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FIBERED 2-KNOTS AND LENS SPACES

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1. Introduction

In 1965, E.C. Zeeman [15] introduced the twisting spun knots. His main theorem states that any twisting spun knot is fibered. In particular, the 2-twist-spun knot of the 2-bridge knot $K(p, q)$ is a fibered 2-knot in S^4 with fiber a punctured lens space $L(p, q)^0$. Throughout the paper, we denote by M a connected sum of lens spaces $\mathop{\#}\limits_{i=1}^{r} L(p_i, q_i)$ $(r \geqq 1)$, and assume that p_i is odd and $p_i > q_i > 0$ (cf. [10], [13]). The punctured manifold M^0 means the space obtained from M by removing an open 3-ball. We consider fibered 2-knots in S^4 with fiber M^0, which we call *M^0-fibered 2-knots*. For example, a connected sum of 2-twist-spun knots of 2-bridge knots is such a fibered 2-knot. We shall determine possible M^0-fibered 2-knots in S^4 (or more generally, a homology 4-sphere Σ) for all M. There are two reasons for selecting these fibered 2-knots:

(1) Any 2-knot with Seifert manifold a connected sum of lens spaces is determined by its exterior [3], [4]. In particular, any fibered 2-knot with fiber a connected sum of lens spaces is determined by its monodromy (precisely, the class of its monodromy in the diffeotopy group of the fiber).

(2) The diffeotopy groups of all lens spaces are computed by Bonahon [1] and Hodgson-Rubinstein [7].

In section 3, we prove that any fibered 2-knot with fiber a punctured lens space $L(p, q)^0$ is the 2-twist-spun knot of $K(p, q)$. In general, we show in section 4 that any M^0-fibered 2-knot with cyclic monodromy is a cable knot about the 2-twist-spun knot of a 2-bridge knot. In section 5, we consider branched covering spaces of cable knots, and observe, for each odd $r > 1$, the existence of a 2-knot which is the same fixed point set of many inequivalent semi-free $\boldsymbol{Z}_r$-actions on S^4 (cf. [2], [5], [14]).

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2. Preliminaries

We work in the smooth category. By using the unique decomposition theorem [12], [6], we can assume that for a diffeomorphism $h: M^0 \to M^0$, h is the identity on ∂M^0 and h permutes lens space factors of M^0. Thus we can consider that h is corresponding to a permutation with respect to lens space factors. Of course, h cannot permute lens space factors which are not diffeomorphic. Hence, it is sufficient to consider the case that M is homogeneous, that is $M = \mathop{\#}\limits_{i=1}^{r} L(p, q)$, (For this case, we write $M = \mathop{\#}\limits_{i=1}^{r} L_i$.) and h corresponds to a cyclic permutation of order r, that is,

$$h(L_i^0) = L_{\sigma(i)}^0 \qquad \text{for} \quad i = 1, 2, \cdots, r,$$

where $\sigma: \{1, 2, \cdots, r\} \to \{1, 2, \cdots, r\}$ is a cyclic permutation of order r. Then we call h *cyclic*. In fact, any two such permutations are conjugate in the symmetric group of degree r. Hence, we may assume that

$$h(L_i^0) = L_{i+1}^0 \qquad \text{for} \quad i = 1, 2, \cdots, r,$$

where $L_{r+1} = L_1$.

The diffeotopy groups of lens spaces are known.

Theorem (Hodgson-Rubinstein [7], Bonahon [1]). *The diffeotopy group of lens space $L(p, q)$ is given by*

$$\pi_0 \operatorname{Diff}(L(p, q)) \cong \begin{cases} \boldsymbol{Z}_2 & \text{if } q^2 \not\equiv \pm 1 \text{ or } q \equiv \pm 1 \pmod{p} \\ \boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2 & \text{if } q^2 \equiv 1 \text{ and } q \not\equiv \pm 1 \pmod{p} \\ \boldsymbol{Z}_4 & \text{if } q^2 \equiv -1 \pmod{p} \end{cases}$$

As the representatives we can take:

$$\begin{array}{lll} \{I, A\} & \text{if } q^2 \not\equiv \pm 1 \text{ or } q \equiv \pm 1 & \pmod{p} \\ \{I, A, C, CA\} & \text{if } q^2 \equiv 1 \text{ and } q \not\equiv \pm 1 & \pmod{p} \\ \{I, D, D^2, D^3\} & \text{if } q^2 \equiv -1 & \pmod{p} \end{array}$$

where I is the identity, and A, C and D are defined in [7]. *Here A and C are involutions, and D has period 4. Note that $D^2 = A$. These induced automorphisms of $H_1(L(p, q)) \cong \boldsymbol{Z}_p$ are given by*:

$$\begin{aligned} A_*&: \text{ multiplication by } -1, \\ C_*&: \text{ multiplication by } -q, \\ D_*&: \text{ multiplication by } -q. \end{aligned}$$

REMARK. A classical knot whose 2-fold branched covering space is $L(p, q)$ is the 2-bridge knot $K(p, q)$. Then A is the canonical covering transformation (see [7], [13]).

3. The special case

In this section, we prove the following.

Lemma 1. *Let K be an M^0-fibered 2-knot in a homology 4-sphere Σ with $M=L(p, q)$. Then $\Sigma\cong S^4$ and K is the 2-twist-spun knot of the 2-bridge knot $K(p, q)$.*

Proof. Let h be the monodromy of the fibered 2-knot $K\subset\Sigma$. For the induced automorphism $t=h_*$ of

$$H_1(M^0)\cong H_1(M)\cong\langle\alpha\rangle\cong \boldsymbol{Z}_p\,,$$

suppose that $t\alpha=m\alpha$. Then $(p, m)=1$. Since $t-1$ is an automorphism, $(p, m-1)=1$. Let

$$\phi\colon H_1(M)\times H_1(M)\to \boldsymbol{Q}/\boldsymbol{Z}$$

be the linking pairing given by

$$\phi(\alpha, \alpha)=q/p\,.$$

Since h is an orientation-preserving diffeomorphism, ϕ is a t-isometry [11]. Then

$$\phi(\alpha, \alpha)=\phi(t\alpha, t\alpha)=\phi(m\alpha, m\alpha)=m^2q/p\,.$$

It implies that $m^2\equiv 1 \pmod p$, hence $(m-1)\,(m+1)\equiv 0 \pmod p$. But p and $m-1$ are relatively prime, so we have $m\equiv -1 \pmod p$.

Any diffeomorphism of M inducing the (-1)-multiplication on $H_1(M)$ is diffeotopic to A, which is just the monodromy of the 2-twist-spun knot of $K(p, q)$. Hence $\Sigma\cong S^4$ and K is the 2-twist-spun knot of $K(p, q)$. This completes the proof.

4. The general case

We consider the case $M=\overset{r}{\#}L(p, q)$. Let J and K be 2-knots in S^4, with K trivial. For an integer $r>0$ and the meridian generator $x\in\pi_1(S^4-K)$, let V be a tubular neighbourhood of a simple closed curve representing x^r in S^4-K. Note that there is a natural diffeomorphism $h\colon S^4-\operatorname{Int}V\to N(J)$, where $N(J)$ is a tubular neighbourhood of J in S^4. Then the 2-knot $h(K)\subset(N(J)\subset)S^4$ is called *the r-cable knot about J* and denote by $\boldsymbol{C}(J; r)$ [8]. In particular, $\boldsymbol{C}(J; 1)=J$.

Proposition ([9; Theorem 2.3]). *If J is a fibered knot with fiber F, then $\boldsymbol{C}(J; r)$ is a fibered knot with fiber a disk sum of r-copies F_i, $i=1, 2, \cdots, r$, of F.*

The proof is direct. (See [9]. Kanenobu's theorem is stated for more general high-dimensional satellite knots.) In the Proposition, we can see easily that the monodromy f of $C(J; r)$ is cyclic, i.e.,

$$f(F_i) = F_{i+1} \qquad \text{for} \quad i = 1, 2, \cdots, r\,,$$

where $F_{r+1}=F_1$.

Lemma 2. *Let K be an M^0-fibered 2-knot in a homology 4-sphere $\sum$ with M the r-fold connected sum of $L(p, q)$ for $r\geqq 2$. If the monodromy h is cyclic, then $\sum\cong S^4$ and K is the r-cable knot about the 2-twist-spun knot of the 2-bridge knot $K(p, q)$.*

Proof. Writing $M=\mathop{\#}\limits_{i=1}^{r} L_i$, we may assume that

$$h(L_i^0) = L_{i+1}^0 \qquad \text{for} \quad i = 1, 2, \cdots, r\,,$$

where $L_{r+1}=L_1$. Let $H_1(L_i)\cong\langle\alpha_i\rangle\cong \boldsymbol{Z}_p$ for $i=1, 2, \cdots, r$. Suppose that the induced automorphism $t=h_*$ of $H_1(M^0)\cong H_1(M)$ is represented by the matrix

$$\begin{vmatrix} 0 & m_2 & & \\ & \ddots & \ddots & \\ & & \ddots & m_r \\ m_1 & & & 0 \end{vmatrix}$$

with respect to the generators $\{\alpha_i\}$. Then $(p, m_i)=1$, $i=1, 2, \cdots, r$. Since $t-1$ is an automorphism, the matrix

$$\begin{vmatrix} -1 & m_2 & & \\ & \ddots & \ddots & \\ & & \ddots & m_r \\ m_1 & & & -1 \end{vmatrix}$$

must be invertible on $\boldsymbol{Z}_p$. Hence the determinant $(-1)^{r-1}(m_1 \cdots m_r-1)$ and p are relatively prime, i.e., $(p, m_1 \cdots m_r-1)=1$. Let

$$\phi\colon\ H_1(M)\times H_1(M) \to \boldsymbol{Q}/\boldsymbol{Z}$$

be the linking pairing given by

$$\begin{aligned} \phi(\alpha_i, \alpha_i) &= q/p \qquad \text{for} \quad i = 1, 2, \cdots, r\,, \\ \phi(\alpha_i, \alpha_j) &= 0 \qquad\ \ \text{for} \quad i, j = 1, 2, \cdots, r \quad (i\neq j)\,. \end{aligned}$$

Since ϕ is t-isometry,

$$\phi(\alpha_i, \alpha_i) = \phi(t\alpha_i, t\alpha_i) = \phi(m_{i+1}\alpha_{i+1}, m_{i+1}\alpha_{i+1}) = m_{i+1}^2 q/p$$

for $i=1, 2, \cdots, r$, where $m_{r+1}=m_1$. It follows that $m_i^2\equiv 1 \pmod p$. Since $m_1^2 m_2^2 \cdots m_r^2 \equiv 1 \pmod p$, we have $(m_1 \cdots m_r - 1)(m_1 \cdots m_r + 1)\equiv 0 \pmod p$. Since $(p, m_1 \cdots m_r - 1)=1$, it implies $m_1 \cdots m_r \equiv -1 \pmod p$.

Next we consider the exterior $E=\sum - \operatorname{Int} N(K)$, which is the mapping torus of h, $(M^0\times I)/h$. We shall show that E is diffeomorphic to the exterior E_1 of the r-cable knot about the 2-twist-spun knot of $K(p, q)$.

Note that E and E_1 are fiber bundles over S^1 with the same fiber M^0 and cyclic monodromies h and f, respectively. Consider the cylinders on lens space factors $L_i^0\times I$ $(i=1, \cdots, r)$ in $M^0\times I$. When we identify them by h, these cylinders form a circular tube in $E=(M^0\times I)/h$, winding r times around S^1-factor of E. We can regard the circular tube as $(L(p, q)^0\times I)/g$, which is embedded in E, where g is the diffeomorphism induced by h. The automorphism g_* induces the $(m_1 \cdots m_r)$-multiplication on H_1 $(L(p, q))$. Since $m_1 \cdots m_r \equiv -1 \pmod p$, g is equal to A which is the unique diffeomorphism of $L(p, q)$ inducing the (-1)-multiplication on H_1 $(L(p, q))$. Considering a similar tube in $E_1=(M^0\times I)/f$, we see that E is diffeomorphic to E_1. Hence $\sum\cong S^4$ and K is the r-cable knot about the 2-twist-spun knot of $K(p, q)$. This completes the proof.

We call the 2-twist-spun knot of a 2-bridge knot *a fibered 2-knot of type L*, and the r-cable knot about a fibered 2-knot of type L with $r\geqq 2$ *a fibered 2-knot of type L**. Let K be an M^0-fibered 2-knot in a homology 4-sphere $\sum$ with $M=\mathop{\#}\limits_{i=1}^{r} L(p_i, q_i)$. We assume that h preserves a disk sum $M_1^0 \natural M_2^0$ of M^0, where h is the monodromy of K. Then $K\subset\sum$ is a connected sum of M_i^0-fibered 2-knots K_i in a homology 4-sphere $\sum_i$ with monodromy $h|M_i^0$, $i=1, 2$. Together with Lemmas 1 and 2, we obtain the following:

Theorem. *Let K be an M^0-fibered 2-knot in a homology 4-sphere $\sum$. Then $\sum\cong S^4$ and K splits, as a connected sum, into fibered 2-knots of type L or L* so that the disk sum of the fibers is M^0.*

5. Branched covering spaces

In this section, we consider the branched covering spaces of r-cable knots, giving counterexamples to the higher-dimensional Smith conjecture. Our next theorem is an extension of a special case of Kanenobu's theorem [8]. In our case, we only observe the monodromies, hence the proof is simple.

Let $M=\mathop{\#}\limits^{r} L(p, q)$ and $K=\boldsymbol{C}(J; r)$ the r-cable knot about the 2-twist-spun knot J of $K(p, q)$. Then as observed before, K is an M^0-fibered 2-knot in S^4 with cyclic monodromy h.

Assertion. *For any odd $n\geqq 1$, let $\sum_{nr}$ be the nr-fold cyclic branched covering* *space of S^4 branched over K, and $\tilde{K}$ the lift of K. Then $\sum_{nr} \cong S^4$ and $\tilde{K}$ is the r-fold connected sum of the 2-twist-spun knot of $K(p, q)$.*

Proof. The monodromy h_* is represented by the matrix

$$\begin{vmatrix} 0 & m_2 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & m_r \\ & & & & \ddots \\ m_1 & & & & 0 \end{vmatrix}.$$

Obviously, $E = \sum_{nr} - \operatorname{Int} N(\tilde{K})$ is the mapping torus of h^{nr}, $(M^0 \times I)/h^{nr}$. The monodromy h_*^r and hence h_{nr}^* are represented by the matrix

$$\begin{vmatrix} -1 & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & -1 \end{vmatrix}.$$

Hence, E is just the exterior of the r-fold connected sum of the 2-twist-spun knot of $K(p, q)$. This completes the proof.

Corollary. *For any integers $n, r > 1$ with r odd, there is a 2-knot in S^4 which is the same fixed point set of n inequivalent semi-free $\boldsymbol{Z}_r$-actions on S^4.*

Proof. Let K be a fibered 2-knot of type L. Let K_i, $i=0, 1, \cdots, n-1$, be the connected sum of i-copies of $\boldsymbol{C}(K; r)$ and $(n-i-1)$ r-copies of K. We take the r-fold cyclic branched covering space of S^4 branched over K_i. By the Assertion, all $K_i \subset S^4$ lift to the same knot, namely, the $(n-1)$ r-fold connected sum of $K \subset S^4$. We show that K_i, $i=0, 1, \cdots, n-1$, belong to mutually distinct knot types. To see this, we use the 1–st Alexander module, A_i, of K_i, which is $\boldsymbol{Z}_p\langle t\rangle$-isomorphic to $(\overset{i}{\oplus} \boldsymbol{Z}_p\langle t\rangle/(t^r+1)) \oplus (\overset{(n-i-1)r}{\oplus} \boldsymbol{Z}_p\langle t\rangle/(t+1))$. We see that $(t+1)A_i$, $i=0, 1, \cdots, n-1$, are not mutually $\boldsymbol{Z}_p\langle t\rangle$-isomorphic. This completes the proof.

This corollary has been suggested by Professor Akio Kawauchi.

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