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CONVERGENCE OF RIEMANNIAN MANIFOLDS AND ALBANESE TORI

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0. Introduction

The purpose of this note is to prove the convergence of the Albanese tori of compact Riemannian manifolds which collapse to a lower dimensional space while keeping their curvatures and diameters bounded.

Given a compact Riemannian manifold M, we denote by $H^1(M)$ the space of harmonic one-forms on M equipped with an inner product $\langle\ ,\ \rangle$ defined by

$$\langle \omega, \eta \rangle = \int_M (\omega, \eta) d\mu_M,$$

where μ_M stands for the normalized Riemannian measure of M with unit mass, $\mu_M = d\mathrm{vol}_M/\mathrm{Vol}(M)$. Let $H^1(M)_Z$ be a lattice of $H^1(M)$ which consists of harmonic one-forms of integral periods. Dividing the dual space $H^1(M)^*$ by the dual lattice $H^1(M)_Z^*$, we obtain a flat torus, called the Albanese torus of M,

$$\mathscr{A}(M) = H^1(M)^*/H^1(M)_Z^*.$$

We may view $\mathscr{A}(M)$ as a map of the set of isometry classes of compact Riemannian manifolds M onto that of flat tori.

Given a positive integer m, a nonnegative number $\varkappa$ and a positive one D, we write $\mathscr{S}(m,\varkappa,D)$ for the set of isometry classes of compact Riemmanian m-manifolds M such that the Ricci curvature of M is bounded from below by $-(m-1)\varkappa^2$ and the diameter of M is bounded from above by D. Then according to Gromov [7], for a Riemannian manifold M in $\mathscr{S}(m, \varkappa, D)$, the dimension of the Albanese torus $\mathscr{A}(M)$, namely, the first Betti number $b_1(M)$, has an upper bound depending only on the dimension m of M and $\varkappa D$. Using this, we shall show the following

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Theorem 1. *Given $m \in Z^+$, $\varkappa \geq 0$ and $D > 0$, there is a constant C depending only on m and $\varkappa D$ such that*

$$\operatorname{diam}(\mathcal{A}(M)) \leq C \operatorname{diam}(M)$$

for any M in $\mathcal{S}(m, \varkappa, D)$.

We would like to ask if the convergence of compact Riemannian manifolds in $\mathcal{S}(m, \varkappa, D)$ with respect to the spectral distance would imply the convergence of their Albanese tori. Here we recall the definition of a *spectral distance* between two compact Riemannian manifolds which was introduced in [9]. Given two compact Riemannian manifolds M and N, a mapping $f: M \to N$ is called an *ε-spectral approximation* if

$$e^{-(1/t+t)} |p_M(t, x, y) - p_N(t, f(x), f(y))| < \varepsilon$$

for all $t > 0$, and for all points x, y of M, where $p_M(t, x, y)$ and $p_N(t, u, v)$ denote respectively the heat kernel of M in $L^2(M, \mu_M)$ and that of N in $L^2(N, \mu_N)$. The spectral distance, $SD(M, N)$, between M and N is by definition the lower bound of the positive numbers ε such that there exist ε-spectral approximations $f: M \to N$ and $h: N \to M$. According to [9], we know that (i) the metric space $(\mathcal{S}(m, \varkappa, D), SD)$ is precompact; (ii) the eigenvalues and eigenfunctions of compact Riemannian manifolds in $\mathcal{S}(m, \varkappa, D)$ are continuous with respect to the spectral distance in a certain sense; (iii) the topology on $\mathcal{S}(m, \varkappa, D)$ induced by the spectral distance is finer than that of measured Hausdorff convergence introduced by Fukaya [3] and hence that of the Gromov-Hausdorff distance. Moreover if we denote by $\mathcal{K}(m, \varkappa, D)$ the set of isometry classes of compact Riemmanian m-manifolds such that the sectional curvatures are bounded by $\varkappa^2$ in the absolute values and the diameters are not greater than D, then the topologies of the spectral distance and the measured Hausdorff convergence coincide on this set $\mathcal{K}(m, \varkappa, D)$. See [3], [8] and [9] for details. We note also that the spectral distance and the Gromov-Hausdorff distance induce the same topology on the set of flat tori.

The following theorem shows that the above question is affirmative if we restrict ourselves to the class $\mathcal{K}(m, \varkappa, D)$ for given numbers m, $\varkappa$, and D.

Theorem 2. *Let $\{M_i\}$ be a sequence in $\mathcal{K}(m, \varkappa, D)$ which converges with respect to the spectral distance. Then the Albanese torus $\mathcal{A}(M_i)$ converges to a flat torus $\mathcal{A}_\infty$ of dimension n with $0 \leq n \leq \liminf_{i \to \infty} b_1(M_i)$.*

Here in our convention, $\mathcal{A}_\infty$ stands for a point when $n = 0$. We remark also that under the assumption of this theorem, we are able to show the convergence of the Albaness maps in a certain sense. See Section 3 for details.

The proofs of Theorems 1 and 2 are respectively given in Sections 2 and 3. For the latter, we shall basically make use of some results in [8]. In this sense, the present paper is a continuation of [8].

1. Albanese Tori

In this section, we shall consider a compact Riemannian manifold M endowed with a certain measure μ and define the Albanese torus and the Albanese map of such a pair (M, μ) (cf. [10], [11]).

Let $M=(M, g_M)$ be a compact Riemannian manifold of dimension m and μ a measure on M with smooth density function $\chi>0$. A one-form ω on M is said to be *μ-harmonic* if ω is closed and co-closed with respect to μ, namely, $d\omega=0$ and

$$\delta_\mu\omega := -\mathrm{trace}(\nabla\omega)-\omega(\nabla\log\chi)=0.$$

In other words, μ-harmonic one-form ω can be expressed locally as the differential of an L_μ-harmonic function f, $\omega=df$. Here a smooth function f defined on an open set in M is called an L_μ-harmonic function if

$$L_\mu f := \frac{1}{\chi}\mathrm{div}(\chi\nabla f)=\Delta f+\nabla\log\chi\cdot f=0.$$

We denote by $H^1(M, \mu)$ the space of μ-harmonic one-forms on M and by $H^1(M, \mu)_Z$ the lattice of $H^1(M, \mu)$ which consists of one-forms with integral periods. The vector space $H^1(M, \mu)$ is endowed with an L^2 inner product $\langle\ ,\ \rangle$ defined by

$$\langle\omega, \eta\rangle_\mu=\int_M(\omega, \eta)d\mu.$$

The norm of $\omega\in H^1(M, \mu)$ is denoted by $\|\omega\|_\mu$.

Similarly, given a compact Riemannian manifold N, we say a smooth mapping ψ of M into N is *μ-harmonic* if it satisfies

$$\tau(\psi)+d\psi(\nabla\log\chi)=0,$$

where $\tau(\psi)$ stands for the tension field of the mapping ψ. A μ-harmonic mapping ψ is a stationary point of the energy functional

$$E_\mu(\psi)=\int_M e(\psi)d\mu.$$

When N is a circle of length 1, $N=R/Z$, we write $\mathcal{H}(M, \mu; R/Z)$ for the set of μ-harmonic mappings of M into R/Z, which forms an additive group in an obvious manner. Since the derivative $d\phi$ of a smooth mapping ϕ of M into R/Z may be considered as an integral one-form on L, we have a natural surjective homomorphism d of $\mathcal{H}(M, \mu; R/Z)$ onto $H^1(M, \mu)_Z$ whose kernel is the set of constant mappings $\theta\in R/Z$. We note that the homomorphism d preserves the norms in the sense that

$$E_\mu(\psi)=\langle d\psi, d\psi\rangle_\mu$$

for $\psi \in \mathcal{H}(M, \mu; R/Z)$.

The Albanese torus $\mathcal{A}(M, \mu)$ of a pair (M, μ) is by definition a flat torus derived from dividing the dual space $H^1(M, \mu)^*$ by the dual lattice $H^1(M, \mu)^*_Z$,

$$\mathcal{A}(M, \mu)=H^1(M, \mu)^*/H^1(M, \mu)^*_Z.$$

Let $\widetilde{M}$ be the universal covering of M and π the projection of $\widetilde{M}$ onto M. If we fix a point p of M and take a point $\tilde{p}$ of $\widetilde{M}$ with $\pi(\tilde{p})=p$, then we have a map $\tilde{J}_{M,\mu}$ of $\widetilde{M}$ into the dual space $H^1(M, \mu)^*$ defined by

$$\tilde{J}_{M,\mu}(\tilde{x})(\omega)=\int_{\tilde{p}}^{\tilde{x}} \pi^* \omega.$$

This map induces a μ-harmonic map $J_{M,\mu}$ of M into $\mathcal{A}(M, \mu)$ (with $J_{M,\mu}(p)=0$). We call $J_{M,\mu}$ the Albanese map of a pair (M, μ).

Let $\Omega=\{\omega_1, \ldots, \omega_r\}$ be a basis of $H^1(M, \mu)_Z$ and $\Omega^*=\{\omega_1^*, \ldots, \omega_r^*\}$ the dual basis. Then a diffeomorphism T_Ω of $\mathcal{A}(M, \mu)$ onto R^r/Z^r is derived from a linear isomorphism of $H^1(M, \mu)^*$ onto R^r:

$$\theta_1\omega_1^*+\cdots+\theta_r\omega_r^* \to (\theta_1, \ldots, \theta_r).$$

If we set a metric g_Ω on R^r/Z^r by

$$g_\Omega=\sum_{\alpha,\beta=1,\cdots,r} \langle \omega_\alpha^*, \omega_\beta^* \rangle_\mu d\theta_\alpha d\theta_\beta,$$

then T_Ω induces an isometry between $\mathcal{A}(M, \mu)$ and $(R^r/Z^r, g_\Omega)$. Moreover if we take a μ-harmonic map $\psi_\alpha : M \to R/Z$ in such a way that

$$\psi_\alpha(p)=0, \quad \omega_\alpha=d\psi_\alpha,$$

we see that

$$T_\Omega \circ J_{M,\mu}(x)=(\psi_1(x), \ldots, \psi_r(x))$$

for $x \in M$. Here we remark that

$$E_\mu(J_{M,\mu})=b_1(M).$$

Given a pair (M, μ), we define a symmetric tensor $Ric_{M,\mu}$ by

$$Ric_{M,\mu}=Ric_M-\frac{1}{\chi}\nabla^2\chi(=Ric_M-d\log\chi\otimes d\log\chi-\nabla^2\log\chi),$$

where Ric_M (resp., χ) stands for the Ricci tensor of M (resp., the density function of μ, $\mu=\chi d\mathrm{vol}_M$). Given m, κ and D as before, we denote by $\mathcal{S}_w^*(m, \kappa, D)$ the set of equivalence classes of pairs (M, μ) such that dim $M=m$, the diameter diam(M) of $M \leq D$, μ has unit mass, and $Ric_{M,\mu} \geq -(m-1)\kappa^2 g_M$. Here we say two pairs (M, μ) and (N, ν) are equivalent when there is an isometry $f: M \to N$ which preserves the measures, $f_*\mu=\nu$. We remark that the spectral distance SD can be defined on the set of equivalence classes of pairs (M,μ). See [9] for some properties of the metric space $(\mathcal{S}_w^*(m, \varkappa, D), SD)$ as mentioned in Introduction.

In what follows, when μ is the canonical Riemannian measure μ_M with unit mass, $\mu_M = d\mathrm{vol}/\mathrm{Vol}(M)$, we omit to indicate the dependency of the measure μ_M in some of the above notations (for example, $\mathcal{A}(M)$ stands for $\mathcal{A}(M, \mu_M)$).

2. Proof of Theorem 1

We recall first that there is a positive constant C' depending only on m and $\varkappa D$ such that

$$e(\psi) \leq C' E_{\mu_M}(\psi) \tag{2.1}$$

for all $\psi \in \mathcal{H}(M; R/Z)$, because the energy density $e(\psi)$ satisfies

$$\Delta e(\psi) \geq -2(m-1)\varkappa^2 e(\psi)$$

(cf. e.g., [8-a, §4]). In addition, we note that if ψ is not constant, the energy density $e(\psi)$ must be greater than or equal to $1/4\ \mathrm{diam}(M)^2$ somewhere on M (otherwise, the distance between $\psi(p)$ and $\psi(q)$ in R/Z for any pair of points p, q of M would be less than $1/2$, and hence the harmonic map ψ should be constant). Therefore we have

$$E_{\mu_M}(\psi) \geq \frac{1}{4C' \mathrm{diam}(M)^2}$$

for all nonconstant $\psi \in \mathcal{H}(M; R/Z)$; in other words,

$$\langle \omega, \omega \rangle_{\mu_M} \geq \frac{1}{4C' \mathrm{diam}(M)^2}$$

for all nonzero $\omega \in H^1(M)_Z$. This implies that the first eigenvalue $\lambda_1(\mathcal{A}(M))$ is bounded from below by $\pi^2/C'\mathrm{diam}(M)^2$. On the other hand, we know that

$$\lambda_1(\mathcal{A}(M)) \leq \frac{C''}{\mathrm{diam}(\mathcal{A}(M))^2}$$

for some constant C'' depending only on the dimension of $\mathcal{A}(M)$, and hence on m and $\varkappa D$, since

$$b_1(M) \leq \frac{v_m(\varkappa^2, 5\mathrm{diam}(M))}{v_m(\varkappa^2, \mathrm{diam}(M))}, \tag{2.2}$$

where $v_m(\varkappa^2, r)$ stands for the volume of a metric ball in the simply connected space form of dimension m with constant curvature $\varkappa^2$ (cf. [7]). Thus the assertion of Theorem 1 is clear. This completes the proof of Theorem 1.

Let (M, μ) be a pair in $\mathcal{S}_w^*(m, \varkappa, D)$. Then it is not hard to see that the above submean value inequality (2.1) holds for any μ-harmonic map $\psi \in \mathcal{H}(M, \mu; R/Z)$ (cf. e.g., [8-a, §4]). Moreover it follows from the same reason as in deriving (2.2) that

$$b_1(M) \leq \frac{v_{m+1}(\varkappa^2,\ 5\mathrm{diam}(M))}{v_{m+1}(\varkappa^2,\ \mathrm{diam}(M))},$$

because we have a Bishop-Gromov type inequality for the pair (M, μ)(cf. [9, §2]). Thus Theorem 1 holds for (M, μ). Namely we have

Theorem 1′. *Given m, $\varkappa$ and D as before, there is a constant C' depending only on m and $\varkappa D$ such that*

$$\mathrm{diam}(\mathcal{A}(M,\ \mu)) \leq C'\mathrm{diam}(M).$$

for all $(M, \mu) \in \mathcal{S}_w^*(m, \varkappa, D)$.

REMARKS. (1) When $\varkappa=0$ in Theorem 1, the classical Bochner theorem says that any harmonic one-form is parallel, so that J_M is a Riemannian submersion with totally geodesic fibers and in particular the diameter of $\mathcal{A}(M)$ is less than or equal to the diameter of M ($C=1$ in Theorem 1). This is also true for Theorem 1' (cf. [12]). (2) A slightly different proof of Theorem 1 is presented in [6].

3. Proof of Theorem 2

The proof of the theorem is divided into 4 steps and the same notations as in the preceding sections will be used.

Step 1. We shall start with recalling the notions of convergence of Gromov-Hausdorff distance and measured Hausdorff topology introduced by Gromov [7] and Fukaya [3] respectively. Given a sequence of compact Riemannian manifolds, $\{M_i\}$, we say that M_i converges to a compact matric space X with respect to the Gromov-Hausdorff distance, if there are a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty} \varepsilon_i=0$ and mappings $f_i: M_i \to X$ such that the ε-neighborhood of $f_i(M_i)$ covers X and $|d_{M_i}(x, y)-d_X(f_i(x), f_i(y))|<\varepsilon$ for all x, y of M_i. Moreover we say that $M_i=(M_i, \mu_{M_i})$ converges to a pair (X, μ) of X and a Borel measure μ on X with respect to the measured Hausdorff topology, if f_i are Borel measurable and the push-forward $f_{i*}\mu_{M_i}$ of the normalized Riemannian measure μ_{M_i} via f_i converges to μ in the weak* topology.

Let M_i be a sequence in $\mathcal{K}(m, \varkappa, D)$ which converges to a compact metric space M_∞ with respect to the Gromov-Haussdorff distance. Then there is a smooth manifold F_∞ with metric of class $C^{1,\alpha}$ (for any $\alpha \in (0, 1)$), on which the orthogonal group $O(m)$ acts by isometries in such a way that the quotient space $F_\infty/O(m)$ is isometric to M_∞. In fact, F_∞ is a limit of the set of the frame bundles FM_i of M_i equipped with a canonical metric so that the action of $O(m)$ is isometric, the projection of FM_i onto M_i is a Riemannian submersion with totally geodesic fibers, and the sectional curvature of FM_i remains bounded uniformly in i. When M_∞ is smooth, there is a fibration $\Phi_i: M_i \to M_\infty$ (for large i) and a sequence of positive numbers $\{\varepsilon_i\}$ with $\lim_{i\to\infty}\varepsilon_i=0$ satisfying

(1) for all $z\in M_\infty$, the diameter of $\Phi_i^{-1}(z)\le\varepsilon_i$;

(2) Φ_i is an ε_i-almost Riemannian submersion, that is, for all $z\in M_\infty$, $x\in\Phi_i^{-1}(z)$ and $X\in T_xM_i$ normal to the fiber $\Phi_i^{-1}(z)$,

$$(1-\varepsilon_i)|d\Phi_i(X)|\le|X|\le(1+\varepsilon_i)|d\phi_i(X)|;$$

(3) the second fundamental form of the submersion Φ_i is bounded uniformly in i.

For these assertions, see [2], [4] and [5].

Now as in Theorem 2, we suppose that M_i converges to (M_∞, μ_∞) with respect to the spectral distance and hence the measured Hausdorff topology. Then we may assume that the push-forward $\Phi_{i*}\mu_{M_i}$ of the canonical Riemannian measure μ_{M_i} of M_i converges to μ_∞ in the weak* topology. In case M_∞ is smooth, the density function χ_∞ of μ_∞ is a positive function of class $C^{1,\alpha}$. Moreover we may assume that the above submersion Φ_i has the following property: for all smooth function h on M_∞,

$$|\Delta_{M_i}\Phi_i^*(h)-\Phi_i^*(L_{\mu_\infty}h)|\le\varepsilon_i\Phi_i^*(|Ddh|+|dh|). \tag{4}$$

See [8-a] for this and further properties of Φ_i.

In the following Steps 2 and 3, we consider the case that the limit metric space M_∞ is a smooth manifold, and assume that the metric of M_∞ and the density χ_∞ are smooth, to avoid some technical argument of approximation. Moreover $\{\varepsilon_i\}$ stands for a sequence of positive constants which tends to zero as i goes to infinity.

Step 2. Given a μ_∞-harmonic one-form $\omega\in H^1(M_\infty, \mu_\infty)$, the pull-back $\Phi_i^*\omega$ can be uniquely expressed as

$$\Phi_i^*\omega=\Gamma_i(\omega)+d\Lambda_i(\omega)$$

according to the orthogonal decomposition of d-closed one-forms $Z(M_i)$ of M_i,

$$Z^1(M_i)=H^1(M_i, \mu_{M_i})\oplus dC^\infty(M_i).$$

Here the function $\Lambda_i(\omega)$ is chosen in such a way that

$$\int_{M_i}\Lambda_i(\omega)d\mu_i=0.$$

Now we claim that

$$|\Phi_i^*\omega-\Gamma_i(\omega)|_{C^0(M_i)}=|d\Lambda_i(\omega)|_{C^0(M_i)}\le\varepsilon_i \tag{3.1}$$

for any $\omega\in H^1(M_\infty, \mu_\infty)$ with unit norm, $\|\omega\|_{\mu_\infty}=1$. Indeed, we fix a sufficiently small a and consider the metric ball $B_\infty(p, 3a)$ of M_∞ around a point p of radius $3a$. Let f be an L_∞-harmonic function on $B_\infty(p, 3a)$ such that $\omega=df$ and

$$\int_{B_\infty(p,3a)}fd\mu_\infty=0.$$

Then applying the Poincaré inequality, we have first

$$\int_{B_\infty(p,3a)}|f|^2 d\mu_\infty \leq C_1 \int_{B_\infty(p,3a)}|df|^2 d\mu_\infty \leq C_1$$

for some constant C_1. Since f is L_∞-harmonic, it follows from the standard elliptic regularity estimates that

$$|f|_{C^{2,\alpha}(B_\infty(p,2a))} \leq C_2$$

for some constant C_2, where $\alpha \in (0, 1)$. Hence in view of the property (4) of Φ_i, we see that

$$|\Delta_{M_i} \Phi_i^* f| \leq \varepsilon_i$$

on $\Phi_i^{-1}(B_\infty(p, a))$. This shows that

$$|\Delta_{M_i} \Lambda_i(\omega)| \leq \varepsilon_i,$$

since $d\Phi_i^* f = \Gamma_i(\omega) + d\Lambda_i(\omega)$ and $\Gamma_i(\omega)$ is harmonic. Finally it follows from the regularity estimates again that

$$|\Lambda_i(\omega)|_{W^{2,p}(M_i,\ \mu_{M_i})} \leq \varepsilon_i,$$

where $p \in (1, \infty)$ (cf. [8-a, Lemma 1.3]), and hence

$$|\Lambda_i(\omega)|_{C^{1,\alpha}(M_i)} \leq \varepsilon_i.$$

This proves (3.1).

Now this estimate (3.1) together with the property (3) of Φ_i implies that

$$(1-\varepsilon_i)\|\omega\|_{\mu_\infty} \leq \|\Gamma_i(\omega)\|_{\mu_i} \leq (1+\varepsilon_i)\|\omega\|_{\mu_\infty} \tag{3.2}$$

For all $\omega \in H^1(M_\infty, \mu_\infty)$. In particular, Γ_i is injective (for large i). We observe further that Γ_i maps the lattice $H^1(M_\infty, \mu_\infty)_Z$ into the lattice $H^1(M_i)_Z$,

$$\Gamma_i(H^1(M_\infty, \mu_\infty)_Z) \subset H^1(M_i)_Z.$$

Step 3. Given any number K, Theorem 4.3 in [8-a] says that for large i , a harmonic one-form ξ on M_i with integral periods must belong to the image $\Gamma_i(H^1(M_\infty, \mu_\infty)_Z)$, whenever the L^2 norm $\|\xi\|_{\mu_i}$ is less than K. In other words, there is a positive constant K_i with $\lim_{i\to\infty} K_i = \infty$ such that

$$\|\xi\|_{\mu_i} \geq K_i$$

for any $\xi \in H^1(M_i)_Z \backslash \Gamma_i(H^1(M_\infty, \mu_\infty)_Z)$ (if it exists).

Let us now take a basis $\Omega = \{\omega_1, \ldots, \omega_r\}$ of $H^1(M_\infty, \mu_\infty)_Z (r = b_1(M_\infty))$ in such a way that an element ω of $H^1(M_\infty, \mu_\infty)_Z$ is a linear combination of $\omega_1, \ldots, \omega_{s-1}$ whenever $\|\omega\|_{\mu_\infty}$ is less than $\|\omega_s\|_{\mu_\infty}$ (cf. [1, Chap. VIII]). Then we choose a basis $\Omega_i = \{\omega_{i,1}, \ldots, \omega_{i,r_i}\}$ of $H^1(M_i)_Z (r_i = b_1(M_i))$ in such a way that

$$\omega_{i,s}=\Gamma_i(\omega_s) \quad (s=1, \ldots, r)$$

and any element ω is linearly dependent of $\omega_{i,1}, \ldots, \omega_{i,s-1}$ whenever $\|\omega\|_{\mu_i}$ is less than $\|\omega_{i,s}\|_{\mu_i}$ for $s>r$. We note that

$$\|\omega_{i,s}\| \geq K_i \tag{3.3}$$

for $s>r$ (if $r_i>r$).

Let $\Omega_i^*=\{\omega_{i,s}^*\}$ $(s=1, \ldots, r_i)$ be the dual basis of Ω_i and Γ_i^*: $H^1(M_i)^* \to H^1(M_\infty, \mu_\infty)^*$ the dual mapping of Γ_i. Then Γ_i^* is surjective and its kernel is spanned by $\omega_{i,s}(s=r+1, \ldots, r_i)$. Hence Γ_i^* induces a surjective homomorphism, denoted by $[\Gamma_i^*]$, from the Albanese torus $\mathscr{A}(M_i)$ of M_i onto $\mathscr{A}(M_\infty, \mu_\infty)$. Then in view of (3.2) and (3.3), $\mathscr{A}(M_i)$ converges via $[\Gamma_i^*]$ to $\mathscr{A}(M_\infty, \mu_\infty)$ with respect to the Gromov-Hausdorff distance. We observe that $[\Gamma_i^*]$ is affine, namely the second fundamental form vanishes identically. Moreover if we take a point p_∞ of M_∞ and choose p_i as a fixed point of M_i in such a way that $\Phi_i(p_i) = p_\infty$, then the mappings J_{M_∞,μ_∞} and $[\Gamma_i^*]\circ J_{M_i}$ are close for large i in the sense that

$$\max_{x\in M_i} \operatorname{dis}(J_{\mu_\infty}\circ \Phi_i(x), [\Gamma_i^*]\circ J_i(x)) \leq \varepsilon_i.$$

To be precise, let T_{Ω_i}: $\mathscr{A}(M_i) \to (R^{r_i}/Z^{r_i}, g_{\Omega_i})$ and T_Ω: $\mathscr{A}(M_\infty, \mu_\infty) \to (R^r/Z^r, g_\Omega)$ respectively be isometries described in Section 1, and let π_i: $R^{r_i}/Z^{r_i} \to R^r/Z^r$ be a canonical projection such that $\pi_i(\theta_1, \ldots, \theta_{r_i})=(\theta_1, \ldots, \theta_r)$. Then $T_{\Omega_i}\circ J_{M_i}$ and $T_\Omega \circ J_{M_\infty,\mu_\infty}$ respectively can be expressed as

$$T_{\Omega_i}\circ J_{M_i}=(\psi_{i,1},\ldots,\psi_{i,r_i})$$

and

$$T_\Omega \circ J_{M_\infty,\mu_\infty}=(\psi_1, \ldots, \psi_r),$$

where $\psi_{i,s}$ is the harmonic mapping of M_i to R/Z corresponding to $\omega_{i,s}$ and also ψ_s is the μ_∞-harmonic mapping of M_∞ to R/Z corresponding to ω_s. We note that for each s, $1 \leq s \leq r$, $\psi_{i,s}$ is homotopic to $\psi_i \circ \Phi_i$ and further that

$$|\psi_{i,s}-\psi_s\circ \Phi_i|_{C^{2,\alpha}(M_i)} \leq \varepsilon_i$$

(cf. [8-a, §4]). Thus the mappings $J_{M_\infty,\mu_\infty}\circ \Phi_i$ and $[\Gamma_i^*]\circ J_{M_i}$ (for large i) are close with respect to the $C^{2,\alpha}$ topology.

Step 4. It remains to prove Theorem 2 in case M_∞ is not smooth. In this case, we consider the frame bundle FM_i of each M_i equipped with a canonical metric $\bar{g}_i$ in such a way that the sectional curvature and the diameter are bounded uniformly in i. We denote by ρ_i the canonical projection of FM_i onto M_i. Observe that the pull-back $\rho_i^*\omega$ of a harmonic one-form ω on M_i is harmonic on FM_i and further that this correspondence preserves the inner products,

$$\langle \rho_i^*\omega, \rho_i^*\omega' \rangle_{\bar{\mu}_i} = \langle \omega, \omega' \rangle_{\mu_i}.$$

For this reason, the space of harmonic one-forms $H^1(M_i)$ on M_i endowed with the L^2 inner product can be considered as a subspace of $H^1(FM_i)$. In the same way, we identify the lattice $H^1(M_i)_Z$ and the group $\mathcal{H}(M_i\,;\,R/Z)$ respectively with a sublattice of $H^1(FM_i)_Z$ and a subgroup of $\mathcal{H}(FM_i\,;\,R/Z)$. Under this identification, an element $\bar{\psi}$ of $\mathcal{H}(FM_i\,;\,R/Z)$ belongs to the subgroup $\mathcal{H}(M_i\,;\,R/Z)$ if and only if $\bar{\psi}$ is $O(m)$-invariant.

In what follows, we suppose that this sequence $\{FM_i\}$ converges in the topology of measured Hausdorff convergence. Let $\bar{M}_\infty$ and $\bar{\mu}_\infty$ be the limit space and measure respectively. Then according to Fukaya [4, 5], $\bar{M}_\infty$ is a smooth manifold with Riemannian metric $\bar{g}_\infty$ of class $C^{1,\alpha}$, on which the orthogonal group $O(m)$ acts as isometries in such a way that the quotient space $\bar{M}_\infty/O(m)$ is isometric to M_∞. Moreover there are $O(m)$-equivariant almost Riemannian submersions $\tilde{\Phi}_i$: $FM_i \to \bar{M}_\infty$ such that $\tilde{\Phi}_{i*}\bar{\mu}_{FM_i}$ converges to $\bar{\mu}_\infty$ in the weak* topology, where $\bar{\mu}_{FM_i}$ stands as before for the normalized Riemannian measure of FM_i. We note that the limit measure μ_∞ on M_∞ coincides with the push-forward $\rho_{\infty*}\bar{\mu}_\infty$ of $\bar{\mu}_\infty$ via the projection ρ_∞: $\bar{M}_\infty \to M_\infty$ and the density $\bar{\chi}_\infty$ of $\bar{\mu}_\infty$ with respect to the Riemannian measure of $\bar{g}_\infty$ is $O(m)$-invariant.

Now perturbing the submersion $\tilde{\Phi}_i$ in the $C^{1,\alpha}$ topology, we can obtain an almost Riemannian submersion $\bar{\Phi}_i$ of FM_i onto $\bar{M}_\infty$, to which we can apply the same arguments as in the preceding steps. To be precise, we write first $\mathcal{H}(M_\infty,\,\mu_\infty\,;\,R/Z)$ for the subgroup of $\mathcal{H}(\bar{M}_\infty,\,\bar{\mu}_\infty\,;\,R/Z)$ consisting of $O(m)$-invariant $\bar{\mu}_\infty$-harmonic maps $\bar{\psi}$: $\bar{M}_\infty \to R/Z$. We note that $\mathcal{H}(M_\infty,\,\mu_\infty\,;\,R/Z)$ *is determined by the pair* $(M_\infty,\,\mu_\infty)$ *itself* (cf. [8-a, §4]). Then we denote by $H^1(M_\infty,\,\mu_\infty)_Z$ and $H^1(M_\infty,\,\mu_\infty)$ respectively the sublattice of $H^1(\bar{M}_\infty,\,\bar{\mu}_\infty)_Z$ corresponding to $\mathcal{H}(M_\infty,\,\mu_\infty\,;\,R/Z)$, $H^1(\bar{M}_\infty,\,\bar{\mu}_\infty)_Z = d\mathcal{H}(M_\infty,\,\mu_\infty\,;\,R/Z)$, and the vector space spanned by $H^1(M_\infty,\,\mu_\infty)_Z$. Set

$$\mathcal{A}(M_\infty,\,\mu_\infty) = H^1(M_\infty,\,\mu_\infty)^*/H^1(M_\infty,\,\mu_\infty)^*_Z.$$

Then we obtain an $O(m)$-invariant $\bar{\mu}_\infty$-harmonic map $\bar{J}_{M_\infty,\mu_\infty}$: $\bar{M}_\infty \to \mathcal{A}(M_\infty,\,\mu_\infty)$, from which a Lipschitz map J_{M_∞,μ_∞}: $M_\infty \to \mathcal{A}(M_\infty,\,\mu_\infty)$ is derived. This map J_{M_∞,μ_∞} is μ_∞-harmonic on the set of regular points of M_∞. Moreover as we have seen in Steps 2 and 3, $\bar{\Phi}_i$ (for large i) induces a surjective homomorphism $[\bar{\Gamma}_i^*]$: $\mathcal{A}(FM_i) \to \mathcal{A}(F_\infty,\,\bar{\mu}_\infty)$ such that $\bar{J}_{M_\infty,\mu_\infty}\circ\bar{\Phi}_i$ and $[\bar{\Gamma}_i^*]\circ J_{FM_i}$ are close in the $C^{2,\alpha}$ topology. Finally we obtain a surjective homomorphism $[\Gamma_i^*]$: $\mathcal{A}(M_i) \to \mathcal{A}(M_\infty,\,\mu_\infty)$ from $[\bar{\Gamma}_i^*]$ such that $J_{M_\infty,\mu_\infty}\circ\Phi_i$ and $[\Gamma_i^*]\circ J_{M_i}$ are close in the C^0 topology for large i, where Φ_i: $M_i \to M_\infty$ is a Lipschitz map derived from the $O(m)$-equivariant submersion $\tilde{\Phi}_i$. As i goes to infinity, the Albanese torus $\mathcal{A}(M_i)$ converges to the torus $\mathcal{A}(M_\infty,\,\mu_\infty)$ via the surjective homomorphism $[\Gamma_i^*]$. This completes the proof of Theorem 2.

It is possible to apply the same arguments as above to a sequence of certain pairs $(M_i,\,\mu_i)$ (cf. [8-a, Remark 3.3]). In fact, we can show the following

Theorem 2. *Let $\{(M_i, \mu_i)\}$ be a sequence in $\mathcal{S}^*_w(m, x, D)$ which converges to (M_∞, μ_∞) with respect to the measured Hausdorff topology. Suppose that the sectional curvature of M_i is bounded uniformly and also the density function χ_i of μ_i satisfies*

$$|Dd\chi_i| \leq C$$

for some constant C. Then for large i, there is a surjective homomorphism Θ_i of the Albanese torus $\mathcal{A}(M_i, \mu_i)$ onto a flat torus $\mathcal{A}(M_\infty, \mu_\infty)$ of dimension n such that $0 \leq n \leq \liminf_{i\to\infty} b_1(M_i)$ and $\mathcal{A}(M_i, \mu_i)$ converges to $\mathcal{A}(M_\infty, \mu_\infty)$ with respect to the Gromov-Hausdorff distance via Θ_i.

Moreover there are a Lipschits map Φ_i of M_i onto M_∞ through which (M_i, μ_i) converges to (M_∞, μ_∞) and a (μ_∞-harmonic) map $J_{M_\infty,\mu_\infty}: M_\infty \to \mathcal{A}(M_\infty, \mu_\infty)$ such that $J_{M_\infty,\mu_\infty} \circ \Phi_i$ and $\Phi_i \circ J_{M_i,\mu_i}$ are close in the C^0 topology, namely,

$$\lim_{i\to\infty} \max_{x\in M_i} \operatorname{dis}(J_{M_\infty,\mu_\infty} \circ \Phi_i(x),\ \Theta_i \circ J_{M_i,\mu_i}(x)) = 0.$$

The convergence holds in the $C^{2,\alpha}$ topology when M_∞ is a smooth manifold.

As an immediate consequence of Theorem 2', we have the following

Corollary 3. *Given numbers m, x and D, and given a flat torus T of dimension n , there is a constant such that the rank of the Albanese map J_M of a manifold M in $\mathcal{K}(m, x, D)$ is greater than or equal to n, if the Gromov-Hausdorff distance between M and T is less than r, and in addition, J_M is a submersion if $b_1(M) = n$.*

Finally we refer the reader to [12] for some results and problems related to this corollary.

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