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A CHARACTERIZATION OF SOME PARTIAL GEOMETRIC SPACES

Dedicated to Professor Hirosi Nagao on his 60th birthday

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1. Introduction

A partial geometric space S of dimension $m\geqq 2$ defined in [2, 6] consists of the sets $\{A_i\}_{i=-1}^{m}$ and the set T such that the following eight axioms are satisfied:

(1) $A_i\cap A_j=\phi$ whenever $i\neq j$ and $-1\leqq i, j\leqq m$.

(2) $|A_{-1}|=|A_m|=1$.

(3) $T\subset \prod_{i=-1}^{m} A_i$.

The elements of A_i, $-1\leqq i\leqq m$, are called i elements of S. The elements of T are called flags of S. There is a property called incidence which is a relation between the elements of S based on the flags.

(4) For each i element x_i there is a flag $(t_{-1}, \cdots, t_m)\in T$ such that $x_i=t_i$, where $-1\leqq i\leqq m$.

(5) Whenever $(y_{-1}, \cdots, y_m)\in T$ and $(z_{-1}, \cdots, z_m)\in T$ and $y_k=z_k$ for some k, $-1\leqq k\leqq m$, then there exists a flag $(t_{-1}, \cdots, t_m)\in T$, where $t_i=y_i$ for $-1\leqq i\leqq k$, and $t_j=z_j$ for $k\leqq j\leqq m$.

(6) If $x_i\in A_i$ and $x_j\in A_j$, then x_i and x_j have an l intersection $x_l\in A_l$ and an s join $x_s\in A_s$. Here x_i and x_j are said to have an l intersection x_l (s join x_s), where $-1\leqq l\leqq \min\{i, j\}$ ($\max\{i, j\}\leqq s\leqq m$) if and only if x_l (x_s) is incident with x_i and x_j such that whenever x_n is an n element of S for $-1\leqq n\leqq \min\{i, j\}$ ($\max\{i, j\}\leqq n\leqq m$) which is incident with x_i and x_j, then x_n is incident with x_l (x_s) and $-1\leqq n\leqq l$ ($s\leqq n\leqq m$). By the definition, x_i and x_j have unique intersection and unique join.

(7) If $x_{i-1}\in A_{i-1}$ and $x_{i+1}\in A_{i+1}$ are incident, then there are $k(i)$ i elements which are incident with x_{i-1} and x_{i+1}, where $2\leqq k(i)<\infty$, for $0\leqq i\leqq m-1$. The number $k(i)$ is independent of the choice of x_{i-1} and x_{i+1}, and depends only on i. $k(0)$, $k(1)$, $\cdots$, $k(m-1)$ are called the configuration parameters of S.

(8) Let $m\geqq 2$. If $x_i\in A_i$ and $x_{i+1}\in A_{i+1}$ have an $(i-1)$ intersection x_{i-1} and an s join x_s, where $0\leqq i\leqq m-2$ and $i+2\leqq s\leqq m$, then there are $t(i, s, k)$ i

elements y_i, which are incident with x_{i-1} and x_{i+1} such that y_i and x_i have an $(i+k)$ join for $1\leqq k\leqq s-i-1$. Also $\sum_{k=1}^{s-i-1} t(i, s, k)\geqq 1$ for $0\leqq i\leqq m-2$ and $i+2\leqq s\leqq m$. The numbers $t(i, s, k)$ are called the geometric parameters of S.

The concept of a partial geometric space of dimension m is an extension of the concept of a partial geometry introduced by R.C. Bose [1]. A partial geometry of dimension three introduced by R. Lasker and J. Dunbar [5] is called an L.D. partial geometric space of dimension three in [6].

We have two examples of partial geometric spaces of dimension m.

EXAMPLE 1 [6]. Let A be a set consisting $(m+1)$ distinct symbols, where $m\geqq 2$. Let $A_{-1}=\{\phi\}$. For $0\leqq j\leqq m$, $A_j=\{B\subset A \mid |B|=j+1\}$. Note that $A_m=\{A\}$. Let $T=\{(t_{-1}, \cdots, t_m)\in \prod_{i=-1}^{m} A_i \mid t_i\subset t_{i+1}$ for $-1\leqq i\leqq m-1\}$. Then $S_1=(\{A_i\}_{i=-1}^{m}, T)$ is a partial geometric space of dimension m. The configuration parameters are $k(i)=2$ for $0\leqq i\leqq m-1$. The geometric parameters are $t(i, i+2, 1)=2$ for $0\leqq i\leqq m-2$ and the rest geometric parameters need not be defined.

EXAMPLE 2. Let $PG(m, q)$ be the finite projective geometry of dimension m and of order q, where $m\geqq 2$ and q is a prime power. Let $A_{-1}=\{\phi\}$. For $0\leqq j\leqq m$, $A_j=\{B \mid B$ is a j dimensional subspace of $PG(m, q)\}$. Let $T=\{(t_{-1}, \cdots, t_m)\in \prod_{i=-1}^{m} A_i \mid t_i\subset t_{i+1}$ for $0\leqq i\leqq m-1\}$. Then $S_2=(\{A_i\}_{i=-1}^{m}, T)$ is a partial geometric space of dimension m. The configuration parameters are $k(i)=q+1$ for $0\leqq i\leqq m-1$. The geometric parameters are $t(i, i+2, 1)=q+1$ for $0\leqq i\leqq m-2$ and the rest geometric parameters need not be defined.

Two partial geometric spaces S_1 and S_2 of dimension m have common property:

$$(\sharp)\quad \begin{cases} \text{(i)} & k(i) \text{ is constant for } 0\leqq i\leqq m-1 \\ \text{(ii)} & t(i, s, k)=k(i) \text{ for } 0\leqq i\leqq m-2, \text{ where } s=i+2 \text{ and } k=1, \\ & \text{and the rest geometric parameters need not be defined.} \end{cases}$$

From (ii) of the property, we note that for any i element and $i+1$ element which have an $(i-1)$ intersection and are not incident, they have an $(i+2)$ join.

In section 2, we shall prove the following theorem.

Theorem. *Let $S=(\{A_i\}_{i=-1}^{m}, T)$ be a partial geometric space of dimension $m\geqq 2$ satisfying property ($\sharp$). Then $S=S_1$ if $k(i)=2$, and $S=S_2$ if $k(i)=\alpha+1>2$ and $m\geqq 3$.*

In section 3, we shall give an another example of partial geometric space of dimension $m\geqq 3$.

2. Proof of Theorem

Let $S=(\{A_i\}_{i=-1}^m, T)$ be a partial geometric space of dimension $m\geqq 2$. Let $x_i\in A_i$ and $x_j\in A_j$, where $-1\leqq i, j\leqq m$. x_i is said to be incident with x_j if and only if there exists a flag $(t_{-1}, \cdots, t_m)\in T$ such that $x_i=t_i$ and $x_j=t_j$. Let $x_j\in A_j$ and $x_k\in A_k$ such that x_j and x_k are incident, where $-1\leqq j<k\leqq m$. $\phi(i, x_j, x_k)$ is the number of i elements of S which are incident with x_j and x_k, where $-1\leqq i\leqq m$. The number $\phi(i, x_j, x_k)$ is a finite positive integer which is independent of the choice of the j element x_j and the k element x_k [2]. Therefore put $\phi(i, j, k)=\phi(i, x_j, x_k)$.

From now on in this section, we assume that S satisfies the property (♯).

Lemma 1. *Let x_i and y_i be two distinct i elements such that they have an $(i-1)$ intersection x_{i-1} for $0\leqq i\leqq m-1$. Then x_i and y_i have an $(i+1)$ join.*

Proof. Let x_{i+l} be a join of x_i and y_i, where $l>1$. Then there exists an $(i+1)$ element y_{i+1} which is incident with x_{i-1} and x_{i+l} and is not incident with x_i. From the property (♯), we have $l=2$ and there are $k(i)$ i elements z_i, which are incident with x_{i-1} and y_{i+1}, such that z_i and x_i have an $(i+1)$ join. Those $k(i)$ i elements are distinct from y_i. Consequently, there are $(k(i)+1)$ i elements which are incident with x_{i-1} and y_{i+1}. This is a contradiction. Therefore $l=1$, i.e. x_i and y_i have an $(i+1)$ join.

Lemma 2. $\phi(i, x_{i-1}, x_k)=\phi(i, i-1, k)=k(i)(k(i)-1)^{k-i-1}+(k(i)-1)^{k-i-2}+\cdots+(k(i)-1)+1$, *where $0\leqq i<k\leqq m$, and $x_{i-1}\in A_{i-1}$ and $x_k\in A_k$.*

Proof. It shall be proved by induction on $k-i+1$, say t. When $t=2$, from the definition, $\phi(i, x_{i-1}, x_{i+1})=\phi(i, i-1, i+1)=k(i)$. Therefore the lemma holds when $t=2$. Suppose that $t>2$ and assume that the lemma holds whenever $k-i+1<t$, where $0\leqq i<k\leqq m$, and $2<t\leqq m+1$. Let x_{i-1} be an $(i-1)$ element and x_k be a k element such that x_{i-1} and x_k are incident in S, where $0\leqq i\leqq m-2$, $i+2\leqq k\leqq m$ and $k-i+1=t$. Count triples (x_i, x_{i+1}, x_{i+2}), where x_l $(i\leqq l\leqq i+2)$ is an l element such that $x_{l'}$ and $x_{l'+1}$ $(i-1\leqq l'\leqq i+1)$, and x_{i+2} and x_k are incident in S.

Given a fixed i element x_i which is incident with x_{i-1} and x_k, there are $(k(i)(k(i)-1)^{k-i-2}+(k(i)-1)^{k-i-3}+\cdots+(k(i)-1)+1)$ $(i+1)$ elements x_{i+1}, which are incident with x_i and x_k, by the induction hypothesis.

Similarly, given a fixed pair (x_i, x_{i+1}), where x_{i-1} and x_i, x_i and x_{i+1}, and x_{i+1} and x_k are incident in S, there are $(k(i)(k(i)-1)^{k-i-3}+(k(i)-1)^{k-i-4}+\cdots+(k(i)-1)+1)$ $(i+2)$ elements, which are incident with both x_{i+1} and x_k.

Therefore the number of triples is

$$(k(i)(k(i)-1)^{k-i-2}+(k(i)-1)^{k-i-3}+\cdots+(k(i)-1)+1)\times$$
$$(k(i)(k(i)-1)^{k-i-3}+(k(i)-1)^{k-i-4}+\cdots+(k(i)-1)+1)\phi(i, x_{i-1}, x_k)\,.$$

On the other hand, count pairs (x_i, x_{i+1}), where x_l $(i \leqq l \leqq i+1)$ is an l element such that $x_{l'}$ and $x_{l'+1}$ $(i-1 \leqq l' \leqq i)$ are incident in S. Let x_i and y_i be distinct two i elements which are incident with x_{i-1} and x_k, then x_i and y_i have an $(i+1)$ join, say y_{i+1}. For y_{i+1}, there are $\binom{k(i)}{2}$ pairs (x_i', y_i') such that an i elements x_i' and y_i' have an $(i+1)$ join y_{i+1} and an intersection x_{i-1}, by the definition of $k(i)$. Consequently there are $\binom{\phi(i,\, x_{i-1},\, x_k)}{2} k(i)/\binom{k(i)}{2}$ pairs (x_i, x_{i+1}) such that x_{i-1} and x_i, x_i and x_{i+1}, and x_{i+1} and x_k are incident in S. The contribution to triples of such a pair (x_i, x_{i+1}) is $(k(i)(k(i)-1)^{k-i-3}+)k(i)-1)^{k-i-4} + \cdots +(k(i)-1)+1)$ by the induction hypothesis. Therefore we get

$$\begin{aligned}
&(k(i)(k(i)-1)^{k-i-2}+(k(i)-1)^{k-i-3}+ \cdots +(k(i)-1)+1)\times \\
&(k(i)(k(i)-1)^{k-i-3}+(k(i)-1)^{k-i-4}+ \cdots +(k(i)-1)+1)\phi(i,\, x_{i-1},\, x_k) \\
&= (k(i)(k(i)-1)^{k-i-3}+(k(i)-1)^{k-i-4}+ \cdots +(k(i)-1)+1)\times \\
&\binom{\phi(i,\, x_{i-1},\, x_k)}{2} k(i)/\binom{k(i)}{2}.
\end{aligned}$$

Consequently we have the lemma.

REMARK. This lemma can be obtained from Theorem 7.1 in [6].

Lemma 3. *If* $k(i)=2$, $\phi(i, i-1, k)=k-i+1$, *and if* $k(i)=\alpha+1>2$, $\phi(i, i-1, k)=(\alpha^{k-i+1}-1)/(\alpha-1)$, *for* $0 \leqq i < k \leqq m$.

Proof. It is obvious from Lemma 2.

Lemma 4. *If* $k(i)=2$, $\phi(i, j, k)=\binom{k-j}{i-j}$ *for* $-1 \leqq j < i < k \leqq m$.

Proof. Let x_j be a j element and x_k be a k element such that x_j and x_k be incident in S. Count $(k-j+1)$-tuples $(x_j, \cdots, x_i, \cdots, x_k)$, where x_l $(j \leqq l \leqq k-1)$ is an l element such that x_l and x_{l+1} are incident. By Lemma 2, there are $(k-j)$ $(j+1)$ elements x_{j+1} which are incident with x_j and x_k. For such x_{j+1}, there are $(k-j-1)$ $(j+2)$ elements which are incident with x_{j+1} and x_k, and so on. Consequently there are $(k-j)!$ $(k-j+1)$-tuples. On the other hand, given a fixed i element x_i which is incident with x_j and x_k, there are $(i-j)!$ $(i-j+1)$-tuples $(x_j, \cdots, x_i)$ where an l element x_l and an $(l+1)$ element x_{l+1} are incident $(j \leqq l \leqq i-1)$, and there are $(k-i)!$ $(k-i+1)$-tuples $(x_i, \cdots, x_k)$ where an l' element $x_{l'}$ and an $(l'+1)$ element $x_{l'+1}$ are incident in S, for $i \leqq l' \leqq k-1$. Therefore we get $\phi(i, x_j, x_k)(k-i)!\,(i-j)!=(k-j)!$. Thus the proof is complete.

Lemma 5. *If* $k(i)=\alpha+1>2$, $\phi(i, j, k) = \prod_{l=1}^{i-j} (\alpha^{k-j+l}-1)/(\alpha^l-1)$ *for* $-1 \leqq j < i < k \leqq m$.

Proof. It is similar to the proof of Lemma 4. So, we shall omit a proof.

We note that $\phi(i, j, k)=\phi(i, -1, j)$ when $i<j$, and $\phi(i, j, k)=\phi(i, k, m)$ when $k<i$. So, $\phi(i, j, k)$ is defined for i, j and k such that $-1\leqq j<k\leqq m$, $-1\leqq i\leqq m$ and $i\neq j, k$.

By the incidence structure in S, an i element x_i can be corresponded to a subset $b(x_i)$ of A_0 consisting of 0 elements which are incident with x_i, where $0\leqq i\leqq m$.

Lemma 6. *The above correspondence of A_i to a family consisting of subsets of A_0 is injective.*

Proof. Assume that $b(x_i)=b(y_i)$ for an i element y_i $(\neq x_i)$. Let x_l be an l intersection of x_i and y_i. Then $l<i$ and $b(x_l)\supseteqq b(x_i)$. On the other hand, $|b(z_j)|=\phi(0, -1, j)$ for every j element z_j. This contradicts $\phi(0, -1, l)<\phi(0, -1, i)$.

REMARK. Similarly we can prove that $b(x_i)\neq b(x_j)$ for $x_i\in A_i$ and $x_j\in A_j$, where $i\neq j$.

Lemma 7. *If $k(i)=2$, then $S=S_1$.*

Proof. $|A_0|=\phi(0, -1, m)=m+1$. Since $\phi(0, -1, i)=i+1$, every element of A_i is a subset of A_0 consisting of $i+1$ elements. By Lemma 4 and Lemma 6, A_i is a family of all subsets of A_0 containing $i+1$ elements. By the definition, for $i<j$, $x_i\in A_i$ and $x_j\in A_j$ are incident if and only if $b(x_i)\subset b(x_j)$. Thus the proof is complete.

Next we assume that $k(i)=\alpha+1\geqq 3$ for $0\leqq i\leqq m-1$. By Lemma 6, an i element x_i is identified with a subset of A_0.

Lemma 8. *A incidence structure $D=(A_0, A_{m-1})$ is a symmetric $2-(v, k, \lambda)$ design, where $v=(\alpha^{m+1}-1)/(\alpha-1)$, $k=(\alpha^m-1)/(\alpha-1)$ and $\lambda=(\alpha^{m-1}-1)/(\alpha-1)$.*

Proof. By the definition, $v=\phi(0, -1, m)$ and $k=\phi(0, -1, m-1)$. Let x_0 and y_0 be two elements of A_0. Then there exists a 1 element x_1 by Lemma 1 which is a join of x_0 and y_0. But every element of A_{m-1} containing x_0 and y_0 has to contain x_1. Thus we have $\lambda=\phi(m-1, 1, m)$. By Lemma 5, we have the lemma.

Elements of A_0 and elements of A_{m-1} are called points and blocks in D, respectively. For $x_i\in A_i$ and $y_j\in A_j$, where $0\leqq i\leqq j\leqq m-1$, we define $\langle x_i, y_j\rangle$

be an intersection of all blocks of D containing x_i and y_j. Especially $\langle x_0, y_0\rangle$ is called a line spanned by x_0 and y_0, where $x_0 \in A_0$ and $y_0 \in A_0$.

Lemma 9. *Let x_1 and y_1 be two elements of A_1. Then there is an element of A_{m-1} which is incident with x_1 and not incident with y_1.*

Proof. Let x_l be an l join of x_1 and y_1. Then $l>1$. By the property of x_l, the number of elements of A_{m-1} which are incident with x_1 and y_1 equals to the number of elements of A_{m-1} which are incident with x_l. This number is $\phi(m-1, l, m)$ which is smaller than $\phi(m-1, 1, m)$ by Lemma 5. This proves the lemma.

Lemma 10. *D is a design such that its points and blocks are points and hyperplanes of a finite projective geometry P of dimension m, respectively.*

Proof. Let x_1 be a 1 join of x_0 and y_0, where $x_0, y_0 \in A_0$. By Lemma 1, x_1 is contained in every block of D which is incident with x_0 and y_0. Therefore $\langle x_0, y_0\rangle \supseteqq x_1$. If $\langle x_0, y_0\rangle \neq x_1$, then there is an element z_0 of $\langle x_0, y_0\rangle$ which is not incident with x_1. Let x_l be an l join of z_0 and x_1, where $l>1$. Let z_1 be an element of A_1 which is incident with x_l and z_0. Then $z_1 \neq x_1$ and z_1 is contained in all blocks which contain x_0 and y_0. But by Lemma 9, there exists a block of D which is incident with x_1 and not incident with z_1, and hence z_1 is not contained in $\langle x_0, y_0\rangle$. Hence $\langle x_0, y_0\rangle = x_1$. Therefore $(v-\lambda)/(k-\lambda) = \alpha+1 = |x_1|$. By using a result in [4], we have the lemma.

Lemma 11. *An i element x_i is a subspace of P of dimension i for $1 \leqq i \leqq m$.*

Proof. We shall prove the lemma by the induction on i. By Lemma 10, the case of $i=1$ is true. Let $i \geqq 2$. Then there exist elements x_{i-1} and y_{i-1} of A_{i-1}, and an element x_{i-2} of A_{i-2} such that they are incident with x_i, and that x_{i-2} is incident with x_{i-1} and y_{i-1}. By Lemma 6, there exists an element y_0 of y_{i-1} which is not contained in x_{i-1}. By the induction hypothesis, $y_{i-2} = \langle x_{i-2}, y_0\rangle$ which is a subspace of P spanned by y_0 and all elements of x_{i-2}. Therefore we have $\langle x_{i-1}, y_0\rangle = \langle x_{i-1}, y_{i-1}\rangle$. Since A_m is a projective space and x_{i-1} is an $i-1$ dimensional subspace, $\langle x_{i-1}, y_0\rangle$ is an i dimensional subspace, and hence $|\langle x_{i-1}, y_0\rangle| = (\alpha^{i+1}-1)/(\alpha-1)$. On the other hand, we have $\langle x_{i-1}, y_{i-1}\rangle \supset x_i$, because x_i is contained in every elements of A_{m-1} containing x_{i-1} and y_{i-1}. By Lemma 3, $|x_i| = |\langle x_{i-1}, y_0\rangle|$. Therefore we have $x_i = \langle x_{i-1}, y_0\rangle$. Thus the proof is complete.

By Lemma 7 and Lemma 11, a proof of Theorem completes.

3. Another example

EXAMPLE 3. Let V be an m dimensional vector space over $GF(2)$ $(m \geqq 3)$, and H the set consisting of all $m-1$ dimensional subspaces of V. Put $A_{-1}=\{\phi\}$, $A_m=\{V-\{0\}\}$ and $A_i=\{M_i^c\cap\cdots\cap M_{m-i}^c\,|\,M_1\supsetneqq M_1\cap M_2\supsetneqq\cdots\supsetneqq\bigcap_{u=1}^{m-i}M_u$, $M_u\in H\}$ for $0\leqq i\leqq m-1$, where $M_u^c=V-M_u$. We say that $x_i\in A_i$ is incident with $x_j\in A_j$ if and only if $x_i\subset x_j$ $(i\leqq j)$. We shall show that $S_3=(\{A_i\}_{i=-1}^{m},\ T)$ is a partial geometric space of dimension m, where

$T=\{(x_{-1},\cdots,x_i,\cdots,x_j,\cdots,x_m)\in\prod_{i=-1}^{m}A_i\,|\,x_i$ is incident with x_j $(-1\leqq i<j\leqq m)\}$.

Lemma 12. *For $x_i\in A_i$, $|x_i|=2^i$ $(i\geqq 0)$.*

Proof. Let $x_i=\bigcap_{u=1}^{m-i}M_u^c$, then $\bigcap_{u=1}^{m-i}M_u$ is a subspace of dimension i. Therefore we have that by the principle of inclusion and exclusion $|x_i|=2^m+\sum_{u=1}^{m-i}\binom{m-i}{u}(-1)^u\,2^{m-u}=2^i(2-1)^{m-i}=2^i$.

REMARK A. Let $x_0\in A_0$ and $M\in H$ $(x_0\notin M)$. Since $V-\{0\}$ is a projective space, M^c is an affine space. Thus $M^c-\{x_0\}$ is a projective space over $GF(2)$.

At first, we define the intersection and the join. For $z_l\in A_l$ $(0\leqq l\leqq m-1)$, put $K(z_l)=\{M\in H\,|\,M^c\supset z_l\}$, and $K(z_{-1})=H$ and $K(z_m)=\phi$, where $z_{-1}\in A_{-1}$ and $z_m\in A_m$. Let x_i and y_j $(-1\leqq i,j\leqq m)$ be elements of A_i and A_j, respectively. Then a set $\bigcap_{u=1}^{m-l}L_u^c$ is defined to be an l intersection of x_i and x_j where elements L_u $(1\leqq u\leqq m-l)$ of $K(x_i)\cup K(y_j)$ satisfy $L_1\supsetneqq L_1\cap L_2\supsetneqq\cdots\supsetneqq\bigcap_{u=1}^{m-l}L_u$ and $\bigcap_{u=1}^{m-l}L_u\subset L$ for any element L of $K(x_i)\cup K(y_j)$. We denote $\bigcap_{u=1}^{m-l}L_u^c$ by $x_i\wedge y_j$. We note that if there exists an element L_{m-l+1} of $K(x_i)\cup K(y_j)$ such that $\bigcap_{u=1}^{m-l}L_u=\bigcap_{u=1}^{m-l+1}L_u$ and $\bigcap_{u=1}^{m-l}L_u^c=\bigcap_{u=1}^{m-l+1}L_u^c$, then x_i and y_j have a -1 intersection. Because let $\bar V=V/L_1\cap\cdots\cap L_{m-l}$ and $\bar L_u=L_u/L_1\cap\cdots\cap L_{m-l}$ $(1\leqq u\leqq m-l)$. By Lemma 12, $|\bar L_1^c\cap\cdots\cap\bar L_{m-l}^c|=1$, and hence $\bar L_1^c\cap\cdots\cap\bar L_{m-l+1}^c=\phi$. This implies $x_i\wedge y_j=\phi$.

Next, a set $\bigcap_{w=1}^{m-s}J_w^c$ is defined to be an s join of x_i and y_j where element J_w $(1\leqq w\leqq m-s)$ of $K(x_i)\cap K(y_j)$ $(\neq\phi)$ are satisfy $J_1\supsetneqq J_1\cap J_2\supsetneqq\cdots\supsetneqq\bigcap_{w=1}^{m-s}J_w$ and $\bigcap_{w=1}^{m-s}J_w\subset J$ for any element J of $K(x_i)\cap K(y_j)$. We denote $\bigcap_{w=1}^{m-s}J_w^c$ or $V-\{0\}$ by $x_i\vee y_j$ according to $K(x_i)\cap K(y_j)\neq\phi$ or $=\phi$. It is obvious that the intersection and the join of x_i and y_j is well-defined.

By the above paragraph, we have the following lemma.

Lemma 13. *Let K be a subset of H. Then $\bigcap_{N\in K}N^c$ is an element of A_l for some l.*

Lemma 14. *Let* $x_i=\bigcap_{w=1}^{m-i} M_w^c$ *and* $x_j=\bigcap_{u=1}^{m-j} N_u^c$ *be elements of* A_i *and* A_j, *respectively. If* $x_i\subset x_j$, *then* $\bigcap_{w=1}^{m-i} M_w\subset \bigcap_{u=1}^{m-j} N_u$.

Proof. Suppose that there exists N_z $(1\leqq z\leqq m-j)$ such that $\bigcap_{w=1}^{m-i} M_w\not\subset N_z$. Then $\bigcap_{w=1}^{m-i} M_w \supsetneqq \bigcap_{w=1}^{m-i} M_w\cap N_z$, so $x_{i-1}=\bigcap_{w=1}^{m-i} M_w^c\cap N_z^c$ is an element of A_{i-1}. Hence $\bigcup_{w=1}^{m-i} M_w\cup N_z\supsetneqq \bigcup_{w=1}^{m-i} M_w$ by Lemma 12. On the other hand, by the hypothesis $x_i\subset x_j$, $\bigcup_{w=1}^{m-i} M_w\supset \bigcup_{u=1}^{m-j} N_u$. Hence $\bigcup_{w=1}^{m-i} M_w\supset N_z$. This is a contradiction.

Lemma 15. *Let* W *be an* i *dimensional subspace of* V. *Then* $|\{x_i\in A_i|$ $x_i=\bigcap_{w=1}^{m-i} M_w^c$, *where* $\bigcap_{w=1}^{m-i} M_w=W\}|=2^{m-i}-1$.

Proof. Put $\bar{V}=V/W$. By Lemma 12, $|\bar{M}_1^c\cap\cdots\cap\bar{M}_{m-i}^c|=1$. Since $GL(m-i, 2)$ acts transitively on $\bar{V}-\{\bar{0}\}$, we have the lemma.

By Lemmas 14 and 15, we have the following:

Lemma 16. $|A_0|=2^m-1$ *and* $|A_i|=\left(\prod_{u=1}^{i}\frac{2^{m+1-u}-1}{2^u-1}\right)(2^{m-i}-1)$ *for* $m>i>0$.

Lemma 17. *Let* $x_i=\bigcap_{u=1}^{m-i} M_u^c$ *be an element of* A_i $(0\leqq i\leqq m-1)$, *then* $|K(x_i)|=2^{m-i-1}$.

Proof. Without loss of generality, we may assume $i=0$. By Lemma 12, put $M_1^c\cap\cdots\cap M_m^c=\{a\}$, that is every elements of H contained in $\bigcup_{u=1}^{m} M_u$ does not contain $\{a\}$. Since the number of hyperplanes of $V/\langle a\rangle$ equals $2^{m-1}-1$, the number in the lemma equals $(2^m-1)-(2^{m-1}-1)=2^{m-1}$.

Lemma 18. $k(0)=k(m-1)=2$ *and* $k(i)=3$ *for* $0<i<m-1$.

Proof. Let x_i be an element of A_i. Since $|A_0|=2^m-1$ by Lemma 16 and $|x_1|=2$ by Lemma 12, we have $k(0)=2$. For $k(m-1)$, consider a factor space. Then we have similarly that $k(m-1)=2$. For $0<i<m-1$, the lemma follows from Remark A and Example 2.

Lemma 19. *Let* a *and* x_i *be elements of* $V-\{0\}$ *and* A_i, *respectively. Assume that there exist elements* M *and* N *of* $K(x_i)$ *such that* $a\in M$ *and* $a\notin N$, *where* $i\geqq 0$. *Then* $|\{L\in K(x_i)\,|\,a\in L\}|=|\{L\in K(x_i)\,|\,a\notin L\}|$.

Proof. Without loss of generality, we may assume $\bigcap_{L\in K(x_i)} L=\{0\}$, that is, $i=0$. Put $X=\{L\in K(x_0)\,|\,a\in L\}$ and $Y=K(x_0)-X$. Let $y_j=\bigcap_{L\in X} L^c$ and $z_l=\bigcap_{L\in Y} L^c$. Since $\bigcap_{L\in X} L\ni a$, $j>0$. Since $z_l\ni a$, $z_l\neq x_0$, and hence $l>0$. Since $|K(x_0)|>|$

$K(y_j)|=2^{m-j-1}\geqq|X|$ and $|K(x_0)|>|K(z_l)|=2^{m-l-1}\geqq|Y|$, we have that $2^{m-1}=|X|+|Y|\leqq 2^{m-j-1}+2^{m-l-1}$. Hence $j=l=1$. This proves the lemma.

Lemma 20. *The geometric parameters are the following*:

(1) $t(i, i+2, 1)=3$ *for* $2<i+2<m$,

(2) $t(i, m, m-1)=1$ *and* $t(i, m, 1)=2$ *for* $0<i\leqq m-2$,

(3) $t(0, 2, 1)=$ *if* $\langle x_0, x_1\rangle$ *is a subspace of dimension* 3 *and* $t(0, m, 1)=2$ *if* $x_0\subset\langle x_1\rangle$, *where* x_u $(u=0, 1)$ *are elements of* A_u *such that* x_0 *is not incident with* x_1. *The rest geometric parameters need not be defined.*

Proof. (1) follows from Example 2 and Remark A. Let x_i and x_{i+1} be elements of A_i and A_{i+1}, respectively, such that they have an $(i-1)$ intersection x_{i-1} and an m join x_m. Considering a factor space, we may assume $i=1$. Put $x_0=\{a\}$, $x_1=\{a, b\}$ and $x_2=\{a, c, d, e\}$ by Lemma 12, where a, b, c, d and e are distinct elements of $V-\{0\}$. Since $x_m=x_1\vee x_2$, there exist elements M and N of $K(x_1)$ and $K(x_2)$, respectively, such that M does not contain a and b, and that N contains b and does not contain a, c, d and e. Let $Y=K(x_0)-K(x_1)$. Then $|Y|=2^{m-2}$ by Lemma 17 and $N\,(\in H)$ is contained in Y if and only if N contains b and does not contain a. Put $y_1=\bigcap_{N\in Y} N^c$, then $x_0\subset y_1\subset x_2$ since $\bigcap_{N\in Y} N\ni b$. Thus y_1 is an element of A_1 and $K(y_1)\cap K(x_1)=\phi$, since $Y=K(y_1)$. Therefore $y_1\vee x_1$ is contained in A_m and $t(1, m, m-1)\geqq 1$. Let $z_1=\{a, c\}$ and $w_1=\{a, d\}$. Since $K(y_1)\cap K(x_1)=\phi$ and $|K(x_i)|=|K(y_1)|=|K(x_0)|/2$, $K(z_1)\cap K(x_1)\neq\phi$ and $K(w_1)\cap K(x_1)\neq\phi$. This implies that there are elements M and N of $K(x_1)$ such that $c\in M$ and $c\notin N$. By Lemma 19, $|K(x_1)\cap K(z_1)|=|K(x_1)|/2$, and hence $x_1\vee z_1\in A_2$. Similarly $x_1\vee w_1\in A_2$. Therefore $t(1, m, 1)\geqq 2$. By the definition, $\sum_{u=1}^{m-1} t(1, m, u)=k(1)=3$. This implies (2). Next assume that $i=0$. Put $x_0=\{a\}$, $x_1=\{b, c\}$ and let $x_s=x_0\vee x_1$. Since $|A_1|=\binom{2^m-1}{2}$ by Lemma 16, $\{a, b\}$ and $\{a, c\}$ are contained in A_1. Thus $t(0, s, 1)=2$. If $\langle a, b\rangle\ni c$, then $|H|-3|\{M\in H\,|\,a\in M\}|+2|\{M\in H\,|\,M\supset\langle a, b\rangle\}|=(2^m-1)-3(2^{m-1}-1)+2(2^{m-2}-1)=0$. Therefore $K(x_0)\cap K(x_1)=\phi$, so $s=m$. If $\langle a, b\rangle\not\ni c$, then

$$\begin{aligned}&|H|-3|\{M\in H\,|\,a\in M \text{ and } b, c\notin M\}|+3|\{M\in H\,|\,a, b\in M\\ &\quad\text{and } c\notin M\}|-|\{M\in H\,|\,a, b, c\in M\}|\\ &\quad=(2^{m-1}-1)-3(2^{m-1}-1)+3(2^{m-2}-1)-(2^{m-3}-1)=2^{m-3}.\end{aligned}$$

Therefore $|\{M\in H\,|\,a, b, c\notin M\}|=|K(x_0\vee x_1)|=2^{m-3}$ and hence $x_0\vee x_1$ is an element of A_2. This completes a proof of the lemma.

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