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# ORTHOGONAL GROUPS AND SYMMETRIC SETS

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Orthogonal groups are considered as automorphism groups of some symmetric sets of vectors. From this point of view, we can prove the well-known theorem of simplicity on orthogonal groups. (The cases for the other classical groups are given in [5].) The proof consists of two steps. The first step which will be given in **1** is to show that a transitive symmetric set of non-isotropic lines (of a certain type) is simple. After a short review on simple symmetric set is given, we will show the above fact. A point here is that it is so when dim $V$ is 3. The second step is to show that the group of displacements of the simple symmetric set is a simple group, which will be given in **2**. A useful supplement to the main theorem on simple symmetric sets will be found, and using it we can show the above fact when dim $V \geq 5$.

## 1. A simple symmetric set of non-isotropic lines

Let $V$ be a vector space over a field of characteristic $\neq 2$ with a non-singular orthogonal metric. Since the following results hold in a stronger sense for a finite field as was shown in [4], we assume in this note that the base field $k$ is infinite. Suppose that dim $V \geq 3$ and that $V$ contains a hyperbolic plane. Then, there exists a vector $v$ such that $v$ is orthogonal to a hyperbolic plane and $(v, v) = \varepsilon \neq 0$. Throughout this note, we fix the element $\varepsilon$. Now we consider $A = \{\bar{u} \mid (u, u) = \varepsilon\}$, where $\bar{u} = \langle u \rangle =$ a subspace generated by $u$. On $A$, we define a binary operation: $\bar{u} \circ \bar{v} = \bar{w}$ with $w = u^{\tau_v}$, where $\tau_v$ is the symmetry with respect to the hyperplane orthogonal to $v$. $A$ is then a symmetric set, i.e., satisfies $\bar{u} \circ \bar{u} = \bar{u}$, $(\bar{u} \circ \bar{v}) \circ \bar{v} = \bar{u}$ and $(\bar{u} \circ \bar{v}) \circ \bar{w} = (\bar{u} \circ \bar{w}) \circ (\bar{v} \circ \bar{w})$.

We summarize some definitions and properties on simple symmetric sets. Let $S = \{a, b, c, \cdots\}$ be a symmetric set. The right multiplication by an element $a$ is an automorphism of $S$, which we denote by $\sigma_a$. Let $G(S) = \langle \sigma_a \mid a \in S \rangle$ and $H(S) = \langle \sigma_a \sigma_b \mid a, b \in S \rangle$. The latter is called the group of displacements of $S$. Let $T$ be another symmetric set. A homomorphism $f$ of $S$ onto $T$ is called proper if it is not one to one and if $T$ contains more than one element. When $G(S) = 1$, we say $S$ is trivial. A non-trivial symmetric set is called simple if it has no proper homomorphism (to some symmetric set). It is important to characterize a simple symmetric set in a different way as follows. Let $f$ be a homomorphism of $S$ to $T$. $f^{-1}(t)$, the set of all inverse images, for an element $t$ in $T$ is called a coset of $f$. Then, $S$ is decomposed into disjoint cosets of $f$: $S=\cup S_i$ with $S_i=f^{-1}(t_i)$ with $S_i \cap S_j=\phi$ if $i \neq j$. In this case, as is easily seen, $\sigma_a$ induces a permutation on $X=\{S_i\}$, the set of cosets $S_i$. It is also clear that $\sigma_a$ and $\sigma_b$ induce the same permutation on $X$ if $a$ and $b$ belong to a same coset. Conversely, suppose that $S=\cup S'_i$ is a disjoint union of subsets $S'_i$ satisfying the above two conditions on $\sigma_a$. Then we can define a symmetric structure on $X=\{S'_i\}$ in such a way that the restriction mapping of $S$ to $X$ is a homomorphism. Thus the concept of homomorphisms is equivalent with that of coset-decompositions. When a homomorphism $f$ is proper, we say that the corresponding coset-decomposition is proper. The simplicity of a non-trivial symmetric set is now characterized by the fact that it has no proper coset-decomposition.

A symmetric set $S$ is called transitive if $G(S)$ (or, equivalently $H(S)$) is a transitive permutation group on $S$. If $S$ is simple, then $S$ is transitive. For, otherwise, $S=\cup S_i$ with $S_i$=orbits of elements in $S$ by $G(S)$ would be a proper coset-decomposition. A symmetric set is called effective if $\sigma_a \neq \sigma_b$ whenever $a \neq b$. The following is the main theorem on simple symmetric set obtained in [2] and [3]. (See also [5].)

**Theorem.** *Suppose that $S$ is a transitive and effective symmetric set. Then, $S$ is simple if and only if $H(S)$ is a minimal normal subgroup of $G(S)$. The latter condition is equivalent with that $H(S)$ is either a simple group or a direct product of simple subgroups $N_1$ and $N_2$ such that $N_2=N_1^{\sigma_a}$ for any element $a$ in $S$. In the latter case, $N_i$ are regular permutation groups on $S$.*

As the last part of Theorem is not explicitly given in the previous papers, we explain it here. It is known that $N_i$ are transitive on $S$. Now suppose that $a^\tau=a$ for an element $a$ in $S$ and $\tau$ in $N_i$. Then, $\tau^{-1}\sigma_a\tau=\sigma_a$, or $\tau$ and $\sigma_a$ are commutative, and so $\tau=\sigma_a^{-1}\tau\sigma_a$, which belongs to $N_1$ as well as to $N_2$. Thus, $\tau=1$.

We now return to $V$ and $A$. Let $O(V)$ be the orthogonal group of $V$ and $\Omega=\Omega(V)$ its commutator subgroup. Elements of $O(V)$ naturally induce automorphisms of $A$. So, there is a natural homomorphism $h$ of $O(V)$ to the group of automorphisms of $A$. The kernel of $h$ is $Z=\{\pm 1\}$, the center of $O(V)$, due to Lemma 5.5, p. 206 [1]. Thus, $PO(V)=O(V)/Z$ is condisered as a group of automorphisms of $A$. In this respect, we want to show that $P\Omega=H(A)$. This is equivalent with $\Omega=h^{-1}(H(A))$. Since $\Omega$ is generated by $\tau_v\tau_w$ with $(v, v)=(w, w)\neq 0$, it is clear that $H(A)\subset P\Omega$. When dim $V=3$, $P\Omega$ is a simple group (Theorem 5.20, [1]). So, in this case, $H(A)=P\Omega$. To show it in a general case and also to show the simplicity of $A$, we use the following

**Lemma.** *If $(u, u)=(v, v)=\varepsilon$, then there exists a hyperbolic plane $P$ such that $(u, P)=0$ and $v\in\langle u\rangle+P$.*

Proof. If $\langle u, v\rangle$ is singular, it is easy to find the above $P$. Suppose that $\langle u, v\rangle$ is non-singular. Then, $v=\alpha u+v'$, where $v'\in\langle u\rangle^{\perp}$ and $(v', v')\neq 0$. (Naturally we are assuming $\dim\langle u, v\rangle=2$.) By the assumption on $\varepsilon$, $\langle u\rangle^{\perp}$ contains a hyperbolic plane. Since a hyperbolic plane contains an element $w$ such that $(w, w)$ is any element in the base field, i.e., is universal, we can find an isometry on $\langle u\rangle^{\perp}$ by Witt theorem which maps $w$ to $v'$. We can let $P$ be the image of the hyperbolic plane under the isometry.

From Lemma, we can conclude that $H(A)=P\Omega$, as we can always restrict our consideration to a 3-dimentional case. We can also conclude that $A$ is transitive. For, $\Omega$ acts on $A$ transitively. Here note that $\langle u\rangle^{\perp}$ is universal and we can insert $\sigma_a$ with no effect on $u$ where $(a, a)$ is any prescribed value. Now we show that $A$ is simple. Assume that $A$ is not simple. Then there exists a proper coset-decomposition $A=\cup A_i$. Let $\bar{a}$ and $\bar{b}$ be two distinct elements in $A_1$. Let $U=\langle a\rangle+P$, where $(a, P)=0$ and $b\in U$ from Lemma. Then $A(U)=\{\bar{u}\,|\,u\in U,\ (u, u)=\varepsilon\}$ is simple by the main theorem. Restrict $A=\cup A_i$ to the elements in $A(U)$. We can conclude that $A(U)\subset A_1$. Now let $Q$ be a hyperbolic plane such that $(a, Q)=0$. Since $O(V)=O(P)\Omega$ as we know, there exists an element in $\Omega$ which fixes $a$ and maps $P$ to $Q$. This implies that $A_1$ contains every element $\bar{w}$ such that $w\in\langle u\rangle+Q$, $(w, w)=\varepsilon$. Using Lemma, we can conclude that $A=A_1$, which is a contradiction. Thus, $A$ is simple.

## 2. Simplicity of the group $H(A)$

Let $S=\{a, b, \cdots\}$ be an effective simple symmetric set. Suppose that $H(S)$ is not a simple group. Then, $H(S)=N_1\times N_2$ with simple subgroups $N_i$ such that $N_2=N_1^{\sigma_a}$ for any element $a$ in $S$. Moreover, $N_i$ are regular permutation groups on $S$. Let $\tau$ be an element in $N_1$ and express it as a product of disjoint cyclic permutations: $\tau=(\cdots, a, b, c, \cdots)$. $(\cdots)\cdots$ We show that $a^{\sigma_b}=c$. Since $N_1$ and $N_2$ are commutative, we have $\tau\tau^{\sigma_b}=\tau^{\sigma_b}\tau$, or $\tau\sigma_b\tau\sigma_b=\sigma_b\tau\sigma_b\tau$. Therefore, $\sigma_b(\tau\sigma_b\tau\sigma_b)=\tau\sigma_b\tau=(\tau\sigma_b\tau\sigma_b)\sigma_b$. So, $\rho^{-1}\sigma_b\rho=\sigma_b$, where $\rho=\tau\sigma_b\tau\sigma_b=\tau\tau^{\sigma_b}$. Since $\rho^{-1}\sigma_b\rho=\sigma_{b^\rho}$ and $S$ is effective, we have $b^\rho=b$, or $b^{\tau\tau^{\sigma_b}}=b$. So, $b^{(\tau^{\sigma_b})^{-1}}=b^{\tau}=c$. On the other hand, $\tau^{\sigma_b}=(\cdots, a^{\sigma_b}, b, c^{\sigma_b}, \cdots)\cdots$. So, $b^{(\tau^{\sigma_b})^{-1}}=a^{\sigma_b}$. Therefore, $a^{\sigma_b}=c$, as required. In the above, $(\cdots, a, b, c, \cdots)$ coincides with a cycle defined in the theory of symmetric set. Note also that $\tau$ is a product of cycles of the same length and every element of $S$ must appear in a cycle. This is a supplement to the main theorem on simple symmetric sets. In the above, especially, $\tau=(a, b)\,(c, d)\cdots$ if $a^{\sigma_b}=a$, $c^{\sigma_d}=c$, etc. Since $N_1$ is regular, such an element $\tau$ exists if there exist $a$ and $b$ such that $a^{\sigma_b}=a$. In this case, we can show that if an element $\sigma$ in $H$ fixes $a$, $b$ and $c$, then $\sigma$ must fix $d$ as well. For, $\tau^{\sigma}=(a, b)(c, d^{\sigma})\cdots$ must coincide with $\tau$ as both are elements in a regular permutation group and move $a$ to $b$.

Now we return to $A$. Assume that dim $V\geq 5$ and that $H(A)$ is not simple. Thus, $H(A)=N_1\times N_2$ as above. Let $u_1$ be an element in $V$ such that $(u_1, u_1)=\varepsilon$ and let $P$ be a hyperbolic plane orthogonal to $u_1$. Select $u_2$ in $P$ such that $(u_2, u_2)=\varepsilon$. Since $N_1$ is transitive, there exists an element $\tau$ in $N_1$ such that $\bar{u}_1^{\tau}=\bar{u}_2$. Then, $\tau=(\bar{u}_1, \bar{u}_2)(\bar{v}, \bar{w})\cdots$, where we assume that $v\in P$. Since $\bar{v}^{\sigma_{\bar{w}}}=\bar{v}$, we have $(v, w)=0$. Also we have that $(u_2, w)=0$, because $\tau$ maps $\bar{u}_1$ and $\bar{v}$ to $\bar{u}_2$ and $\bar{w}$ respectively and $(u_1, v)=0$. Thus, $(w, P)=0$ as $P=\langle u_2, v\rangle$. Let $W=P^{\perp}$. $W$ contains $u_1$ and $w$ and dim $W\geq 3$. There exists an element $\rho$ in $\Omega(W)$ such that $\bar{u}_1^{\rho}=\bar{u}_1$ and $\bar{w}^{\rho}\neq\bar{w}$. For example, let $\rho$ be a rotation around $u_1$ in some non-singular subspace of dim 3 containing $u_1$ and $w$. Then, $\rho$ (extended as an element in $\Omega(V)$ in a natural way) fixes $\bar{u}_1$, $\bar{u}_2$ and $\bar{v}$ but moves $\bar{w}$, which contradicts the above argument. Thus, we have shown that $H(A)$ must be a simple group if dim $V\geq 5$.

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