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ON WEAKLY TRANSITIVE TRANSLATION PLANES

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1. Introduction

Let π^{l_∞} be a translation plane of order p^r with p a prime. Let G be a subgroup of the translation complement and Δ a subset of l_∞ with $|\Delta|=p+1$. π is said to be Δ-transitive if the following conditions are satisfied (V. Jha [4]):

(i) G leaves Δ invariant and acts transitively on l_∞-Δ.

(ii) G fixes at least two points of Δ.

(iii) G has a normal Sylow p-subgroup.

On Δ-transitive planes, V. Jha has proved the following theorem.

Theorem (V. Jha [4]). *If π^{l_∞} is Δ-transitive with $|\Delta|=p+1$, then π has order p^2 and $\Delta=\pi_0\cap l_\infty$ where π_0 is a subplane of order p.*

If $(\pi^{l_\infty}, \Delta, G)$ satisfies the conditions (i) and (ii) above, π is said to be weakly transitive.

In his paper [4], V. Jha has conjectured that weakly transitive planes are the Hall planes of order p^2, the Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16.

In this paper we prove the following theorems on weakly transitive planes.

Theorem 1. *Let π^{l_∞} be a translation plane of order p^r with p a prime and Δ a subset of l_∞ with $|\Delta|=p+1$. If a subgroup G of the translation complement of π leaves Δ invariant and acts transitively on $l_\infty-\Delta$, then one of the following holds.*

(i) *$O_p(G)$ is semiregular on $\Delta-\{A\}$ for some point $A\in\Delta$.*

(ii) *π has order p^2.*

(iii) *π has order p^3 and G is transitive on Δ.*

The Lorimer-Rahilly plane of order 16 and the Johnson-Walker plane of order 16 are examples of the case (i). The Hall planes of order p^2 and the plane of order 25 constructed by M.L. Narayana Rao and K. Satyanarayana in [6] are examples of the case (ii). The desarguesian plane of order 27 is an example of the case (iii).

As an immediate corollary we have the following.

Theorem 2. *Suppose* $(\pi^{l_\infty}, \Delta, G)$ *with* $|\Delta|=p+1$ *is weakly transitive. If* $O_p(G)\neq 1$, *then* π *has order* p^2 *and* $\Delta=F(O_p(G))\cap l_\infty$.

We note that if π^{l_∞} is Δ-transitive, then it satisfies the assumption of Theorem 2.

2. Proof of Theorem 1

We prove Theorem 1 by way of contradiction. Assume that $(\pi^{l_\infty}, \Delta, G)$ is a counterexample such that $p^r+|G|$ is minimal. Therefore $r\geq 3$ and $O_p(G)\neq 1$.

Throughout the paper we use the following notations.

T: the group of translations of π

$M(=O_p(G))$: the maximal normal p-subgroup of G

$F(H)$: the fixed structure consisting of points and lines of π fixed by a nonempty subset H of G.

n_p: the highest power of a prime p dividing a positive integer n

Γ: $l_\infty-\Delta$.

Other notations are taken from [1] and [2].

Lemma 1. *$F(M)$ is a subplane of π of order p and $\Delta=F(M)\cap l_\infty$.*

Proof. Let K be the pointwise stabilizer of Δ in G and assume that $M\not\leq K$. We denote by $\bar{G}$ the restriction of G on Δ. Clearly $\bar{G}\rhd\bar{M}\neq 1$ and as $|\Delta|=p+1$, $\bar{M}$ is a Sylow p-subgroup of $\bar{G}$. By the Schur-Zassenhaus' theorem (Theorem 6.2.1 of [1]), there is a subgroup $\bar{L}$ of $\bar{G}$ such that $K<L$ and $|\bar{G}:\bar{L}|=p$, $\bar{G}=\bar{M}\bar{L}$.

Set $N=M\cap K$. We have $N\neq 1$, for otherwise π satisfies (i) of Theorem 1, contrary to the minimality of π. As $G\rhd K$, $G\rhd N$. It follows from the transitivity of G on Γ that N is $\frac{1}{2}$-transitive on Γ.

Let Ψ be the set of N-orbits on Γ. Since there is no nontrivial homology of order p, N acts faithfully on Γ. As $N\neq 1$ and $|\Gamma|_p=p$, $|\Psi|=|\Gamma|/p=p^{r-1}-1$. Hence Ψ coincides with the set of M-orbits on Γ.

Since $G=ML$, L is transitive on Ψ by the last paragraph. Hence L is transitive on Γ as $N<L$. From this $(\pi^{l_\infty}, \Delta, L)$ satisfies (ii) or (iii) of Theorem 1 by the minimality of $(\pi^{l_\infty}, \Delta, G)$. Therefore $(\pi^{l_\infty}, \Delta, G)$ also satisfies (ii) or (iii) of Theorem 1. This is a contradiction. Thus $M\leq K$.

Since $F(M)\cap\Gamma=\phi$, $F(M)\cap l_\infty=\Delta$, so that $F(M)$ is a subplane of π of order p.

Lemma 2. *If $p=2$, then r is even.*

Proof. Assume $p=2$. Let x be an involution in M. Since $F(x)$ contains Δ by Lemma 1, $F(x)$ is a subplane of π. By a Baer's theorem (Thoerem 4.3 of [2]), $F(x)$ is of order $\sqrt{2^r}$. Thus r is even.

Lemma 3. *Let t be a prime p-primitive divisor of $p^{r-1}-1$ and let x be a nontrivial t-element of G. If x centralizes M, then $F(x)\cap\Delta=\phi$.*

Proof. Let $A\in F(x)\cap\Delta$ and set $U=T(A)$, the set of translations of T with center A. Clearly $|U|=p^r$. By Lemma 1, $|C_U(M)|=p$ as U is regular on the set of affine points on the line OA. Set $R=\langle x\rangle$. Since R normalizes $C_U(M)$ and $t\not| p-1$, $C_U(R)$ contains $C_U(M)$.

If $C_U(R)\neq C_U(M)$, R acts trivially on $U/C_U(R)$ as $|U/C_U(R)|<p^{r-1}$ and t is a p-primitive divisor of $p^{r-1}-1$. Hence $[R, U]=1$ by Theorem 5.3.2 of [1]. Therefore x is a homology with axis OA and so $t|(p^{r-1}-1, p^r-1)=p-1$, a contradiction. Thus $C_U(R)=C_U(M)$.

By Theorem 5.2.3 of [1], $U=C_U(R)\times[U, R]$. Since M centralizes R and normalizes U, it also normalizes $[U, R]$. Hence $1\neq C_{[U,R]}(M)\leq C_U(M)=C_U(R)$, a contradiction. Thus $F(x)\cap\Delta=\phi$.

Lemma 4. *If $r=3$, then $p\equiv-1 \pmod 4$.*

Proof. By a Baer's theorem and Lemma 1, $p\neq2$ and $|M|=p$ as $r=3$. Assume $p\equiv1 \pmod 4$ and let t be an odd prime dividing $p+1$. Clearly t is a prime p-primitive divisor of $p^{r-1}-1=p^2-1$. Since $|M|=p$ and $t\not| p-1$, a Sylow t-subgroup R of G centralizes M. Applying Lemma 3, R is semiregular on Δ. As $p+1||G|$ and t is arbitrary, the length of each G-orbit on Δ is divisible by $(p+1)/2$. Since π is a counterexample of Theorem 1, G has two orbits of length $(p+1)/2$ on Δ.

Let S be a Sylow 2-subgroup of G and let $X\in F(S)\cap\Delta$. Set $\pi_0=F(M)$, $S_0=S_{(0,l_\infty)}$ and $K=G_\Delta$, the pointwise stabilizer of Δ in G. Since M is a nontrivial normal subgroup of G, π_0 is G-invariant and isomorphic to $PG(2, p)$. The restriction of $\mathrm{Aut}(PG(2, p))$ on the line at infinity is isomorphic to $PGL(2, p)$ in its usual 2-transitive permutation representation. Hence G/K is isomorphic to a subgroup of $PGL(2, p)$. As $|G/K|$ is divisible by $(p+1)/2$, G/K is isomorphic to a subgroup of the dihedral group of order $2(p+1)$ by a Dickson's theorem (Theorem 14.1 of [5]). Since G/K is not transitive on Δ, $|G/K|=(p+1)/2$ or $p+1$. Therefore $|S: S\cap K|=1$ or 2. Hence $S\cap K$ is semiregular on $F(M)\cap(OX-\{O, X\})$ and so $|S\cap K|\,|(p-1)_2$. From this, $|S|\leq2(p-1)_2$. But, as $S\cap K\neq1$, $S_0\neq1$ and so $|S/S_0|\geq|\Gamma|_2=2(p-1)_2$. This implies $|S|\geq4(p-1)_2$, a contradiction.

Lemma 5. *Let S be a 2-group acting faithfully on an elementary abelian p-group W of order p^r with $p^r\equiv-1 \pmod 4$. If an element $x\in S$ inverts W, then $S=\langle x\rangle\times S_1$ for a subgroup S_1 of S.*

Proof. We may assume that $S\leq GL(r, p)$ and $x=-I$, where I is the unit matrix of degree r. Since r is odd, $\det(x)=(-1)^r=-1$ and so $x\notin SL(r, p)$.

Since $2|p-1$ and $4\not| p-1$, $\langle x\rangle\times SL(r,p)$ is a normal subgroup of $GL(r,p)$ of odd index. Thus $S=\langle x\rangle\times S_1$, where $S_1=S\cap SL(r,p)$.

Lemma 6. *Let S be a Sylow 2-subgroup of G. If $r=3$, then the length of every S-orbit on Δ is divisible by $|\Delta|_2$.*

Proof. By Lemma 4, $p\equiv-1 \pmod 4$. Since G is transitive on Γ, $|\Gamma|=p(p^2-1)|\,|G|$ and so $2(p+1)_2|\,|S/S_0|$, where S is a Sylow 2-subgroup of G and $S_0=S_{(0,l_\infty)}$. Hence $|S_X|\geq 2\times|S_0|$ for some point $X\in\Delta$. Here S_X denotes the stabilizer of X in S. Let $Y\in F(S_X)\cap(\Delta-\{X\})$.

First we show that $S_0\neq 1$. Assume that $S_0=1$ and let u be an involution in $Z(S_X)$. By a Baer's theorem, any involution in S is a homology. Hence either u is a (X, OY)-homology or u is a (Y, OX)-homology. In either case $C_S(u)\leq S_X$. As $u\in Z(S_X)$, $C_S(u)=S_X$. In particular $|S_X|\geq 4$.

We note that either $S_{(X,OY)}=1$ or $S_{(Y,OY)}=1$, for otherwise $S_0\neq 1$ by Lemma 4.22 of [2]. Let $A\in\{X, Y\}$ such that $S_{(B,OA)}=1$, where $\{B\}=\{X, Y\}-\{A\}$. Then S_X acts faithfully on $T(A)$. In particular every involution in S_X fixes no affine point on $OA-\{O\}$. Therefore every involution in S_X inverts $T(A)$. From this S_X has exactly one involution. But, by Lemma 5, S_X contains a subgroup isomorphic to $Z_2\times Z_2$, a contradiction. Thus $S_0\neq 1$.

Let z be an involution in S_0. Since O is the only affine fixed point of z, z inverts T. As $(p-1)_2=2$, $\langle z\rangle$ is a unique Sylow 2-subgroup of $G_{(0,l_\infty)}$.

Set $V=S_X$. If $V_{(X,OY)}=1$, then V acts faithfully on $T(Y)$ and moreover z inverts $T(Y)$. By Lemma 5, V contains a subgroup U such that $z\in U$ and U is isomorphic to $Z_2\times Z_2$. By Lemma 4.22 of [2], we obtain a contradiction. Hence $V_{(X,OY)}\neq 1$.

Let u be an involution in $V_{(X,OY)}$. Then, as $u\in Z(V)$, we have $C_S(u)=V$. Assume $|V|>4$. $\bar{V}=V/\langle u\rangle$ normalizes $T(Y)$ and z inverts $T(Y)$. Hence $\bar{V}=\langle\bar{z}\rangle\times\bar{L}$ for a subgroup L of V with $u\in L$ by Lemma 5. Since $L_{(0,l_\infty)}=1$ and $u\in L$, L acts faithfully on $T(X)$ and u inverts $T(X)$. Hence $L=\langle u\rangle\times Z$ for a subgroup Z of L by Lemma 5. As $|L|\geqq 4$, Z contains an involution. Therefore $Z_{(0,l_\infty)}\neq 1$ or $Z_{(Y,OX)}\neq 1$, a contradiction. Thus $|V|=4$.

As $V\leq S_Y$ and $F(V)\cap l_\infty=\{X, Y\}$, we have $V=S_Y$. Since V is isomorphic to $Z_2\times Z_2$ and $C_S(u)=V$, S is dihedral or semidihedral by a lemma of [7]. Therefore any involution in S is S-conjugate to an involution in V. Hence, if $S_Q\neq 1$ for some $Q\in\Delta$, then $Q=X^s$ or Y^s for some $s\in S$. Thus $|S_Q|=|V|=4$. Therefore $|Q^S|\geq 2|\Gamma|_2/4=(p+1)_2$ for all $Q\in\Delta$.

Lemma 7. $r\neq 3$.

Proof. Assume that $r=3$. Let t be an odd prime dividing $p+1$. Then

t is a prime p-primitive divisor of p^2-1. Let R be a Sylow t-subgroup of G. Since G is transitive on Γ, $p(p^2-1)=|\Gamma|\,|\,|G|$ and so $R\neq 1$. By Lemma 1, $|M|=p$ as $r=3$. Hence R centralizes M. Applying Lemma 3, R acts semiregularly on Δ. Since t is arbitrary, using Lemma 6 we have that G acts transitively on Δ. As π is a counterexample, this is a contradiction. Thus we we have the lemma.

Lemma 8. *There exists a prime p-primitive divisor t of $p^{r-1}-1$ such that $t\,|\,|G|$ and $t\nmid |C_G(M)|$.*

Proof. $|G|$ is divisible by $p^{r-1}-1$ as $|\Gamma|\,|\,|G|$. By Lemmas 1 and 7, $r-1\geq 3$ and by Lemma 2, $(p, r-1)\neq(2, 6)$. It follows from a Zsigmondy's theorem (Theorem 6.2 of [5]) that there exists a prime p-primitive divisor t of $p^{r-1}-1$.

Assume $t\,|\,|C_G(M)|$ and let R be a Sylow t-subgroup of $C_G(M)$. By Lemma 3, R is semiregular on Δ. Hence $t|p+1$ and so $t|p^2-1$. Since t is a p-primitive divisor of $p^{r-1}-1$, we have $r-1=2$, contrary to Lemma 7.

Lemma 9. *Each M-orbit on Γ is of length p.*

Proof. Since $p\,|\,|\Gamma|$, $p^2\nmid|\Gamma|$ and M is $\frac{1}{2}$-transitive on Γ, using Lemma 1 each M-orbit on Γ has length p.

Proof of Theorem 1.

Let t be a prime as in Lemma 8 and let R be a Sylow t-subgroup of G. By Lemma 8, $R\neq 1$ and acts faithfully on M. Since t is a p-primitive divisor of $p^{r-1}-1$, we have $|M|\geq p^{r-1}$. Hence, by Proposition 6.12 of [3], $p^r=16$. From this, $p=2$, $t=7$ and $|M|\geq 8$.

Let $A\in\Gamma$ and set $N=M_A$. By Lemma 1, $F(N)\supset\Delta\cup\{A\}$. Therefore $F(N)$ is a subplane of order 4. Let $B\in l_\infty-F(N)\cap l_\infty$. Clearly $F(N_B)=\pi$ and so $N_B=1$. By Lemma 9, $|M\colon N|=2$ and $|N\colon N_B|=2$. Hence $|M|=4$, a contradiction. Thus we have Theorem 1.

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