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NON-LINEARIZABLE REAL ALGEBRAIC ACTIONS OF $O(2, R)$ ON R^4

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0. Introduction

In algebraic transformation groups, one of the important problems is the following.

Linearization problem ([6]). *Let G be a reductive complex algebraic group. Is any algebraic G action on affine space $\boldsymbol{C}^n$ linearizable, i.e. isomorphic to some G module as G variety?*

Some positive answers to this problem have been given (see [1] for a survey article) but in 1989, G.W. Schwarz [17] constructed counterexamples for many noncommutative groups with $O(2,\boldsymbol{C})$ being the most explicit case (in the case that the acting group is commutative, any counterexample have never found, and see [7], [9], [11], [12] for further recent results).

In this paper, we consider the analogous problem in the real algebraic category, which was posed in [15]. Then it would be appropriate to take a compact Lie group as acting group since there is a one-to-one correspondence between the family of compact Lie groups and that of reductive complex algebraic groups through the complexification (see [14] p.247).

Schwarz used the properties of complex algebraic geometry to find the counterexamples, so it is not clear whether his argument works in the real algebraic category because $\boldsymbol{R}$ is not algebraically closed. We use the methods of Masuda-Petrie [11] to obtain the following result.

Theorem. *There is a continuous family of algebraically inequivalent, non-linearizable real algebraic $O(2,\boldsymbol{R})$ actions on $\boldsymbol{R}^4$.*

Let G be a compact real algebraic group and G_C be the reductive complex algebraic group obtained from G via the complexification. Let $ACT(G,\boldsymbol{R}^n)$ (resp. $ACT(G_C,\boldsymbol{C}^n)$) be the set of equivalence classes of real algebraic G actions on $\boldsymbol{R}^n$ (resp. complex algebraic G_C actions on $\boldsymbol{C}^n$), where the equivalence relation is defined by G variety (resp. G_C variety) isomorphism. Then there is a complexification map

$$c_a : ACT(G,\boldsymbol{R}^n) \to ACT(G_C,\boldsymbol{C}^n).$$

It is natural to ask that c_a is injective, but it turns out that the examples in the theorem above give a negative answer to this question.

Proposition. *The map c_a is not injective.*

This paper is organized as follows. We consider the relation between the linearization problem and algebraic G vector bundles in section 1 and construct non-trivial real (affine) algebraic $O(2,\boldsymbol{R})$ vector bundles in section 2. In section 3 we consider the complexification of real algebraic G vector bundles and that of algebraic actions. In section 4 we prove the theorem above using vector bundles constructed in section 2, and apply the complexifications to the examples in the theorem. We give an explicit description of a non-linearizable real algebraic $O(2,\boldsymbol{R})$ action in the appendix. Most of the results in this paper are from the author's master thesis [13].

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1. Algebraic G vector bundles and non-linearizable actions

Let $\boldsymbol{K}$ be the real numbers $\boldsymbol{R}$ or the complex numbers $\boldsymbol{C}$. We say that X ($\subset \boldsymbol{K}^n$) is *an affine variety* if X is the set of the zeros of a map from $\boldsymbol{K}^n$ to some $\boldsymbol{K}^m$ whose coordinate functions are polynomials, and we say that $f: X \to Y$, where X ($\subset \boldsymbol{K}^n$) and Y ($\subset \boldsymbol{K}^m$) are affine varieties, is an *algebraic map* if f extends to a map from $\boldsymbol{K}^n$ to $\boldsymbol{K}^m$ whose coordinate functions are polynomials. A group G is an *algebraic group* if G is an affine variety and the map $\varphi: G \times G \to G$ defined by $(g_1,g_2) \mapsto g_1 g_2^{-1}$ is algebraic, X is an *(affine) G variety* if X is an affine variety and the action map $\phi: G \times X \to X$ is algebraic, and $f: X \to Y$ is an *algebraic G map* (here X and Y are G varieties) if f is algebraic and G equivariant. An algebraic G map is an *algebraic G isomorphism* if it is bijective and its inverse is also an algebraic G map. Two G varieties are *isomorphic* if there is an algebraic G isomorphism between them.

Let G denote an algebraic group over $\boldsymbol{K}$ and let B, F, S denote G modules over $\boldsymbol{K}$ whose representation maps ($: G \times B \to B$ etc.) are algebraic.

DEFINITION 1.1. Let $Vec(B,F;S)$ be the set of algebraic G vector bundles E over B such that $E \oplus \boldsymbol{S}$ is isomorphic to $\boldsymbol{F} \oplus \boldsymbol{S}$ as algebraic G vector bundle, where $\boldsymbol{F} = B \times F$ and $\boldsymbol{S} = B \times S$ are product bundles over B. We define $VEC(B,F;S)$ to be the set of isomorphism classes of elements in $Vec(B,F;S)$ as algebraic G vector bundles.

We recall some results about $Vec(B,F;S)$ from [11]. The following results are established in [11] when $\boldsymbol{K}=\boldsymbol{C}$. But the same argument works when $\boldsymbol{K}=\boldsymbol{R}$.

DEFINITION 1.2. Let $sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$ be the set of algebraic G vector bundle surjections $L:\boldsymbol{F}\oplus\boldsymbol{S}\to\boldsymbol{S}$ which allow an algebraic G splitting map from $\boldsymbol{S}$ to $\boldsymbol{F}\oplus\boldsymbol{S}$, and let $aut(\boldsymbol{F}\oplus\boldsymbol{S})$ be the group of algebraic G vector bundle automorphisms τ of $\boldsymbol{F}\oplus\boldsymbol{S}$.

REMARK. In the complex category, any algebraic G vector bundle surjection from $\boldsymbol{F}\oplus\boldsymbol{S}$ to $\boldsymbol{S}$ has a splitting (see [2]). But in the real category, this is not the case. For example, $f:\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}\times\boldsymbol{R}$ defined by $(a,b)\mapsto(a,(a^2+1)b)$ has no splitting, where $\boldsymbol{R}\times\boldsymbol{R}$ is viewed as a trivial bundle with the projection on the first factor $\boldsymbol{R}$.

The group $aut(\boldsymbol{F}\oplus\boldsymbol{S})$ acts on $sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$ by $L\xrightarrow{\tau}L\circ\tau$ and $L\in sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$ defines an element $ker\,L$ in $Vec(B,F;S)$.

Theorem 1.3 ([11]). *The map sending* $L\in sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$ *to* $ker\,L\in Vec(B,F;S)$ *induces a bijection*

$$sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})\,/\,aut(\boldsymbol{F}\oplus\boldsymbol{S})\cong VEC(B,F;S).$$

Because of the solution of the Serre conjecture (see[16], [19]), any vector bundle $E\in Vec(B,F;S)$ is trivial if we forget the actions. So E gives an algebraic G action on some $\boldsymbol{K}^n$. We consider the classification of (the total spaces of) elements in $Vec(B,F;S)$ as G varieties.

DEFINITION 1.4. Let $VAR(B,F;S)$ be the set of isomorphism classes of elements in $Vec(B,F;S)$ as G varieties. Let $Aut(B)^G$ be the group of G variety automorphisms of B.

The group $Aut(B)^G$ acts on $VEC(B,F;S)$ by taking pull back bundles and the trivial element in $VEC(B,F;S)$ is fixed under the action. One easily sees that the natural map from $VEC(B,F;S)$ to $VAR(B,F;S)$ factors through the map

$$VEC(B,F;S)\,/\,Aut(B)^G\to VAR(B,F;S).$$

This map is often (but not always) bijective ([11]). We recall a sufficient condition for the above map to be bijective.

DEFINITION 1.5. Let $E_1,E_2\in Vec(B,F;S)$ and let $f:E_1\to E_2$ be a G variety isomorphism. We say that *f maps B as graph* if the composition $pfs:B\to B$ is in $Aut(B)^G$, where $p: E_2 \to B$ is the projection and $s: B \to E_1$ is the zero-section.

Theorem 1.6 ([11]). *Suppose that any G variety isomorphism between elements in $Vec(B,F;S)$ maps B as graph. Then the natural map: $VEC(B,F;S) \to VAR(B,F;S)$ induces a bijection*

$$VEC(B,F;S)\,/\,Aut(B)^G \cong VAR(B,F;S).$$

In particular, if $E \in Vec(B,F;S)$ is non-trivial, then the G action on E is non-linearizable.

2. Non-trivial $O(2,\boldsymbol{R})$ vector bundles

In this section we show that $VEC(B,F;S)$ can be non-trivial. Let $O(2,\boldsymbol{R})$ be the real orthogonal group. We identify it with $S^1 \quad \boldsymbol{Z}_2$. Define a two dimensional real $O(2,\boldsymbol{R})$ module $W_n = \{(a,\bar{a})\,; a \in \boldsymbol{C}\}$ $(n \in \boldsymbol{N})$ as follows (here $\bar{a}$ denotes the complex conjugate of a). For $g \in S^1$ and $1 \neq J \in \boldsymbol{Z}_2$, the representation map is defined by

$$g \mapsto \begin{pmatrix} g^n & 0 \\ 0 & \bar{g}^n \end{pmatrix}, \qquad J \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem 2.1. *There exists a bijection: $VEC(W_1, W_m; \boldsymbol{R}) \cong \boldsymbol{R}^{m-1}$.*

In order to prove this theorem, we use Theorem 1.3. We first calculate $sur(W_m \oplus \boldsymbol{R}, \boldsymbol{R})$ and $aut(W_m \oplus \boldsymbol{R})$.

Lemma 2.2. (1) *Any surjection $L \in sur(W_m \oplus \boldsymbol{R}, \boldsymbol{R})$ is of the following form on the fiber over $(a,\bar{a}) \in W_1$;*

$$L(a,\bar{a}) = (f\bar{a}^m, fa^m, h),$$

where f, h are relatively prime polynomials of $t = |a|^2$ with real coefficients and $h(0) \neq 0$.

(2) *Any automorphism $\tau \in aut(W_m \oplus \boldsymbol{R})$ is of the following form on the fiber over $(a,\bar{a}) \in W_1$;*

$$\tau(a,\bar{a}) = \begin{pmatrix} u & a^{2m}l & a^m s \\ \bar{a}^{2m}l & u & \bar{a}^m s \\ \bar{a}^m r & a^m r & w \end{pmatrix},$$

where u, w, l, r, s are polynomials of $t = |a|^2$ and u, w are congruent to non-zero constants modulo t^m.

Proof. (1) L is linear relative to each coordinate of W_m and $\boldsymbol{R}$, so one can write

$$L(a,\bar{a}) = (L_1(a,\bar{a}),\ L_2(a,\bar{a}),\ L_3(a,\bar{a})),$$

where L_i is a polynomial for $i=1,2,3$. The S^1 equivariance of L means that

$$L_1(ga,\overline{ga})=\bar{g}^m L_1(a,\bar{a}),\qquad L_2(ga,\overline{ga})=g^m L_2(a,\bar{a}),\qquad L_3(ga,\overline{ga})=L_3(a,\bar{a}).$$

An elementary computation shows that these imply

$$L_1(a,\bar{a})=f_1(t)\bar{a}^m,\qquad L_2(a,\bar{a})=f_2(t)a^m,\qquad L_3(a,\bar{a})=h(t)$$

for some polynomials f_1, f_2 and h with real coefficients. The $\boldsymbol{Z}_2$ equivariance shows that f_1 coincides with f_2, which we denote by f. The property that f and h are relatively prime follows from the existence of a splitting of L and that $h(0)$ is non-zero follows from the surjectivity of L.

(2) Because of $O(2,\boldsymbol{R})$ equivariance, one can check that τ is of the form in the statement. Since τ is an automorphism,

$$\det(\tau(a,\bar{a}))=(u-t^m l)(uw-2t^m rs+t^m lw)$$

must be a unit polynomial, which is a non-zero constant. So each factor at the right hand side is also a non-zero constant. It follows that u and uw are congruent to non-zero constants modulo t^m, hence so is w. □

NOTATION. Let $L_{f,h}$ denote L in Lemma 2.2 (1) and $E(f,h)$ denote the kernel of $L_{f,h}$. We abbreviate $E(1,h)$ as $E(h)$. Then the vector bundle $E(h)$ (with the obvious projection on W_1) is written as follows;

$$E(h)=\{(a,\bar{a},x,\bar{x},z)\in W_1\times W_m\times\boldsymbol{R};\ \bar{a}^m x+a^m\bar{x}+h(t)z=0\}.$$

Note that if h is a non-zero constant, $E(h)$ is isomorphic to W_m through the correspondence $(a,\bar{a},x,\bar{x},z)\mapsto(a,\bar{a},x,\bar{x})$.

Lemma 2.3. *There are three vector bundle isomorphisms.*
(1) $E(f,h)\cong E(f,h/h(0))$.
(2) $E(f,h)\cong E(h)$.
(3) $E(h_1)\cong E(h_2)$ *if and only if there is a non-zero constant* c *such that* $h_1\equiv ch_2$ *modulo* t^m.

Proof. (1) $(x,\bar{x},z)\mapsto(x,\bar{x},h(0)z)$ is the required isomorphism.

(2) By Theorem 1.3 and Lemma 2.2 (2), it suffices to show the existence of polynomials u, w, l, r, s such that

$$(\bar{a}^m\ \ a^m\ \ h)\ =\ (f\bar{a}^m\ \ fa^m\ \ h)\begin{pmatrix} u & a^{2m}l & a^m s\\ \bar{a}^{2m}l & u & \bar{a}^m s\\ \bar{a}^m r & a^m r & w\end{pmatrix}$$

and that the determinant of the above 3×3 matrix is a non-zero constant. Choose polynomials ξ and η of t such that $f\xi+h\eta=1$ (this is possible since f and h are relatively prime by Lemma 2.2 (1)) and polynomials r' and r'' of t such that $hr'=(1-f)-t^m r''$ (this is possible since $h(0)\neq 0$ by Lemma 2.2 (1)). Then one can check that

$$u=1+t^m l,\quad w=1-2t^m fl,\quad s=hl,\quad l=\xi r''/2,\quad r=r'+t^m\eta r''$$

satisfies the required conditions.

(3) If $E(h_1)\cong E(h_2)$ there is $\tau\in aut(\boldsymbol{W}_m\oplus\boldsymbol{R})$ such that $L_{1,h_1}=L_{1,h_2}\circ\tau$, i.e.

$$(\bar{a}^m\ a^m\ h_1)\ =\ (\bar{a}^m\ a^m\ h_2)\begin{pmatrix} u & a^{2m}l & a^m s\\ \bar{a}^{2m}l & u & \bar{a}^m s\\ \bar{a}^m r & a^m r & w\end{pmatrix},$$

where the determinant of the above 3×3 matrix is a non-zero constant. Hence $h_1=h_2 w+2t^m s$. Since w is a non-zero constant modulo t^m by Lemma 2.2 (2), the necessity is clear. Conversely if $h_1=ch_2+t^m h_0$ for some polynomial h_0 of t, then $\tau\in aut(\boldsymbol{W}_m\oplus\boldsymbol{R})$ defined by

$$\tau(a,\bar{a})\ =\begin{pmatrix} 1 & 0 & a^m h_0/2\\ 0 & 1 & \bar{a}^m h_0/2\\ 0 & 0 & c\end{pmatrix}$$

is the isomorphism between $E(h_1)$ and $E(h_2)$. □

Proof of Theorem 2.1. By Theorem 1.3 and Lemma 2.2 (1), any element in $VEC(W_1,W_m;\boldsymbol{R})$ is of the form $[E(f,h)]$, where [ ] denotes the isomorphism class. Then Lemma 2.3 implies that the correspondence

$$\boldsymbol{R}^{m-1}\ni(a_1,\cdots,a_{m-1})\mapsto[(E(h)],$$

where $h(t)=1+a_1t+\cdots+a_{m-1}t^{m-1}$, gives the bijection. □

3. Complexification

In this section, we assume that G is a real algebraic group and B, F, S are real G modules. We first define the complexification of real affine verieties and algebraic maps and prove some properties.

DEFINITION 3.1. Let $X(\subset\boldsymbol{R}^n)$ be a real affine variety and let $I(X)$ be the ideal of polynomial maps from $\boldsymbol{R}^n$ to $\boldsymbol{R}$ which vanish on X. We define the complex affine variety X_C to be the common zeros of all the elements in $I(X)$ regarded as maps from $\boldsymbol{C}^n$ to $\boldsymbol{C}$, and we call X_C the *complexification* of X.

Here are some elementary properties about the complexification.

Proposition 3.2. (1) *Let $I(X_C)$ be the ideal of polynomial maps from $\boldsymbol{C}^n$ to $\boldsymbol{C}$ which vanish on X_C. Then $I(X_C)=I(X)\otimes\boldsymbol{C}$.*

(2) $(X\times Y)_C=X_C\times Y_C$.

(3) *Any algebraic map $f: X\to Y$ extends to a unique algebraic map $f_C: X_C\to Y_C$.*

Proof. (1) It is clear that $I(X_C)\supset I(X)\otimes\boldsymbol{C}$ by definition. We prove the opposite inclusion. For $f\in I(X_C)$, we express $f=f_1+if_2$, where f_1 and f_2 are polynomials with real coefficients. Then $f_1|_X+if_2|_X=f|_X=0$, so f_1 and f_2 are in $I(X)$. This means that $I(X_C)\subset I(X)\otimes\boldsymbol{C}$.

(2) The ideal $I(X\times Y)$ is generated by the elements $f_t h_s$, where $f_t\in I(X)$ and $h_s\in I(Y)$. This together with (1) shows that the ideal $I((X\times Y)_C)$ is generated by the elements $\tilde{f}_t\tilde{h}_s$, where $\tilde{f}_t\in I(X_C)$ and $\tilde{h}_s\in I(Y_C)$. This implies (2).

(3) Suppose $X\subset\boldsymbol{R}^n$ and $Y\subset\boldsymbol{R}^m$ and let $F:\boldsymbol{R}^n\to\boldsymbol{R}^m$ be an extension of f. We regard F as a map from $\boldsymbol{C}^n$ to $\boldsymbol{C}^m$. One easily checks that F maps X_C to Y_C. Therefore $F|_{X_C}: X_C\to Y_C$ is an extension of f. Now we prove the uniqueness. Suppose that two maps f_1, $f_2: X_C\to Y_C$ are extensions of f. Let $F_j:\boldsymbol{C}^n\to\boldsymbol{C}^m$ be an extension of f_j $(j=1,2)$. Then F_1-F_2 is algebraic and vanishes on X. Therefore F_1-F_2 vanishes on X_C by (1). Hence $f_1-f_2=(F_1-F_2)|_{X_C}=0$, i.e. $f_1=f_2$. □

We call f_C the *complexification* of f. By Proposition 3.2, we obtain the following.

Corollary 3.3. (1) *The complexification of a real algebraic group is a complex algebraic group.*

(2) *If G is a real algebraic group and X is a real G variety, X_C is a complex G_C variety.*

(3) *If X and Y are real G varieties and $f: X\to Y$ is G equivariant, then $f_C: X_C\to Y_C$ is G_C equivariant.*

(4) *If $f: X\to Y$ and $h: Y\to Z$ are algebraic G maps between real G varieties, then $(f\circ h)_C=f_C\circ h_C$.*

Now we define a complexification of elements in $VEC(B,F;S)$ and an involution on $VEC(B_C,F_C;S_C)$. Note that the usual complexification of vector bundles means to complexify only fibers, but our definition means to complexify also base space. Let L be an element in $sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$. The map $L_C:(\boldsymbol{F}\oplus\boldsymbol{S})_C\to\boldsymbol{S}_C$ is G_C equivariant and has a splitting because if P is an algebraic G splitting of L then P_C is an algebraic G_C splitting of L_C. Hence L_C is in $sur((\boldsymbol{F}\oplus\boldsymbol{S})_C,\boldsymbol{S}_C)$. Let L' be another element of $sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$. If $L'=L\circ\tau$ for some $\tau\in aut(\boldsymbol{F}\oplus\boldsymbol{S})$, then $L'_C=L_C\circ\tau_C$ and $\tau_C\in aut(\boldsymbol{F}\oplus\boldsymbol{S})_C)$. Therefore the following definition makes sense, i.e. it does not depend on the choice of L.

DEFINITION 3.4. Let $[E]\in VEC(B,F;S)$ and let $L\in sur(\boldsymbol{F}\oplus\boldsymbol{S},\boldsymbol{S})$ represent E, i.e.

$E = ker\ L$. Then we define the *complexification* of $[E]$ by $[ker\ L_C] \in VEC(B_C, F_C; S_C)$.

Let $X (\subset \boldsymbol{R}^n)$ be a real G variety. For $x \in X_C$ $(\subset \boldsymbol{C}^n)$, the complex conjugation $\bar{x}$ is also in X_C since $f(\bar{x}) = 0$ for any $f \in I(X)$. Hence X_C has an involution defined by $x \mapsto \bar{x}$. Similarly, G_C has an involution. Since the action map: $G \times X \to X$ is real algebraic, we have $\overline{g \cdot x} = \bar{g} \cdot \bar{x}$ for any $g \in G_C$ and $x \in X_C$.

DEFINITION 3.5. For $L \in sur((\boldsymbol{F} \oplus \boldsymbol{S})_C, \boldsymbol{S}_C)$, we define $\bar{L}: (\boldsymbol{F} \oplus \boldsymbol{S})_C \to \boldsymbol{S}_C$ by

$$\bar{L}(b, f, s) = \overline{L(\bar{b}, \bar{f}, \bar{s})}.$$

One can check that $\bar{L}$ is in $sur((\boldsymbol{F} \oplus \boldsymbol{S}_C, \boldsymbol{S}_C)$. So the correspondence $L \mapsto \bar{L}$ induces an involution on $VEC(B_C, F_C; S_C)$. Since $\overline{L_C} = L_C$ for $L \in sur(\boldsymbol{F} \oplus \boldsymbol{S}, \boldsymbol{S})$, the complexification in Definition 3.4 induces a map

$$c_b : VEC(B, F; S) \to VEC(B_C, F_C; S_C)^{Z_2}.$$

We ask

Complexification problem (vector bundle case). *Is the above map c_b bijective?*

We turn to the complexification of actions. Let $ACT(G, \boldsymbol{R}^n)$ (resp. $ACT(G_C, \boldsymbol{C}^n)$) be the set of the equivalence classes of real algebraic G actions on $\boldsymbol{R}^n$ (resp. complex algebraic G_C actions on $\boldsymbol{C}^n$), where the equivalence relation is defined by G variety (resp. G_C variety) isomorphism. By the complexification of real G varieties, we obtain a map

$$c_a : ACT(G, \boldsymbol{R}^n) \to ACT(G_C, \boldsymbol{C}^n).$$

Complexification problem (action case). *Is the above map injective?*

We deal with these problems in the next section.

4. Non-linearizable actions and the complexification problems

We first classify the elements in $Vec(W_1, W_m; \boldsymbol{R})$ as $O(2, \boldsymbol{R})$ varieties, i.e. we calculate $VAR(W_1, W_m; \boldsymbol{R})$. We show that the assumption of Theorem 1.6 is satisfied.

Lemma 4.1. *Any $O(2, \boldsymbol{R})$ variety isomophism between elements in $Vec(W_1, W_m; \boldsymbol{R})$ maps W_1 as graph.*

Proof. Let E_1, E_2 be elements in $Vec(W_1, W_m; \boldsymbol{R})$ and $f: E_1 \to E_2$ be an $O(2, \boldsymbol{R})$ variety isomorphism. We show that pfs is in $Aut(W_1)^{O(2, \boldsymbol{R})}$, where $p: E_2 \to W_1$ is the projection and $s: W_1 \to E_1$ is the zero-section. Take the complexification

$f_C:(E_1)_C \to (E_2)_C$, which is an $O(2,C)$ variety isomorphism. According to [11], f_C maps $(W_1)_C$ as graph, in fact, $p_C f_C s_C:(W_1)_C \to (W_1)_C$ is a non-zero scalar multiplication. We recall the proof. The map $f_C s_C$ is $O(2,C)$ equivariant, so it is of the form

$$(W_1)_C \ni (a,b) \mapsto (af_0, bf_0, a^m h_0, b^m h_0, k_0),$$

where f_0, h_0 and k_0 are polynomials of $t=ab$. If f_0 is not a non-zero constant, f_0 has some zero t_0. Let ζ be a primitive m-th root of 1. Then $f_C s_C$ maps $(t_0,1)$ and $(\zeta t_0, \zeta^{-1})$ to the same element $(0,0,a^m h_0(t_0), b^m h_0(t_0), k_0(t_0))$, which contradicts to the injectivity of $f_C s_C$. Hence f_0 must be a non-zero constant. Finally since $p_C f_C s_C$ is the complexification of pfs, it preserves W_1. This proves that $pfs \in Aut(W_1)^{O(2,R)}$.

□

We can check $Aut(W_1)^{O(2,R)} = R^*$ using the $O(2,R)$ equivariance. Suppose that $E(h_1)$ is isomorphic to $E(h_2)$ as $O(2,R)$ varieties. Then $E(h_1)$ is isomorphic to $c^*E(h_2)$ as $O(2,R)$ vector bundles for some $c \in Aut(W_1)^{O(2,R)} = R^*$ by Theorem 1.6 and Lemma 4.1. The fiber of $c^*E(h_2)$ over $(a,\bar{a})$ is the set of points satisfying the equation; $c^m(\bar{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0$. Then

$$\begin{aligned} c^*E(h_2) &= \{(a,\bar{a},x,\bar{x},z)\,;\, c^m(\bar{a}^m x + a^m \bar{x}) + h_2(c^2 t)z = 0\} \\ &\cong \{(a,\bar{a},x,\bar{x},z)\,;\, \bar{a}^m x + a^m \bar{x} + h_2(c^2 t)z = 0\} \end{aligned}$$

by Lemma 2.3 (1). Hence $h_1(t)$ is congruent to $h_2(c^2 t)$ modulo t^m by Lemma 2.3 (3) and we obtain the following bijection.

Theorem 4.2. *$VAR(W_1, W_m; R) \cong R^{m-1}/R^*$, where the R^* action on R^{m-1} is defined as follows. For $c \in R^*$ and $(a_1, \cdots, a_{m-1}) \in R^{m-1}$,*

$$(a_1, \cdots, a_{m-1}) \overset{c}{\mapsto} (c^2 a_1, c^4 a_2, \cdots, c^{2(m-1)} a_{m-1}).$$

Proof of the Theorem (in introduction). By Theorem 1.6, it suffices to show that the set $VAR(W_1, W_m; R)$ can be continuous density, but Theorem 4.2 says that the case $m \geq 3$ satisfies this condition. □

Next we apply the complexification defined in section 3 to the $O(2,R)$ case. We recall Schwarz's [17] and Masuda-Petrie's [11] results in the complex category. Here $O(2,C) = C^* \quad Z_2$ and its action on $(W_m)_C = \{(a,b) \in C^2\}$ is defined as follows. For $g \in C^*$, $1 \neq J \in Z_2$ and $(a,b) \in (W_m)_C$,

$$(a,b) \overset{g}{\mapsto} (g^m a, g^{-m} b) \qquad (a,b) \overset{J}{\mapsto} (b,a).$$

Theorem 4.3 ([11],[17]). *$VEC((W_1)_C, (W_m)_C; C) \cong C^{m-1}$, where the correspon-*

dence is defined similarly to Theorem 2.1.

Theorem 4.4 ([11]). *$VAR((W_1)_C,(W_m)_C;\boldsymbol{C}) \cong \boldsymbol{C}^{m-1}/\boldsymbol{C}^*$, where the $\boldsymbol{C}^*$ action on $\boldsymbol{C}^{m-1}$ is defined similarly to Theorem* 4.2.

We study the involution on $VEC((W_1)_C,(W_m)_C;\boldsymbol{C})$. Any element of $VEC((W_1)_C,(W_m)_C;\boldsymbol{C})$ is represented by $L \in sur((\boldsymbol{W}_m \oplus \boldsymbol{R})_C,\boldsymbol{C})$ of the form;

$$L(a,b,x,y,z) = b^m x + a^m y + f(t)z,$$

where $t = ab$ and f is a polynomial with real coefficients. Then

$$\bar{L}(a,b,x,y,z) = \overline{L(\bar{a},\bar{b},\bar{x},\bar{y},\bar{z})} = b^m x + a^m y + \bar{f}(t)z,$$

where $\bar{f}$ is a polynomial whose coefficients are complex conjugate of those of f. So the involution on $VEC((W_1)_C,(W_m)_C;\boldsymbol{C})$ coincides with the complex conjugate on $\boldsymbol{C}^{m-1}$ through the bijection in Theorem 4.3. This together with Theorem 2.1 shows that the complexification map

$$c_b : VEC(W_1,W_m;\boldsymbol{R}) \to VEC((W_1)_C,(W_m)_C;\boldsymbol{C})^{Z_2}$$

is bijective.

Now we turn to the case of actions. Remember that we have the complexification map

$$c_a : ACT(O(2,\boldsymbol{R}),\boldsymbol{R}^4) \to ACT(O(2,\boldsymbol{C}),\boldsymbol{C}^4).$$

The sets $VAR(W_1,W_m;\boldsymbol{R})$ and $VAR((W_1)_C,(W_m)_C;\boldsymbol{C})$ are subsets of $ACT(O(2,\boldsymbol{R}),\boldsymbol{R}^4)$ and $ACT(O(2,\boldsymbol{C}),\boldsymbol{C}^4)$ respectively and c_a maps $VAR(W_1,W_m;\boldsymbol{R})$ into $VAR((W_1)_C,(W_m)_C;\boldsymbol{C})$. Through the bijections in Theorems 4.2 and 4.4, one can see that the map c_a restricted to $VAR(W_1,W_m;\boldsymbol{R})$ is nothing but the map from $\boldsymbol{R}^{m-1}/\boldsymbol{R}^*$ to $\boldsymbol{C}^{m-1}/\boldsymbol{C}^*$ induced from the natural inclusion $\boldsymbol{R}^{m-1} \subset \boldsymbol{C}^{m-1}$. An elementary observation shows that the map from $\boldsymbol{R}^{m-1}/\boldsymbol{R}^*$ to $\boldsymbol{C}^{m-1}/\boldsymbol{C}^*$ is not injective, in fact, the inverse image of an element in $\boldsymbol{C}^{m-1}/\boldsymbol{C}^*$ consists of one or two elements. This gives a negative answer to the complexification problem in the action case. However $c_a^{-1}([0]) = [0]$, where $[0]$ denotes the element in $\boldsymbol{R}^{m-1}/\boldsymbol{R}^*$ or $\boldsymbol{C}^{m-1}/\boldsymbol{C}^*$ represented by 0. Since $[0]$ corresponds to a linear action, we pose

Weak complexification problem. *If the complexification of a real algebraic action on $\boldsymbol{R}^n$ is linearizable, then is the action itself linearizable?*

Appendix

We give an explicit description of a non-linearizable real algebraic $O(2,\boldsymbol{R})$ action on $\boldsymbol{R}^4$ obtained from Theorem 4.2. For example, we take $E(1-t^2) \in Vec(W_1,W_4;\boldsymbol{R})$.

The following (nonequivariant) algebraic vector bundle automorphism of $W_4 \oplus \boldsymbol{R}$ gives a trivialization of $E(1-t^2) \cong W_4 \subset W_4 \oplus \boldsymbol{R}$.

$$\tau(a,\bar{a}) = \begin{pmatrix} 1+it & 0 & -a^4/2 \\ 0 & 1-it & -a^{-4}/2 \\ a^{-4} & a^4 & 1-t^2 \end{pmatrix}.$$

We define $\sigma: \boldsymbol{R}^4 \to W_4$ by $(a,b,x,y) \mapsto (a+ib, a-ib, x+iy, x-iy)$. Then it suffices to calculate the correspondence of the composition map in the following;

$$\boldsymbol{R}^4 \xrightarrow{\sigma} W_4 \xrightarrow{\tau^{-1}} E(1-t^2) \xrightarrow{action} E(1-t^2) \xrightarrow{\tau} W_4 \xrightarrow{\sigma^{-1}} \boldsymbol{R}^4.$$

It turns out that the actions on $\boldsymbol{R}^4$ of $g = \cos\theta + i\sin\theta \in S^1$ and $1 \neq J \in \boldsymbol{Z}_2$ ($\subset O(2,\boldsymbol{R})$) are as follows.

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \overset{g}{\mapsto} \left(\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} \cos 4\theta & -\sin 4\theta \\ \sin 4\theta & \cos 4\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

$$\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right) \overset{J}{\mapsto} \left(\begin{pmatrix} a \\ -b \end{pmatrix}, \begin{pmatrix} -f_2t+2t^4-2t^2+1 & f_1t+t^5-2t^3+2t \\ -f_1t+t^5-2t^3+2t & -f_2t-2t^4+2t^2-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right),$$

where $t = a^2+b^2$, and f_1, f_2 are polynomials of a, b with the real coefficients such that $(a+ib)^8 = f_1 + if_2$.

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