left\|\sum_{i=0}^{n+1}\nabla_x^i(R(\xi_t,\xi_x)\nabla_x^{n+1-i}\xi_x)\right\|$$

$$\leq C\left(\||\xi_t||\nabla_x^{n+1}\xi_x|\| + \sum_{i=0}^{n+1}\|\nabla_x\xi_t\|\right) \leq C\,(\|\xi_t\|_1\|\xi_x\|_{n+1} + \|\xi_t\|_{n+1})$$

$$\leq C\|\xi_t\|_{n+1},$$

$$\|w\|_{n+2} \leq C(1+\|\rho(x,|\xi_x|^2)\|_n^N)\||\xi_t|^2 - |\nabla_x\xi_x|^2 + |\xi_x|^4\|_n$$

$$\leq C\left(\sum_{i+j\leq n}\|\xi_t\|_i\|\xi_t\|_{j+1} + \sum_{i+j\leq n,\ i\leq j}\|\nabla_x\xi_x\|_i\|\nabla_x\xi_x\|_{j+1} + 1\right)$$

$$\leq C(\|\xi_t\|_{n+1} + \|\xi_x\|_1\|\xi_x\|_{n+2} + 1) \leq C(\|\xi_t\|_{n+1} + \|\xi_x\|_{n+2} + 1).$$

Put

$$f := (w - |\xi_x|^2)\nabla_x\xi_x + 2w_x\xi_x - \frac{3}{2}\partial_x|\xi_x|^2\xi_x.$$

Then,

$$\|f\|_{n+1} \leq C(1 + \|\xi_x\|_{n+2} + \|\xi_x\|_{n+2}\|\xi_x\|_1 + \|w\|_{n+2}\|\xi_x\|_1 + \|w\|_2\|\xi_x\|_{n+1})$$

$$\leq C(1 + \|\xi_x\|_{n+2} + \|w\|_{n+2}) \leq C(1 + \|\xi_x\|_{n+2} + \|\xi_t\|_{n+1}).$$

Using these, we have

$$\frac{1}{2}\frac{d}{dt}\left\{\|\nabla_x^{n+1}\xi_t\|^2 + (1+\varepsilon^2)\|\nabla_x^{n+2}\xi_x\|^2\right\}$$

$$= \langle\nabla_x^{n+1}\xi_t, \nabla_t\nabla_x^{n+1}\xi_t\rangle + (1+\varepsilon^2)\langle\nabla_x^{n+2}\xi_x, \nabla_t\nabla_x^{n+2}\xi_x\rangle$$

$$\leq \langle\nabla_x^{n+1}\xi_t, \nabla_x^{n+1}\nabla_t\xi_t\rangle + (1+\varepsilon^2)\langle\nabla_x^{n+2}\xi_x, \nabla_x^{n+3}\xi_t\rangle$$

$$+ C(\|\xi_t\|_{n+1} + \|\xi_x\|_{n+2})(1 + \|\xi_t\|_{n+1})$$

$$\leq \langle\nabla_x^{n+1}\xi_t, \nabla_x^{n+1}(f + 2\varepsilon\nabla_x^2\xi_t - \mu\xi_t)\rangle + C(1 + \|\xi_t\|_{n+1}^2 + \|\xi_x\|_{n+2}^2)$$

$$\leq \langle\nabla_x^{n+1}\xi_t, 2\varepsilon\nabla_x^{n+3}\xi_t\rangle + C(1 + \|\xi_t\|_{n+1}^2 + \|\xi_x\|_{n+2}^2)$$

$$\leq C\{1 + \|\nabla_x^{n+1}\xi_t\|^2 + (1+\varepsilon^2)\|\nabla_x^{n+2}\xi_x\|^2\}. \qquad \square$$

Lemma 3.8. *For any smooth initial data* $\{\xi_0, \xi_1\}$, $K > 0$, $T > 0$ *and* m, $n \geq 0$, *there is* $M > 0$ *with the following property*:

Let ξ *is a solution of* $(\mathrm{EW}^{\xi_\varepsilon})$ *with* $\varepsilon \in [0, 1]$ *on* $[0, T)$. *If* $\|\xi_t\|$, $\|\xi_x\|_1 \leq K$ *on* $0 \leq t < T$, *then* ξ *is smooth on* $S^1 \times [0, T)$, *and the derivatives are bounded as* $\|\nabla_t^m \xi\|_{(n)} \leq M$.

Proof. By Lemma 3.7, the claim holds for $m \leq 1$. Suppose that the claim holds up to m. In particular, we have C_x^∞ bounds of ξ and $\nabla_t^{m-1}\xi_t$. Therefore, using

$$-(\partial_t^j w)_{xx} + \partial_t^j w = \partial_t^j f - \sum_{0<i\leq j} \binom{j}{i} \partial_t^i \rho\, \partial_t^{j-i} w$$

for $0 \leq j \leq m-1$, we have C_x^∞ bounds of $\partial_t^{m-1} w$. Since $\nabla_t^{m+1}\xi_t$ is expressed as a polynomial of these lower derivatives, we get the result. □

Proposition 3.9. *The equation* (EW^ξ) *has a short time solution for any smooth initial data.*

Proof. We put $K := \|\xi_1\|^2 + \|\xi_{0x}\|_1^2$ and take $T > 0$ in Lemma 3.5. Then, by Lemma 3.8, any solution has a priori estimate on $0 \leq t < T$.

Let $[0, T_\varepsilon)$ be the maximal interval such that a solution exists for ε. If $T_\varepsilon < T$, then ξ is smoothly and uniformly bounded on $[0, T_\varepsilon)$, hence can be continued beyond T_ε. This contradicts to the definition of T_ε, therefore we see that $T_\varepsilon \geq T$. We conclude that a solution ξ exists on the interval $[0, T)$ for each $\varepsilon > 0$, and these ξ's have smooth uniform bounds on $S^1 \times [0, T)$.

Therefore, taking a sequence $\varepsilon_i \to 0$, we get a solution of

$$\begin{cases} \nabla_t \xi_t + \nabla_x^3 \xi_x + \mu \xi_t = (w - |\xi_x|^2)\nabla_x \xi_x + 2w_x \xi_x - \dfrac{3}{2}\partial_x |\xi_x|^2 \xi_x, \\ -w_{xx} + \rho(x, |\xi_x|^2) w = |\xi_t|^2 - |\nabla_x \xi_x|^2 + |\xi_x|^4, \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \xi_1(x). \end{cases}$$

Since $\rho(x, |\xi_x|^2) = |\xi_x|^2$ when ξ_x is sufficiently close to ξ_{0x}, we have a solution ξ of (EW^ξ) on some time interval. Once we have a short time solution ξ of (EW^ξ), we can estimate the solution as Lemma 3.8, and the solution ξ can be continued to the interval $[0, T)$. □

Proposition 3.10. *Let* ξ *and* $\tilde{\xi}$ *be solutions of* (EW^ξ) *on* $[0, T)$. *If* ξ *and* $\tilde{\xi}$ *have same smooth initial data, then they identically coincide.*

Proof. To express the difference of two solutions, we use local coordinates. We fix the initial value $\{\xi_0, \xi_1\}$, and take a local coordinate U which contains the initial value ξ_0. In U, (EW$^\xi$) is expressed as:

$$\begin{cases} \xi^p_{tt} + \partial^4_x \xi^p + 4\Gamma_q{}^p{}_r(\xi)\xi^q_x \partial^3_x \xi^r = F^p[\xi_{xx}, w_x, \xi_t], \\ -w_{xx} + g_{qr}(\xi)\xi^q_x \xi^r_x w = G[\xi_{xx}, \xi_t], \end{cases}$$

where $F^p[\xi_{xx}, w_x, \xi_t]$ is a polynomial of ξ^q_x, ξ^q_{xx}, w, w_x, ξ^q_t, functions of ξ^q, and $G[\xi_{xx}, \xi_t]$ is a polynomial of ξ^q_x, ξ^q_{xx}, ξ^q_t, functions of ξ^q. (We only note highest derivatives.)

Let $\{\widetilde{\xi}, \widetilde{w}\}$ be another solution of (EW$^\xi$) on $[0, t_1)$ ($t_1 \le T$). Applying Lemma 3.5 and Lemma 3.8 with $\varepsilon = 0$, we have smooth bounds of ξ and $\widetilde{\xi}$. We put $\zeta := \widetilde{\xi} - \xi$, $u := \widetilde{w} - w$. Then, we see that

$$\zeta^p_{tt} + \partial^4_x \zeta^p + 4\Gamma_q{}^p{}_r(\xi)\xi^q_x \partial^3_x \zeta^r$$

equals to a sum of terms containing at least one of ζ_x, ζ_{xx}, u, u_x, ζ_t or the difference of the values of a function at $\widetilde{\xi}$ and ξ. Similarly,

$$-u_{xx} + g_{qr}(\xi)\xi^q_x \xi^r_x u$$

equals to a sum of terms containing at least one of ζ_x, ζ_{xx}, ζ_t or the difference of the values of a function at $\widetilde{\xi}$ and ξ.

Therefore, we can estimate ζ and u linearly:

$$\begin{aligned} \left|\zeta^p_{tt} + \partial^4_x \zeta^p + 4\Gamma_q{}^p{}_r(\xi)\xi^q_x \partial^3_x \zeta^r\right| &\le C\big(|\zeta| + |\zeta_x| + |\zeta_{xx}| + |u| + |u_x| + |\zeta_t|\big), \\ \left|-u_{xx} + g_{qr}(\xi)\xi^q_x \xi^r_x u\right| &\le C\big(|\zeta| + |\zeta_x| + |\zeta_{xx}| + |\zeta_t|\big). \end{aligned}$$

Regarding ζ as a vector field along ξ, these inequalities can be written using covariant derivation along ξ:

$$\begin{aligned} \|\nabla_t^2 \zeta + \nabla_x^4 \zeta\| &\le C\{\|\zeta\|_2 + \|u\|_1 + \|\nabla_t \zeta\|\}, \\ \| - u_{xx} + |\xi_x|^2 u\| &\le C\{\|\zeta\|_2 + \|\nabla_t \zeta\|\}. \end{aligned}$$

Thus we have $\|u\|_1 \le C(\|\zeta\|_2 + \|\nabla_t \zeta\|)$, and

$$\begin{aligned} &\frac{d}{dt}\{\|\nabla_t \zeta\|^2 + \|\zeta\|_2^2\} \\ &= 2\langle \nabla_t \zeta, \nabla_t^2 \zeta\rangle + 2\langle \zeta, \nabla_t \zeta\rangle + 2\langle \nabla_x \zeta, \nabla_t \nabla_x \zeta\rangle + 2\langle \nabla_x^2 \zeta, \nabla_t \nabla_x^2 \zeta\rangle \\ &\le 2\langle \nabla_t \zeta, \nabla_t^2 \zeta + \nabla_x^4 \zeta\rangle + 2\langle \nabla_x \zeta, \nabla_x \nabla_t \zeta\rangle + C(\|\zeta\|_2^2 + \|\nabla_t \zeta\|^2) \\ &\le C_1(\|\nabla_t \zeta\|^2 + \|\zeta\|_2^2), \end{aligned}$$

from which we see that $(\|\nabla_t \zeta\|^2 + \|\zeta\|_2^2)e^{-C_1 t}$ is non-increasing, hence identically vanishes.

This proof applies at any time t_0 such that $\widetilde{\xi}(t_0) = \xi(t_0)$. Therefore, the set $\{t \mid \widetilde{\xi}(t) = \xi(t)\}$ is open and closed in $[0, T)$, hence agrees to $[0, T)$. $\square$

Combining Proposition 3.9 and Proposition 3.10, we get the following

Theorem 3.11. *The equation* (EW$^{\xi}$) *has a unique short time solution for any smooth initial data.*

Remark 3.12. To show this theorem, we did not assume that $\mu \geq 0$. Hence the result is time-invertible. That is, a unique solution exists on some open time interval $(-T, T)$ containing $t = 0$.

Corollary 3.13. *The equation* (EW) *has a unique short time solution for any smooth initial data.*

4. Singular perturbation

In this section, we assume that $\mu > 0$ and change the time variable t of (EW$^{\xi}$) to $\mu^{-1}t$.

$$(\mathrm{EW}^{\xi\mu}) \qquad \begin{cases} \mu^{-2}\nabla_t\xi_t + \nabla_x^3\xi_x + \xi_t = (w - |\xi_x|^2)\nabla_x\xi_x + 2w_x\xi_x - \dfrac{3}{2}\partial_x|\xi_x|^2\xi_x, \\ -w_{xx} + |\xi_x|^2 w = \mu^{-2}|\xi_t|^2 - |\nabla_x\xi_x|^2 + |\xi_x|^4, \\ \xi(x,0) = \xi_0(0), \quad \xi_t(x,0) = \mu\xi_1(x), \quad \displaystyle\int_0^1 \xi_0\,dx = \int_0^1 \xi_1\,dx = 0. \end{cases}$$

First, we show uniform existence and boundedness of solutions with respect to large μ. Constants T, M below are independent of μ.

Lemma 4.1. *For any $K > 0$, there are $T > 0$ and $M > 0$ with the following property*:

If ξ is a solution of (EW$^{\xi\mu}$) *on an interval $[0, t_1) \subset [0, T)$ and if its initial value satisfies $\|\xi_0\|$, $\|\xi_1\| \leq K$, then $\|\xi_x\|_1$, $\mu^{-1}\|\xi_t\| \leq M$ holds on $0 \leq t < t_1$.*

Proof. It is similar to the proof of Lemma 3.5. We put

$$f = (w - |\xi_x|^2)\nabla_x\xi_x + 2w_x\xi_x - \frac{3}{2}\partial_x|\xi_x|^2\xi_x,$$

and we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\mu^{-2}\|\xi_t\|^2 + \|\nabla_x\xi_x\|^2\} + \|\xi_t\|^2 = \langle\xi_t, f\rangle + \langle\nabla_x\xi_x, R(\xi_t, \xi_x)\xi_x\rangle \\ &\leq \left(\frac{1}{4} + \frac{1}{4}\right)\|\xi_t\|^2 + \|f\|^2 + C(\|\xi_x\|_1^2\|\nabla_x\xi_x\|)^2. \end{aligned}$$

Here, $\|f\|^2$ is bounded by a polynomial of $X := \mu^{-2}\|\xi_t\|^2 + \|\nabla_x \xi_x\|^2 + \|\xi_x\|^2$. Combining it with $d\|\xi_x\|^2/dt \le \|\xi_t\|^2 + \|\nabla_x \xi_x\|^2$, we have a μ-independent estimate of time derivative of X by a polynomial of X. Therefore, there is a μ-independent time $T > 0$ such that $\|\xi_t\| \le C\mu$ and $\|\xi_x\|_1 \le C$ on $[0, T)$. $\square$

Lemma 4.2. *For any $K > 0$ and $n > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:*

Let ξ be a solution of ($\mathrm{EW}^{\xi\mu}$) *on $[0, T)$ with $\mu \ge \mu_0$. If its initial value satisfies $\|\xi_0\|_{n+1}$, $\|\xi_1\|_n \le K$ and if it satisfies $\|\xi_x\|_1$, $\mu^{-1}\|\xi_t\| \le K$ on $[0, T)$, then it holds that $\|\xi_x\|_{n+1}$, $\|w\|_{n+1}$, $\mu^{-1}\|\xi_t\|_n \le M$ on $[0, T)$.*

Proof. It is similar to the proof of Lemma 3.7. Suppose that we have bounds: $\|\xi_x\|_{n+1}$, $\mu^{-1}\|\xi_t\|_n \le M$. They imply that $\|\xi_x\|_{(n)}$, $\mu^{-1}\|\xi_t\|_{(n-1)} \le C$, and,

$$\begin{aligned}\|w\|_{n+2},\ \|f\|_{n+1} &\le C(1 + \mu^{-1}\|\xi_t\|_{n+1} + \|\xi_x\|_{n+2})\\ &\le C(1 + \mu^{-1}\|\nabla_x^{n+1}\xi_t\| + \|\nabla_x^{n+2}\xi_x\|).\end{aligned}$$

Using this, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left\{\mu^{-2}\|\nabla_x^{n+1}\xi_t\|^2 + \|\nabla_x^{n+2}\xi_x\|^2\right\} + \|\nabla_x^{n+1}\xi_t\|^2\\ &= \langle \nabla_x^{n+1}\xi_t, \mu^{-2}\nabla_t\nabla_x^{n+1}\xi_t\rangle + \langle \nabla_x^{n+2}\xi_x, \nabla_t\nabla_x^{n+2}\xi_x\rangle + \|\nabla_x^{n+1}\xi_t\|^2\\ &\le \langle \nabla_x^{n+1}\xi_t, \mu^{-2}\nabla_x^{n+1}\nabla_t\xi_t\rangle + \langle \nabla_x^{n+2}\xi_x, \nabla_x^{n+3}\xi_t\rangle + \|\nabla_x^{n+1}\xi_t\|^2\\ &\quad + C\mu^{-2}\|\nabla_x^{n+1}\xi_t\| \cdot \mu\|\xi_t\|_{n+1} + C\|\xi_x\|_{n+2}\|\xi_t\|_{n+1}\\ &\le \langle \nabla_x^{n+1}\xi_t, \nabla_x^{n+1}f\rangle + \left(C\mu^{-1} + \frac{1}{8}\right)(\|\nabla_x^{n+1}\xi_t\|^2 + \|\xi_t\|^2) + C\|\xi_x\|_{n+2}^2\\ &\le \left(C_1\mu^{-1} + \frac{1}{4}\right)(\|\nabla_x^{n+1}\xi_t\|^2 + \|\xi_t\|^2) + C(1 + \|\nabla_x^{n+2}\xi_x\|^2).\end{aligned}$$

Assuming that $\mu \ge 4C_1$ and combining it with the first estimation:

$$\frac{1}{2}\frac{d}{dt}\left\{\mu^{-2}\|\xi_t\|^2 + \|\nabla_x\xi_x\|^2\right\} \le -\frac{1}{2}\|\xi_t\|^2 + C,$$

we can estimate

$$X(t) := \mu^{-2}(\|\nabla_x^{n+1}\xi_t\|^2 + \|\xi_t\|^2) + (\|\nabla_x^{n+2}\xi_x\|^2 + \|\nabla_x\xi_x\|^2)$$

by $X'(t) \le C(1 + X(t))$. Hence we have $\|\xi_x\|_{n+2} \le C$, $\|\xi_t\|_{n+1} \le C\mu$. Substituting it to the estimate of $\|w\|_{n+2}$, we get $\|w\|_{n+2} \le C$. $\square$

Proposition 4.3. *For any initial data ξ_0 and ξ_1, there is $T > 0$ such that* ($\mathrm{EW}^{\xi\mu}$) *has a solution on $[0, T)$ for each $\mu > 0$. Moreover, for any $n \ge 0$, there are $\mu_0 > 0$*

and $M > 0$ such that the solution with $\mu \geq \mu_0$ satisfies $\|\xi_x\|_n$, $\|w\|_n \leq M$ and $\|\xi_t\|_n \leq M\mu$ on $[0, T)$.

Proof. Using Lemma 4.1 and Lemma 4.2, the proof is similar to that of Proposition 3.9. □

Let $\{\eta, v\}$ be a solution of the limiting equation ($\mu \to \infty$) of ($\mathrm{EW}^{\xi\mu}$) omitting initial data $\xi_t(x, 0)$.

$$(\mathrm{EP}^\eta) \qquad \begin{cases} \eta_t + \nabla_x^3 \eta_x = (v - |\eta_x|^2)\nabla_x \eta_x + 2v_x \eta_x - \dfrac{3}{2}\partial_x |\eta_x|^2 \eta_x, \\ -v_{xx} + |\eta_x|^2 v = -|\nabla_x \eta_x|^2 + |\eta_x|^4, \\ \eta(x, 0) = \xi_0(0). \end{cases}$$

In [4] (Theorem 7.5), we know that the corresponding equation for closed curves in the euclidean space has a unique all time solution. Therefore, (EP^η) has a unique all time solution, via Lemma 2.2.

We regard function η as the 0-th approximation of ξ for $\mu \to \infty$. To compare ξ and η, we divide the interval $[0, \infty)$ so that the image $\eta(S^1 \times I)$ of each subinterval I is contained in a local coordinate U of S^2. For a solution ξ and an interval $[t_0, t_1) \subset I$ such that $\xi\big(S^1 \times [t_0, t_1)\big)$ is contained in U, we denote by $\{\zeta, u\}$ the difference between ξ and η in the local coordinate, i.e., $\zeta^p := \xi^p - \eta^p$, $u := w - v$. We use the local expression of ($\mathrm{EW}^{\xi\mu}$):

$$\begin{cases} \mu^{-2}\big(\xi_{tt}^p + \Gamma_q{}^p{}_r(\xi)\xi_t^q \xi_t^r\big) + \partial_x^4 \xi^p + 4\Gamma_q{}^p{}_r(\xi)\xi_x^q \partial_x^3 \xi^r + \xi_t^p = F^p[\xi_{xx}, w_x], \\ -w_{xx} + g_{qr}(\xi)\xi_x^q \xi_x^r w = \mu^{-2} g_{qr}(\xi)\xi_t^q \xi_t^r + G[\xi_{xx}], \\ \xi(x, 0) = \xi_0(0), \quad \xi_t(x, 0) = \mu\xi_1(x), \quad \displaystyle\int_0^1 \xi_0 \, dx = \int_0^1 \xi_1 \, dx = 0, \end{cases}$$

where $F^p[\xi_{xx}, w_x]$ are polynomials of ξ_x, ξ_{xx}, w, w_x, functions of ξ, and $G[\xi_{xx}]$ is a polynomial of ξ_x, ξ_{xx}, functions of ξ. (We only note highest derivatives.) Since the local expression of (EP^η) is given by the above equations substituting $\mu^{-1} = 0$, $\{\zeta, u\}$ satisfies

$$\begin{cases} \mu^{-2}\big(\zeta_{tt}^p + 2\Gamma_q{}^p{}_r(\eta)\eta_t^q \zeta_t^r\big) + \partial_x^4 \zeta^p + 4\Gamma_q{}^p{}_r(\eta)\eta_x^q \partial_x^3 \zeta^r + \zeta_t^p \\ \quad = F^p[\xi_{xx}, w_x] - F^p[\eta_{xx}, v_x] - 4\Gamma_q{}^p{}_r(\xi)\zeta_x^q \partial_x^3 \xi^r - 4\big(\Gamma_q{}^p{}_r(\xi) - \Gamma_q{}^p{}_r(\eta)\big)\eta_x^q \partial_x^3 \xi^r \\ \qquad - \mu^{-2}\{\eta_{tt}^p + \Gamma_q{}^p{}_r(\xi)\eta_t^q \eta_t^r + \Gamma_q{}^p{}_r(\xi)\zeta_t^q \zeta_t^r + 2(\Gamma_q{}^p{}_r(\xi) - \Gamma_q{}^p{}_r(\eta))\eta_t^q \zeta_t^r\}, \\ -u_{xx} + g_{qr}(\xi)\xi_x^q \xi_x^r u = \mu^{-2} g_{qr}(\xi)\xi_t^q \xi_t^r + G[\xi_{xx}] - G[\eta_{xx}], \\ \zeta(x, 0) = 0, \quad \zeta_t(x, 0) = \mu\xi_1(x). \end{cases}$$

We regard ζ as a vector field along η. Then, we can rewrite the above expression as

(EW$^\zeta$)

$$\begin{cases} \mu^{-2}\nabla_t^2\zeta + \nabla_x^4\zeta + \nabla_t\zeta \\ \quad = L_1[\nabla_x^2\zeta, u_x] + Q_1[\nabla_x^2\zeta, u_x; \nabla_x^3\zeta, u_x] - \mu^{-2}\{\nabla_t\eta_t + L_2[\zeta] + Q_2[\nabla_t\zeta; \nabla_t\zeta]\}, \\ -u_{xx} + |\xi_x|^2 u \\ \quad = \mu^{-2}\{|\eta_x|^2 + L_3[\nabla_t\zeta] + Q_3[\nabla_t\zeta; \nabla_t\zeta]\} + L_4[\nabla_x^2\zeta] + Q_4[\nabla_x^2\zeta; \nabla_x^2\zeta], \\ (|\xi_x|^2 = |\eta_x|^2 + L_5[\nabla_x\zeta] + Q_5[\nabla_x\zeta; \nabla_x\zeta]), \\ \zeta(x,0) = 0, \quad \nabla_t\zeta(x,0) = \mu\xi_1(x), \end{cases}$$

where L_i are linear, $|Q_i(\alpha;\beta)| \leq C|\alpha|\,|\beta|$. (We only note highest derivatives.)

To get estimate of $\{\zeta, u\}$, we need following

Lemma 4.4 ([5] Lemma 1.5). *For any K_1, $K_2 > 0$ and any $T > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:*

If $\mu \geq \mu_0$ and $X(t)$, $Y(t)$ and $Z(t)$ are non-negative functions on $[0, T)$ such that

$$X(0) \leq K_1\mu^{-2}, \quad |X'(0)| \leq K_1, \quad Y(0) \leq K_1, \quad Z(0) \leq K_1\mu^2,$$

and that

$$\begin{aligned} \mu^{-2}X''(t) + X'(t) &\leq K_1\big(X(t) + \mu^{-2}Z(t) + \mu^{-2}\big) - K_2Y(t), \\ Y'(t) + \mu^{-2}Z'(t) &\leq K_1\big(Y(t) + 1\big) - K_2Z(t), \end{aligned}$$

on $[0, T)$, then they satisfy

$$X(t) < M\mu^{-2}, \quad Y(t) < M \quad \textit{and} \quad Z(t) < M\mu^2$$

on $[0, T)$.

Lemma 4.5. *For any $n \geq 0$ and any $K > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:*

Let $\{\zeta, u\}$ be the solution of (EW$^\zeta$) *with $\mu \geq \mu_0$, defined on $[t_0, t_1) \subset [0, T)$. If $\|\zeta\|_n \leq K\mu^{-1}$ at $t = t_0$, then $\|\zeta\|_n \leq M\mu^{-1}$ holds on $[t_0, t_1)$.*

Proof. Note that we have bounds of $\{\xi, w\}$ and $\{\eta, v\}$ by Proposition 4.3. Therefore, we know $\|\zeta\|_n \leq C$, $\|\nabla_t\zeta\|_n \leq C\mu$ and $\|u\|_n \leq C$. We may assume that $\mu \geq \mu_0 \geq 1$. For

$$h := \mu^{-2}(|\eta_x|^2 + L_3[\nabla_t\zeta] + Q_3[\nabla_t\zeta; \nabla_t\zeta]) + L_4[\nabla_x^2\zeta] + Q_4[\nabla_x^2\zeta; \nabla_x^2\zeta],$$

we have

$$\begin{aligned}\|h\|_n &\le C\{\mu^{-2}(1+\|\nabla_t\zeta\|_n+\|\nabla_t\zeta\|_1\|\nabla_t\zeta\|_n)+\|\zeta\|_{n+2}+\|\zeta\|_3\|\zeta\|_{n+2}\}\\ &\le C(\mu^{-2}+\mu^{-1}\|\nabla_t\zeta\|_n+\|\zeta\|_{n+2}),\end{aligned}$$

and, $\|u\|_{n+2}\le C\|h\|_n\le C(\mu^{-2}+\mu^{-1}\|\nabla_t\zeta\|_n+\|\zeta\|_{n+2})$. And, for

$$f := L_1[\nabla_x^2\zeta, u_x]+Q_1[\nabla_x^2\zeta, u_x;\nabla_x^3\zeta, u_x]-\mu^{-2}(\nabla_t\eta_t+L_2[\zeta]+Q_2[\nabla_t\zeta;\nabla_t\zeta]),$$

we have

$$\begin{aligned}\|f\|_n &\le C\{\|\zeta\|_{n+2}+\|u\|_{n+1}+\mu^{-2}(1+\|\nabla_t\zeta\|_1\|\nabla_t\zeta\|_n)\}\\ &\le C\{\|\zeta\|_{n+2}+\mu^{-2}+\mu^{-1}\|\nabla_t\zeta\|_n\}.\end{aligned}$$

Put $X_n(t) := \|\nabla_x^n\zeta\|$ and $Z_n(t) := \|\nabla_x^n\nabla_t\zeta\|$. Then, we see that

$$\begin{aligned}&({X_0}^2)' = 2\langle\zeta, \nabla_t\zeta\rangle \le 2X_0Z_0,\\ &({X_1}^2)' = 2\langle\nabla_x\zeta, \nabla_t\nabla_x\zeta\rangle \le -2\langle\nabla_x\zeta, \nabla_x\nabla_t\zeta\rangle + C\|\zeta\|_1\|\zeta\|\\ &\quad \le 2X_2Z_0 + C({X_0}^2+{X_1}^2),\\ &\mu^{-2}({Z_i}^2)' + 2{Z_i}^2 + ({X_{i+2}}^2)'\\ &\quad = 2\langle\nabla_x^i\nabla_t\zeta, \mu^{-2}\nabla_t\nabla_x^i\nabla_t\zeta+\nabla_t\nabla_x^i\zeta\rangle + 2\langle\nabla_x^{i+2}\zeta, \nabla_t\nabla_x^{i+2}\zeta\rangle\\ &\quad \le 2\langle\nabla_x^i\nabla_t\zeta, \nabla_x^i f\rangle + C\|\nabla_x^i\nabla_t\zeta\|(\mu^{-2}\|\nabla_t\zeta\|_{i-1}+\|\zeta\|_{i-1}) + C\|\nabla_x^{i+2}\zeta\|\,\|\zeta\|_{i+1}\\ &\quad \le CZ_i\{X_{i+2}+X_0+\mu^{-2}+\mu^{-1}(Z_i+Z_0)\} + C({X_{i+2}}^2+{X_0}^2).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mu^{-2}(\|\nabla_t\zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' + 2\|\nabla_t\zeta\|_n^2\\ &\quad \le C\|\zeta\|_{n+2}^2 + C\mu^{-1}\|\nabla_t\zeta\|_n^2 + C\mu^{-2} + C\sum_{i=0}^n Z_i(X_{i+2}+X_0)\\ &\quad \le \frac{1}{2}\|\nabla_t\zeta\|_n^2 + C\|\zeta\|_{n+2}^2 + C_1\mu^{-1}\|\nabla_t\zeta\|_n^2 + C\mu^{-2},\\ &\mu^{-2}(\|\nabla_t\zeta\|_n^2)' + (\|\zeta\|_{n+2}^2)' \le C(\|\zeta\|_{n+2}^2+\mu^{-2}) - \|\nabla_t\zeta\|_n^2\end{aligned}$$

if $\mu \ge 2C_1$.

We also have,

$$\begin{aligned}&\mu^{-2}({X_i}^2)'' + ({X_i}^2)' + 2{X_{i+2}}^2\\ &\quad = 2\mu^{-2}\|\nabla_t\nabla_x^i\zeta\|^2 + 2\langle\nabla_x^i\zeta, \mu^{-2}\nabla_t^2\nabla_x^i\zeta + \nabla_t\nabla_x^i\zeta + \nabla_x^{i+4}\zeta\rangle\end{aligned}$$

$$
\begin{aligned}
&\le 3\mu^{-2}\|\nabla_x^i\nabla_t\zeta\|^2 + 2\langle\nabla_x^i\zeta, \nabla_x^i f\rangle \\
&\quad + C\mu^{-2}\|\zeta\|_{i-1}^2 + C\|\nabla_x^i\zeta\|\{\mu^{-2}(\|\nabla_t\zeta\|_{i-1} + \|\zeta\|_{i-2}) + \|\zeta\|_{i-1}\} \\
&\le 3\mu^{-2}Z_i{}^2 + CX_i\{X_{i+2} + X_0 + \mu^{-2} + \mu^{-1}(Z_i + Z_0)\} \\
&\quad + C\mu^{-2}(X_i{}^2 + X_0{}^2) + CX_i\{\mu^{-2}(Z_i + Z_0) + X_i + X_0\} \\
&\le X_{i+2}{}^2 + C\{X_i{}^2 + X_0{}^2 + \mu^{-2}(Z_i{}^2 + Z_0{}^2) + \mu^{-4}\},
\end{aligned}
$$

$$
\mu^{-2}(\|\zeta\|_n^2)'' + (\|\zeta\|_n^2)' \le C\{\|\zeta\|_n^2 + \mu^{-2}\|\nabla_t\zeta\|_n^2 + \mu^{-4}\} - \|\zeta\|_{n+2}^2.
$$

Setting $X := \|\zeta\|_n^2$, $Y := \|\nabla_x^{n+2}\zeta\|^2$ and $Z := \|\nabla_t\zeta\|_n^2$ in Lemma 4.4, we have $\|\zeta\|_n \le C\mu^{-1}$. □

Lemma 4.6. *For any n, $m \ge 0$ and $K > 0$, there are $M > 0$ and $\mu_0 > 0$ with the following property:*

Let $\{\zeta, u\}$ be the solution of (EW$^\zeta$) *with $\mu \ge \mu_0$, defined on $[t_0, t_1) \subset [0, T)$. If $\|\nabla_t^m\zeta\|_n \le K\mu^{2m-1}$ at $t = t_0$, then*

$$
\begin{aligned}
\|\nabla_t^m\zeta\|_{(n)} &\le M(\mu^{-1} + \mu^{2m-1}e^{-\mu^2 t/2}), \\
\|\partial_t^m u\|_{(n)} &\le M(\mu^{-1} + \mu^{2m}e^{-\mu^2 t/2})
\end{aligned}
$$

hold on $[t_0, t_1)$.

Proof. We put $V_j := \mu^{-1} + \mu^j e^{-\mu^2 t/2}$. Note the log-convexity:

$$
V_j{}^2 \le V_{j-1}V_{j+1} \quad \text{and} \quad V_jV_k \le V_0V_{j+k} \le (1 + \mu_0^{-1})V_{j+k} \quad \text{for} \quad j, k \ge 0.
$$

We know that $\|\nabla_t\zeta\|_{(n)} \le C\mu$, $\|u\|_{(n)} \le C$ by Proposition 4.3, and $\|\zeta\|_{(n)} \le C\mu^{-1}$ by Lemma 4.5. In particular, $\|\zeta\|_{(n)} \le CV_{-1}$ holds. We prove the estimate of $\partial_t^m u$ and the estimate of $\nabla_t^{m+1}\zeta$, assuming the estimate of $\partial_t^j u$ and $\nabla_t^{j+1}\zeta$ for $j < m$.

First, we estimate $\partial_t^m u$. Put

$$
h := \mu^{-2}(|\eta_x|^2 + L_3[\nabla_t\zeta] + Q_3[\nabla_t\zeta; \nabla_t\zeta]) + L_4[\nabla_x^2\zeta] + Q_4[\nabla_x^2\zeta; \nabla_x^2\zeta].
$$

It is estimated as

$$
\begin{aligned}
\|\partial_t^m h\|_{(n)} &\le C\{\mu^{-2}(1 + \|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m-1} \\
&\qquad + \|\nabla_t\zeta\|_{(n)}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_3^* V_{2m-1}) + V_{2m-1}\} \\
&\le C\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\},
\end{aligned}
$$

where V_3^* appears only if $m \geq 2$. Therefore, we have

$$\begin{aligned}\|\partial_t^m u\|_{(n+2)} &\leq \|\partial_t^m h\|_{(n)} + C\sum_{j=1}^{m} \|\partial_t^j |\xi_x|^2\|_{(n)} \|\partial_t^{m-j} u\|_{(n)} \\ &\leq C\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\} + C\sum_{j=1}^{m}(1 + V_{2j-1})V_{2(m-j)} \\ &\leq C\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\}.\end{aligned}$$

Now, we estimate $\nabla_t^{m+1}\zeta$. Put

$$f := L_1[\nabla_x^2\zeta, u_x] + Q_1[\nabla_x^2\zeta, u_x; \nabla_x^3\zeta, u_x] - \mu^{-2}(\nabla_t\eta_t + L_2[\zeta] + Q_2[\nabla_t\zeta; \nabla_t\zeta]).$$

Then,

$$\begin{aligned}\|\nabla_t^m f\|_{(n)} &\leq C\{V_{2m-1} + \|\partial_t^m u\|_{(n+1)} + \|u\|_{(n+1)}\|\partial_t^m u\|_{(n+1)} \\ &\qquad + \mu^{-2}(1 + V_{2m-1} + \|\nabla_t\zeta\|_{(n)}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_3^* V_{2m-1})\} \\ &\leq C\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\},\end{aligned}$$

where V_3^* appears only if $m \geq 2$. Therefore,

$$\begin{aligned}\|\nabla_t^m(\mu^{-2}\nabla_t^2\zeta + \nabla_t\zeta)\|_{(n)} &\leq \|\nabla_t^m\zeta\|_{(n+4)} + \|\nabla_t^m f\|_{(n)} \\ &\leq C\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\}.\end{aligned}$$

Thus,

$$\begin{aligned}&\mu^{-2}\frac{\partial}{\partial t}|\nabla_x^n\nabla_t^{m+1}\zeta|^2 + 2|\nabla_x^n\nabla_t^{m+1}\zeta|^2 \\ &\quad = 2\left(\nabla_x^n\nabla_t^{m+1}\zeta, \mu^{-2}\nabla_t\nabla_x^n\nabla_t^{m+1}\zeta + \nabla_x^n\nabla_t^{m+1}\zeta\right) \\ &\quad \leq 2\left(\nabla_x^n\nabla_t^{m+1}\zeta, \nabla_x^n(\mu^{-2}\nabla_t^{m+2}\zeta + \nabla_t^{m+1}\zeta)\right) \\ &\qquad + C\mu^{-2}|\nabla_x^n\nabla_t^{m+1}\zeta|\|\nabla_t^{m+1}\zeta\|_{(n-1)} \\ &\quad \leq C|\nabla_x^n\nabla_t^{m+1}\zeta|\{\mu^{-1}\|\nabla_t^{m+1}\zeta\|_{(n)} + V_{2m}\}.\end{aligned}$$

From this, for $X(t) := \|\nabla_t^{m+1}\zeta\|_{(n)}^2$, we have

$$\mu^{-2}X'(t) + 2X(t) \leq C_1\mu^{-1}X(t)^2 + CV_{2m}X(t) \leq \left(\frac{1}{2} + C_1\mu^{-1}\right)X(t)^2 + CV_{2m}^2,$$

where $X'(t) = \limsup_{\delta\to+0}\{X(t+\delta) - X(t)\}/\delta$.

We set $\mu_0 \leq 2C_1$. Then,

$$\mu^{-2}X'(t) + X(t) \leq C_2(\mu^{-2} + \mu^{4m}e^{-\mu^2 t}),$$

$$\begin{aligned} X(t) &\leq X(t_0)e^{-\mu^2 t} + C_2(\mu^{-2} + \mu^{4m+2}e^{-\mu^2 t}) \\ &\leq C(\mu^{-2} + \mu^{4m+2}e^{-\mu^2 t}), \end{aligned}$$

that is, $\|\nabla_t^{m+1}\zeta\|_{(n)}^2 \leq CV_{2m+1}$.

Substituting it to the estimate of $\|\partial_t^m u\|_{(n+2)}$, we get the estimation of $\partial_t^m u$. □

Proposition 4.7. *For any initial data $\{\xi_0, \xi_1\}$, any interval $[t_0, t_1) \subset [0, T)$ and any local coordinate U of S^2 such that the image $\eta(S^1 \times [t_0, t_1))$ is contained in U, there exists $\mu_0 > 0$ with the following property*:

If ξ is a solution of (EW$^{\xi\mu}$) *on $[0, T)$, then the image $\xi\big(S^1 \times [t_0, t_1)\big)$ is contained in U. Moreover, ξ uniformly converges to η on $[0, T)$ when $\mu \to \infty$.*

Proof. We divide the interval $[0, T)$ so that the image $\eta(S^1 \times I)$ of each subinterval I is included to a local coordinate U_I. □

Note that ζ is defined only on each short time interval.

Starting from $t = 0$ and applying this Lemma on each time interval where $\{\zeta, u\}$ is defined, we see that $\|\zeta\|_n$ is small for large μ.

We sum up these results, and get the following

Theorem 4.8. *For any non-negative integers m, n and any positive number T, there are positive numbers μ_0 and M with the following properties*:

For each $\mu \geq \mu_0$, there exists a solution ξ of (EW$^{\xi\mu}$) *on $[0, T)$, and ξ uniformly converges to η when $\mu \to \infty$. More precisely,*

$$|\partial_t^m \partial_x^n (\xi^p - \eta^p)| \leq M(\mu^{-1} + \mu^{2m-1}e^{-\mu^2 t/2})$$

holds on each local coordinate.

REMARK 4.9. In general, we cannot expect uniform estimation on the whole time $[0, \infty)$. The limit $\eta(\infty)$ can be an unstable elastic curve, and in that case, $\xi(\infty)$ and $\eta(\infty)$ discontinuously depend on the initial data.

Corollary 4.10. *For any positive number T, there exists a unique solution γ of* (EW$^\tau$) *on $[0, T)$ for sufficiently large $\mu > 0$. Moreover, the solution converges to a solution η of* (EP) *when $\mu \to \infty$.*

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