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ON THE SEQUENCES INDUCED FROM AUSLANDER-REITEN SEQUENCES

Dedicated to Professor Hirosi Nagao on his 60th birthday

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0. Introduction

Let kG be the group algebra of a finite group G over an algebraically closed field k of characteristic p, $p \neq 0$. Fix a normal subgroup N of G and a non-projective indecomposable kN-module V. Let SV: $0 \to \Omega^2 V \to X \to V \to 0$ be the Auslander-Reiten sequence terminating at V. Here Ω denotes the Heller operator. In this paper, we study the induced sequence $0 \to (\Omega^2 V)^G \to X^G \to V^G \to 0$. We shall decompose it according to the decomposition of V^G and investigate the relation between the sequences appearing in the decomposition and the Auslander-Reiten sequences terminating at the indecomposable direct summands of V^G. For example, we shall give a condition which guarantees that some Auslander-Reiten sequences appear in the decomposition of the induced sequence. This result is related to the work of Knörr [6].

Notation is standard. All the kG-modules considered here are finite dimensional right modules. For kG-modules W and W', we use $(W, W')^G$ to denote $\mathrm{Hom}_{kG}(W, W')$. An element f of $(W, W')^G$ is said to be projective if there are a projective kG-module P and maps $\alpha \in (W, P)^G$ and $\beta \in (P, W')^G$ such that $f = \beta \circ \alpha$. We denote by $(W, W')^{1,G}$ the factor space of $(W, W')^G$ divided by the subspace consisting of projective homomorphisms. Note that $(W, W')^{1,G}$ is an $\mathrm{End}_{kG}(W')$-$\mathrm{End}_{kG}(W)$-bimodule. For any k-algebra R, we denote its radical by JR. Unless otherwise noted, $\otimes$ means $\otimes_{kN}$.

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1. Decomposition of the induced sequence

Throughout this paper except Theorem 2.5, we deal with the situation in the Introduction. Let $E = \mathrm{End}_{kG}(V^G)$ and $E_1 = \mathrm{End}_{kN}(V)$. Then E_1 can naturally be considered as a subalgebra of E by the injection ι: $E_1 \to E$ defined by $\iota(f)=f\otimes \mathrm{Id}_{kG}$ for all $f\in \underline{E}_1$. We denote $(V^G, V^G)^{1,G}$ and $(V, V)^{1,N}$ by $\underline{E}$ and $\underline{E}_1$, respectively.

We begin with the following lemma, which is well-known and easy to see.

Lemma 1.1. *$\Omega^n(V^G)\cong(\Omega^n V)^G$ for all $n=1, 2, \cdots$.*

Henceforth we write the above modules without parentheses.

Let P be the projective cover of V^G. For any $f\in \underline{E}$, we can take $f_1\in \mathrm{End}_{kG}(P)$ and $f'\in \mathrm{End}_{kG}(\Omega V^G)$ such that the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega V^G & \longrightarrow & P & \longrightarrow & V^G & \longrightarrow & 0 \quad \text{(exact)} \\ & & {\scriptstyle f'}\big\downarrow & & {\scriptstyle f_1}\big\downarrow & & {\scriptstyle f}\big\downarrow & & \\ 0 & \longrightarrow & \Omega V^G & \longrightarrow & P & \longrightarrow & V^G & \longrightarrow & 0 \quad \text{(exact)} \end{array}$$

In this case, f' corresponds to f under the isomorphism $\underline{E}\cong(\Omega V^G, \Omega V^G)^{1,G}$. (See the discussion following [1, 2.17.2].) Likewise we can find $f''\in \mathrm{End}_{kG}(\Omega^2 V^G)$ such that it corresponds to f' via $(\Omega^2 V^G, \Omega^2 V^G)^{1,G}\cong(\Omega V^G, \Omega V^G)^{1,G}$. Define left actions of $\underline{E}$ on $(V^G, \Omega V^G)^{1,G}$ and on $\mathrm{Ext}_{kG}(V^G, \Omega^2 V^G)$ via the above isomorphisms. Recall that we have the following. ([1, 2.17.5])

(1.2.a) $\quad \underline{E}_1^*\cong \mathrm{Ext}_{kN}(V, \Omega^2 V)\cong(V, \Omega V)^{1,N}$ as $\underline{E}_1$–$\underline{E}_1$-bimodules

(1.2.b) $\quad \underline{E}^*\cong \mathrm{Ext}_{kG}(V^G, \Omega^2 V^G)\cong(V^G, \Omega V^G)^{1,G}$ as $\underline{E}$–$\underline{E}$-bimodules

Here $\underline{E}^*$ is the dual $\underline{E}$–$\underline{E}$-bimodule $\mathrm{Hom}(\underline{E}, k)$.

The next lemma is also easy to show.

Lemma 1.3. *Let H be a subgroup of G, V_1 and V_2 kH-modules, and let $f\in(V_1, V_2)^H$. Then f is projective if and only if $f\otimes_{kH}\mathrm{Id}_{kG}\in(V_1{}^G, V_2{}^G)^G$ is projective.*

By the above lemma $\underline{E}_1$ can be regarded as a subalgebra of $\underline{E}$. Thus $\underline{E}_1^*$ is a submodule of $\underline{E}^*$. Likewise we can and will regard the modules in (1.2.a) as submodules of the modules in (1.2.b).

Lemma 1.4. *Let $\gamma\in \mathrm{Ext}_{kN}(V, \Omega^2 V)$ represent an extension $0\to\Omega^2 V\to Y\to V\to 0$. Then considering γ as an element of $\mathrm{Ext}_{kG}(V^G, \Omega^2 V^G)$, it represents the induced sequence.*

Proof. Take an element f of $(V, \Omega V)^N$ whose image in $(V, \Omega V)^{1,N}$ corresponds to γ under the isomorphism (1.2.a). Then we have the following pullback diagram.

$$\begin{array}{ccc} Y & \xrightarrow{\alpha} & V \\ {\scriptstyle \beta}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ P' & \longrightarrow & \Omega V \end{array}$$

Here P' denotes the projective cover of ΩV. The above induces the following diagram, which is also pullback.

$$\begin{array}{ccc} Y^G & \xrightarrow{\alpha\otimes \mathrm{Id}_{kG}} & V^G \\ {\scriptstyle \beta\otimes\mathrm{Id}_{kG}}\downarrow & & \downarrow{\scriptstyle f\otimes\mathrm{Id}_{kG}} \\ P'^G & \longrightarrow & \Omega V^G \end{array}$$

Note that P'^G is the projective cover of ΩV^G. Thus $f\otimes\mathrm{Id}_{kG}$ gives the sequence $0\to\Omega^2V^G\to Y^G\to V^G\to 0$. This completes the proof.

As E_1 is local, the $\underline{E}_1$–$\underline{E}_1$-bimodules in (1.2.a) have irreducible socles which are of 1-dimensional over k. We denote the socles of those modules by L. Note that $J\underline{E}_1$ annihilates L from the both sides. A nonzero element γ of $\mathrm{Ext}_{kN}(V, \Omega^2V)$ represents the Auslander-Reiten sequence if and only if γ lies in L. (See the proof of [1, 2.17.7].)

Lemma 1.5. *$xl=lx$ for all $l\in L$ and $x\in\underline{E}$.*

Proof. We fix representatives G/N of cosets of N in G containing 1. Let T be the inertial subgroup of V in G. For any $t\in T/N$, there is a kN-isomorphism ϕ_t: $V\to V\otimes t$. This gives a unit $u_t=\phi_t\otimes\mathrm{Id}_{kT}$ of $E_T=\mathrm{End}_{kT}(V^T)$. Let $\underline{E}_T$ be $(V^T, V^T)^{1,T}$. Note that $\underline{E}_T$ is naturally a subalgebra of $\underline{E}$. (See Lemma 1.3.) We first claim that;

(1.5.a) $\qquad xl = lx \qquad$ for all $l\in L$ and $x\in\underline{E}_T$.

Recall that $E_1/JE_1\simeq k$. For all $m\in E_1/JE_1$ and $t\in T/N$, we have $u_t^{-1}mu_t=m$ in E_1/JE_1. Since L is dual to E_1/JE_1, we have $\bar{u}_tl=l\bar{u}_t$ for all $l\in L$ and $t\in T/N$, where $\bar{u}_t$ is the image of u_t in $\underline{E}_T$. We also have $ul=lu$ for all $l\in L$ and $u\in\underline{E}_1$. Thus (1.5.a) holds since $\underline{E}_T$ is generated by $\underline{E}_1$ and $\{\bar{u}_t\}_{t\in T/N}$.

Now note that $V^G{}_N=\oplus_{g\in G/N}V\otimes g$ as kN-modules. So by the Frobenius reciprocity, we have the following isomorphisms.

$$E \xrightarrow{\sim} (V, V^G{}_N)^N \xrightarrow{\sim} \oplus_{g\in G/N}(V, V\otimes g)^N.$$

Letting E_g be the inverse image of $(V, V\otimes g)^N$ in E, we obtain $E=\oplus_{g\in G/N}E_g$. (Note: our previous E_1 coincides with the new one.) Then it is easy to check that $E_gE_{g'}\subseteq E_{gg'}$ for all $g, g'\in G/N$. Since $E=E_T\oplus(\oplus_{g\in G/N\setminus T/N}E_g)$ as k-spaces, to complete the proof, it suffices to show that

(1.5.b) $\quad \bar{x}l = l\bar{x} = 0 \qquad$ for all $l\in L$ and $x\in E_g$ with $g\notin T/N$,

where $\bar{x}$ is the image of x in $\underline{E}$.

Fix $g\in G/N\setminus T/N$ and $x\in E_g$. Then for any $g'\in G/N$ and any $y\in E_{g'}$, it follows that

(1.5.c) xy and yx lie in $J\underline{E}_1$, if $Ng' = Ng^{-1}$, and
xy and yx lie in $\bigoplus_{g\neq 1} E_g$, otherwise.

Now consider $l \in L$ as an element of $\underline{E}^*$, i.e., as a k-linear map from $\underline{E}$ into k. Since $l \in \underline{E}_1^*$, for all $z \in \bigoplus_{g\neq 1} E_g$, l takes $\bar{z}$ to zero. Further, l vanishes on $J\underline{E}_1$. Hence by (1.5.c) we can conclude that, for all $y \in E$, l maps both $\bar{x}\bar{y}$ and $\bar{y}\bar{x}$ to zero. By the definition of the action of $\underline{E}$ on $\underline{E}^*$, this means that the elements $l\bar{x}$ and $\bar{x}l$ of $\underline{E}^*$ both send $\bar{y}$ to zero for all $y \in E$. Therefore, we can conclude that (1.5.b) holds. This completes the proof.

Now we decompose the sequence $0 \to \Omega^2 V^G \to X^G \to V^G \to 0$. Let $e_1, \cdots, e_n$ be orthogonal primitive idempotents of E with $\mathrm{Id}_{V^G} = e_1 + \cdots + e_n$. We can find orthogonal primitive idempotents $e_1'', \cdots, e_n''$ of $\mathrm{End}_{kG}(\Omega^2 V^G)$ such that each $\bar{e}_i''$ corresponds to $\bar{e}_i$ via $\underline{E} \cong (\Omega^2 V^G, \Omega^2 V^G)^{1,G}$. Remark that the left actions of $\bar{e}_i$ and $\bar{e}_i''$ on the modules in (1.2.b) are equal to each other.

Theorem 1.6. *For each i, $1 \le i \le n$, there exists a non-sprit exact sequence S_i: $0 \to e_i''\Omega^2 V^G \to Y_i \to e_i V^G \to 0$ such that their direct sum $0 \to \Omega^2 V^G \to \bigoplus_i Y_i \to V^G \to 0$ is equivalent to the induced sequence $(SV)^G$: $0 \to \Omega^2 V^G \to X^G \to V^G \to 0$. Moreover, this gives the unique (up to equivalence) decomposition of $(SV)^G$ with respect to $e_1, \cdots, e_n$.*

Proof. It follows from Lemma 1.5 that $\bar{e}_i l = l\bar{e}_i$ for all $l \in L$ and i, $1 \le i \le n$. Hence we have

$$l = (\sum_i \bar{e}_i) l (\sum_j \bar{e}_j) = l \sum_{i,j} \bar{e}_i \bar{e}_j = \sum_i \bar{e}_i l \bar{e}_i$$

for all $l \in L$. For each i, the element $\bar{e}_i l \bar{e}_i$ gives an extension S_i: $0 \to e_i''\Omega^2 V^G \to Y_i \to e_i V^G \to 0$ and their sum $\sum_i \bar{e}_i l \bar{e}_i$ corresponds to the direct sum of those sequences. Hence it follows by Lemma 1.4 that the direct sum $0 \to \Omega^2 V^G \to \bigoplus_i Y_i \to V^G \to 0$ is equivalent to $(SV)^G$ if l represents SV.

Now suppose that some S_i splits, i.e., $l\bar{e}_i = 0$. Then we have $l\underline{E}\bar{e}_i = 0$ by Lemma 1.5. This implies that the following sequence is exact.

$$0 \to (e_i V^G, \Omega^2 V^G)^G \to (e_i V^G, X^G)^G \to (e_i V^G, V^G)^G \to 0$$

By the Frobenius reciprocity law, there holds

$$0 \to (e_i V^G, \Omega^2 V)^N \to (e_i V^G, X)^N \to (e_i V^G, V)^N \to 0 \quad \text{(exact)}.$$

Since V is isomorphic to a direct summand of $(e_i V^G)_N$, the above contradicts our assumption that SV is an Auslander-Reiten sequence. Therefore each S_i does not split.

To see that this gives the unique decomposition, note that if we have

$l=\sum_i \bar{e}_i x_i \bar{e}_i$ for some $x_i \in \mathrm{Ext}_{kG}(V^G, \Omega^2 V^G)$, then $\bar{e}_i x_i \bar{e}_i = \bar{e}_i l \bar{e}_i$ for all i, $1 \leq i \leq n$. Now the proof is complete.

2. The sequences appearing in the decomposition of $(SV)^G$

In this section, we shall discuss how S_i in Theorem 1.6 is far from $S(e_i V^G)$, the Auslander-Reiten sequence terminating at $e_i V^G$.

For any subgroup H of G and any kG-module W, let Tr_H^G: $(W, W)^H \to (W, W)^G$ denote the trace map. We begin with the following general result.

Lemma 2.1. *For an indecomposable kG-module W, suppose that* $J(\mathrm{End}_{kG}(W)) = \sum_{H <_G \mathrm{vtx}(W)} \mathrm{Im}\mathrm{Tr}_H^G$. *Then a short exact sequence* S: $0 \to \Omega^2 W \to Z \xrightarrow{\sigma} W \to 0$ *is an Auslander-Reiten sequence if and only if the following two conditions hold.*

(i) *S does not split.*

(ii) *S splits on the restriction to H for all $H <_G vtx(W)$.*

Proof. It is well known that the above two hold if S is an Auslander-Reiten sequence ([1, 2.17.10]). To see the converse, we first prove that any map f in $J(\mathrm{End}_{kG}(W))$ factors through σ. By the assumption, we may assume that $f = \mathrm{Tr}_H^G(h)$ for some $H <_G vtx(W)$ and $h \in \mathrm{End}_{kH}(W)$. We can take $h' \in (W_H{}^G, W)^G$ corresponding to h by the Frobenius reciprocity law. Also, let ξ be the element of $(W, W_H{}^G)^G$ corresponding to $\mathrm{Id}_W \in \mathrm{End}_{kH}(W)$. Then it is routine to check that $f = \mathrm{Tr}_H^G(h) = \mathrm{Tr}_H^G(h \circ \mathrm{Id}_W) = h' \circ \xi$. Since $W_H{}^G$ is H-projective, the condition (ii) yields that there exists $\phi \in (W_H{}^G, Z)^G$ such that $\sigma \circ \phi = h'$. Thus we obtain $f = \sigma \circ \phi \circ \xi$. Therefore f factors through σ. Now by (i), the only elements of $\mathrm{End}_{kG}(W)$ that factor through σ are precisely those that lie in $J(\mathrm{End}_{kG}(W))$.

Let γ be the element of $\mathrm{Ext}_{kG}(W, \Omega^2 W)$ corresponding to S. Then the above shows that $J(\mathrm{End}_{kG}(W))$ annihilates γ from the right. Hence γ generates a semisimple module. Because $\mathrm{Ext}_{kG}(W, \Omega^2 W)$ has a simple socle, γ must be a generator of the socle. This completes the proof. (See also the proof of [1, 2.17.7].)

The above lemma implies the following.

Theorem 2.2. *Suppose that* $J(\mathrm{End}_{kG}(e_i V^G)) = \sum_{H <_G \mathrm{vtx}(V)} \mathrm{Im}\mathrm{Tr}_H^G$. *Then the sequence S_i is an Auslander-Reiten sequence.*

Proof. Note that $vtx(V) = vtx(e_i V^G)$. If $H <_G vtx(V)$, then since $vtx(V) \leq N$, it easily follows from [1, 2.17.10] that $0 \to (\Omega^2 V^G)_H \to X^G{}_H \to V^G{}_H \to 0$ splits. (Note that $0 \to \Omega^2 V \otimes g \to X \otimes g \to V \otimes g \to 0$ is an Auslander-Reiten sequence for all $g \in G$.) Thus by Theorem 1.6 each S_i is H-split. Moreover,

S_i itself does not split. Therefore the result follows from Lemma 2.1.

For any exact sequence S: $0\to\Omega^2W\to Z\xrightarrow{\sigma}W\to 0$, let $V^G\cdot S$ denote the cokernel of σ_*: $(V^G, Z)^G\to(V^G, W)^G$. So $V^G\cdot S$ is naturally a right E-module.

Let I be the two-sided ideal of E generated by JE_1. In the case where V is G-invariant, $\bar{E}=E/I$ is isomorphic to a twisted group algebra of G/N over k. Now we have;

Proposition 2.3. *Suppose that V is G-invariant. Then;*

(i) *For each i, $1\leq i\leq n$, $V^G\cdot S_i$ is a projective indecomposable right $\bar{E}$-module.*

(ii) *A sequence S: $0\to e_i''\Omega^2V^G\to Z\to e_iV^G\to 0$ is an Auslander-Reiten sequence if and only if $V^G\cdot S$ is a simple E-module. Hence in this case $V^G\cdot S$ is a simple $\bar{E}$-module.*

Proof. (i) We first claim that $V^G\cdot(SV)^G$ is isomorphic to $\bar{E}$. By the Frobenius reciprocity law, we have $V^G\cdot(SV)^G\cong(V^G{}_N)\cdot SV$ as E_1–E-bimodules. Since V is G-invariant, $(V^G{}_N)\cdot SV\cong(V\cdot SV)^{|G:N|}$ as E_1–E_1-bimodules. Thus JE_1 annihilates $V^G\cdot(SV)^G$ from the right, and hence $V^G\cdot(SV)^G$ is an $\bar{E}$-module. Since it is a factor module of E having the dimension $|G: N|$ over k, it must coincide with $\bar{E}$. Now Theorem 1.6 yields that $V^G\cdot S_i$ is a direct summand of $V^G\cdot(SV)^G$. Therefore $V^G\cdot S_i$ is projective. Since $I\subseteq JE$, the image of e_i in $\bar{E}$ is a nonzero idempotent of $\bar{E}$ for all i, $1\leqq i\leqq n$. Hence $V^G\cdot S_i$ is an indecomposable $\bar{E}$-module.

(ii) Note that $V^G\cdot S_i$ is a factor module of a projective indecomposable E-module $e_iE=(V^G, e_iV^G)^G$. On the other hand, S is an Auslander-Reiten sequence if and only if e_iJEe_i is contained in the kernel of the epimorphism $e_iE\to V^G\cdot S$ and $V^G\cdot S\neq 0$. These hold if and only if $V^G\cdot S$ is simple. Now the proof is complete.

Now we give an application of the above results, which is related to the work of Knörr [6].

Corollary 2.4. *Suppose that N is a p-group. Let $H=NC_G(N)$ and let B_i be the block of kG containing e_iV^G. Then, if S_i is an Auslander-Reiten sequence, the blocks of H covered by B_i have N as their defect groups.*

Proof. By [5, Satz 2.2] and [2, § 6, Exercise 14], we may assume that each e_i lies in $E_T=\mathrm{End}_{kT}(V^T)$, where T is the inertial subgroup of V in G. Let S_i' be the sequence $0\to\Omega^2e_iV^T\to Y_i'\to e_iV^T\to 0$ appearing in the decomposition of $(SV)^T$. (By Theorem 1.6, S_i' is determined uniquely up to equivalence.) We claim that S_i' is also an Auslander-Reiten sequence. Let $\bar{e}_il\bar{e}_i\in\mathrm{Ext}_{kT}(e_iV^T, \Omega^2e_iV^T)$ represent S_i'. By the proof of Theorem 1.6, $\bar{e}_il\bar{e}_i$ also represents S_i. Now $\mathrm{End}_{kT}(e_iV^T)$ is naturally considered as a subalgebra of $\mathrm{End}_{kG}(e_iV^G)$, and hence $J(\mathrm{End}_{kT}(e_iV^T))\subseteq J(\mathrm{End}_{kG}((e_iV^G))$. Since $J(\mathrm{End}_{kG}(e_iV^G))$

annihilates $\bar{e}_i I \bar{e}_i$ by the assumption, so does $J(\mathrm{End}_{kT}(e_i V^T))$. This implies that S'_i is an Auslander-Reiten sequence. Thus by Proposition 2.3, $V^T \cdot S'_i$ is a simple projective E_T/I'-module, where I' is the ideal of E_T generated by JE_1. Therefore the result follows by [6, Cor. 2.2].

Our final result concerns relative projectivity of Auslander-Reiten sequences. Recall that each Auslander-Reiten sequence gives a (finitely presented) simple object of the category $\mathrm{MMod}(kG)$ of contravariant k-linear functors from the category of kG-modules into the category of k-spaces. (See [4, §1], for example.) In [4], Green defined relative projectivity of finitely presented objects of $\mathrm{MMod}(kG)$ and showed that each of those indecomposable objects S has vertex $vtx(S)$, which is a p-subgroup of G determined uniquely up to G-conjugate. (See [4, § 4] for detail.) He also proved that for any non-projective indecomposable kG-module W, there holds $vtx(SW) \geq_G vtx(W)$, [4, Theorem 5.12]. Here we identify the sequence SW with the corresponding simple object. The following was suggested by the referee.

Theorem 2.5. *Let W be a non-projective indecomposable kG-module. Suppose that $J(\mathrm{End}_{kG}(W)) = \sum_{H <_G \mathrm{vtx}(W)} \mathrm{ImTr}_H^G$. Then $vtx(W) =_G vtx(SW)$. In particular, if W is simple, then $vtx(W) =_G vtx(SW)$.*

Proof. Let P be a vertex of W, $M = N_G(P)$, and W' the Green correspondent of W with respect to (G, P, M). Since $J(\mathrm{End}_{kM}(W')) = \sum_{H <_M \mathrm{vtx}(W')} \mathrm{ImTr}_H^M$ by [3, Chap. III, Lemma 5.10 (i)], Theorem 2.2 yields that $S(W')$ appears in the decomposition of $(SV_0)^M$, where V_0 is the P-source of W. This shows that $S(W')$ is P-projective. On the other hand, it follows from [4, Theorem 7.8] that $vtx(SW) \leq_G vtx(S(W'))$. Hence we have $vtx(SW) \leqq_G P = vtx(W)$.

Therefore, the proof is completed by [4, Theorem 5.12].

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