Kakutani, S.
Osaka J. Math.
18 (1981), 533–554

EQUIVARIANT *KO*-RINGS AND *J*-GROUPS OF SPHERES WHICH HAVE LINEAR PSEUDOFREE S^1-ACTIONS

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(Received January 30, 1980)

1. Introduction

In this paper, we consider the equivariant *KO*-rings and *J*-groups of spheres which have linear pseudofree circle actions.

Let S^1 be the circle group consisting of complex numbers of absolute value one. For a sequence $p=(p_1, p_2, \cdots, p_m)$ of positive integers, we define the S^1-action φ_p on the complex m-dimensional vector space C^m by

$$\varphi_p(s, (z_1,z_2, \cdots, z_m)) = (s^{p_1}z_1, s^{p_2}z_2, \cdots. s^{p_m}z_m)$$

and denote by

$$S^{2m-1}(p_1, p_2, \cdots, p_m)$$

the unit sphere S^{2m-1} in C^m with this action φ_p. Then the S^1-action on $S^{2m-1}(p_1, p_2, \cdots, p_m)$ is said to be *pseudofree* (resp. *free*) if $(p_i, p_j)=1$ for $i \neq j$ and $p_i>1$ for some $1\leqq i\leqq m$ (resp. $p_1=p_2=\cdots=p_m=1$) (see Montgomery-Yang [19], [20]).

The main results of our paper are as follows:

Theorem 4.7. *Let p_i $(1\leqq i\leqq m)$ be positive odd integers such that $(p_i, p_j)=1$ for $i\neq j$. Then there is a monomorphism of rings:*

$$\Phi\colon KO_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m)) \to KO(CP^{m-1})\oplus \bigoplus_{i=1}^{m} RO(Z_{p_i})\,.$$

(For details see §4.)

Let G_i $(i\geqq 1)$ denote the stable homotopy group $\pi_{n+i}(S^n)$ $(n\geqq i+2)$. We define $s(k)=\prod_{i=1}^{k}|G_i|$ for $k>0$, where $|G_i|$ denotes the order of the group G_i and put $s(-1)=1$.

Theorem 5.4. *Let p_i $(1\leqq i\leqq m)$ be positive odd integers such that $(p_i, p_j)=1$ for $i\neq j$ and $(p_i, s(2m-3))=1$ for $1\leqq i\leqq m$. Then there is a monomorphism of groups:*

$$\tilde{\Phi}\colon J_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m)) \to J(CP^{m-1}) \oplus \bigoplus_{i=1}^{m} J_{Z_{p_i}}(*)\,.$$

(For details see §5.)

The paper is organized as follows:

In §§2 and 3, we consider a generalization of the results due to Folkman [9] and Rubinsztein [23] and prove some preliminary results. In §§4 and 5, we study an isomorphism and an S^1-fiber homotopy equivalence of real S^1-vector bundles over the pseudofree S^1-manifold $S^{2m-1}(p_1, p_2, \cdots, p_m)$ respectively. In §6, we consider the problem on quasi-equivalence posed by Meyerhoff and Petrie ([18], [21]).

2. Equivariant homotopy

Let n be a positive integer. Denote by $\boldsymbol{Z}_n$ the cyclic group $\boldsymbol{Z}/n\boldsymbol{Z}$ of order n. If V is a real representation space of $\boldsymbol{Z}_n$, we denote by $S(V)$ its unit sphere with respect to some $\boldsymbol{Z}_n$-invariant inner product. Denote by $[X, Y]$ the set of homotopy classes of maps from X to Y. In this section, we shall prove the following theorem (cf. Folkman [9; Proposition 2.3] and Rubinsztein [23; Corollary 5.3]).

Theorem 2.1. *Let V be a complex $\boldsymbol{Z}_n$-representation space such that $\boldsymbol{Z}_n$ acts freely on $S(V)$ and $\dim_{\boldsymbol{R}} V=2m$. Let X be a $\boldsymbol{Z}_n$-space which satisfies the following conditions:*

(i) *X is path-connected and q-simple for $1 \leqq q \leqq 2m-1$,*

(ii) *the map of X into itself given by the action of a generator of $\boldsymbol{Z}_n$ is homotopic to the identity,*

(iii) $\begin{cases} \mathrm{Hom}(\boldsymbol{Z}_n, \pi_{2i-1}(X))=0 & \text{for } 1 \leqq i \leqq m\,, \\ \mathrm{Ext}(\boldsymbol{Z}_n, \pi_{2i}(X))=0 & \text{for } 1 \leqq i \leqq m-1\,. \end{cases}$

If there exist $\boldsymbol{Z}_n$-maps $f_0, f_1\colon S(V) \to X$ such that $[f_0]=[f_1] \in [S^{2m-1}, X]$, then f_0 and f_1 are $\boldsymbol{Z}_n$-homotopic.

Before beginning the proof of Theorem 2.1, we require some notations and lemmas.

Let M be a $\boldsymbol{Z}_n$-space $S(V) \times [0, 1]$, where $[0, 1]$ is the unit interval with the trivial $\boldsymbol{Z}_n$-action. Then M is a compact smooth $\boldsymbol{Z}_n$-manifold with a free $\boldsymbol{Z}_n$-action. Let x_0 be a point of $S(V)$. We put $N=S(V) \times \{0, 1\} \cup \{x_0\} \times [0, 1]$ and $M'=M/\boldsymbol{Z}_n$. Let $\pi\colon M \to M'$ be the natural projection. We put $N'=\pi(N)$.

Let R be an arbitrary abelian group. By the universal-coefficient theorem, we have the following lemmas.

Lemma 2.2. *There are isomorphisms:*

$$H^q(M, N; R) = 0 \qquad \text{for } 0 \leqq q \leqq 2m-1\,,$$
$$H^{2m}(M, N; R) \cong R\,.$$

Lemma 2.3. *There are isomorphisms:*

$$\begin{aligned} &H^0(M', N'; R) = H^1(M', N': R) = 0\,, \\ &H^{2q-1}(M', N'; R) \cong \mathrm{Ext}(\boldsymbol{Z}_n, R) \qquad \textit{for } 2\leqq q\leqq m\,, \\ &H^{2q}(M', N'; R) \cong \mathrm{Hom}(\boldsymbol{Z}_n, R) \qquad \textit{for } 1\leqq q\leqq m-1\,, \\ &H^{2m}(M', N'; R) \cong R\,. \end{aligned}$$

Since the $\boldsymbol{Z}_n$-action on M is free and orientation-preserving, we have

Lemma 2.4. *Assume that* $\mathrm{Hom}(\boldsymbol{Z}_n, R)=0$. *Then the homomorphism*

$$\pi^*\colon H^{2m}(M', N'; R) \to H^{2m}(M, N; R)$$

is injective.

Proof of Theorem 2.1. In order to prove Theorem 2.1, it suffices to show that there exists a $\boldsymbol{Z}_n$-map $F\colon M\to X$ such that $F\,|\,S(V)\times\{0\}=f_0$ and $F\,|\,S(V)\times\{1\}=f_1$.

Since $[f_0]=[f_1]\in[S^{2m-1}, X]$, there exists a continuous map $F'\colon M\to X$ such that $F'\,|\,S(V)\times\{0\}=f_0$ and $F'\,|\,S(V)\times\{1\}=f_1$. Since M is a compact smooth $\boldsymbol{Z}_n$-manifold and $\boldsymbol{Z}_n$ acts freely on M, we can consider the fiber bundle $\mathscr{B}$:

$$X \to M\underset{\boldsymbol{Z}_n}{\times} X \to M'\,.$$

A cross-section s_0 of the part of $\mathscr{B}$ over N' $(=\pi(N))$ is defined by

$$s_0(\pi(z)) = [z, F'(z)]\in M\underset{\boldsymbol{Z}_n}{\times} X \qquad \text{for } z\in N\,.$$

To prove Theorem 2.1, it suffices to show that the cross-section s_0 defined on N' is extendable to a full cross-section of $\mathscr{B}$. Because there is a one-to-one correspondence between $\boldsymbol{Z}_n$-maps from M to X and cross-sections of $\mathscr{B}$.

Let K be a simplicial complex. Denote by K^q the q-skelton. Denote by $|K|$ the geometric realization of K in the weak topology. It is easy to see that there exist finite simplicial complexes K_1 and K_2 which satisfy the following:
(2.5) $|K_1|=M$ and $|K_2|=M'$,
(2.6) there exist subcomplexes $L_1\subset K_1$ and $L_2\subset K_2$ such that $|L_1|=N$ and $|L_2|=N'$,
(2.7) there exists a simplicial map $\tau\colon(K_1, L_1)\to(K_2, L_2)$ such that $|\tau|=\pi\colon(|K_1|, |L_1|)\to(|K_2|, |L_2|)$.

Let $\mathscr{B}(\pi_{q-1})$ $(1\leqq q\leqq 2m)$ be the bundles of coefficients associated with $\pi_{q-1}(X)$ (see Steenrod [27; §30]). By the assumption (ii), $\mathscr{B}(\pi_{q-1})$ $(1\leqq q\leqq 2m)$ are product bundles. Therefore the cohomology groups $H^q(M', N'; \mathscr{B}(\pi_{q-1}))$ are isomorphic to the ordinary cohomology groups $H^q(M', N'; \pi_{q-1}(X))$ for $1\leqq q\leqq 2m$. By the assumption (iii) and Lemma 2.3, we have

$$H^q(M', N'; \pi_{q-1}(X)) = 0 \qquad \text{for } 1 \leqq q \leqq 2m-1 .$$

It follows from Steenrod [27; 34.2] that there exists a cross-section of $\mathcal{B}$ defined on $|K_2^{2m-1}|(\supset |L_2|)$:

$$s_1\colon |K_2^{2m-1}| \to M \underset{Z_n}{\times} X$$

such that $s_1 | |L_2| = s_0$. There exists an obstruction cohomology class

$$\bar{c}(s_1) \in H^{2m}(M', N'; \pi_{2m-1}(X))$$

such that its vanishing is a necessary and sufficient condition for $s_1 | |K_2^{2m-2} \cup L_2|$ to be extendable over M'. Thus we shall show that $\bar{c}(s_1)=0$. Consider the product bundle $\mathcal{B}'$:

$$X \to M \times X \to M .$$

Let $\mathcal{B}'(\pi_{q-1})$ $(1 \leqq q \leqq 2m)$ be the bundles of coefficients associated with $\pi_{q-1}(X)$. Since $\mathcal{B}'$ is a product bundle, $\mathcal{B}'(\pi_{q-1})$ $(1 \leqq q \leqq 2m)$ are also product bundles. The natural projection $M \times X \to M \underset{Z_n}{\times} X$ induces the bundle maps $\bar{\pi}\colon \mathcal{B}' \to \mathcal{B}$ and $\bar{\pi}_{q-1}\colon \mathcal{B}'(\pi_{q-1}) \to \mathcal{B}(\pi_{q-1})$ $(1 \leqq q \leqq 2m)$ covering $\pi\colon (M, N) \to (M', N')$. Let $s_2\colon |K_1^{2m-1}| \to M \times X$ be the cross-section of $\mathcal{B}'$ induced by s_1 and $\bar{\pi}$. It follows from (2.7) that we have

$$\pi^*(\bar{c}(s_1)) = \bar{c}(s_2) \in H^{2m}(M, N; \pi_{2m-1}(X)) .$$

By the assumption (iii) and Lemma 2.4, π^* is a monomorphism. Hence $\bar{c}(s_1)=0$ if and only if $\bar{c}(s_2)=0$. Let $s_3\colon M=|K_1| \to M \times X$ be a cross-section of $\mathcal{B}'$ defined by

$$s_3(z) = (z, F'(z)) \in M \times X \qquad \text{for } z \in M .$$

We put

$$s_4 = s_3 | |K_1^{2m-1}|\colon |K_1^{2m-1}| \to M \times X .$$

Then s_2 and s_4 are cross-sections of $\mathcal{B}'$ defined on $|K_1^{2m-1}|(\supset |L_1|)$ such that $s_2 | |L_1| = s_4 | |L_1|$. By Lemma 2.2, we have

$$H^q(M, N; \pi_q(X)) = 0 \qquad \text{for } 0 \leqq q \leqq 2m-2 .$$

It follows from Steenrod [27; 35.9] that

$$\bar{c}(s_2) = \bar{c}(s_4) \in H^{2m}(M, N; \pi_{2m-1}(X)) .$$

It is obvious that $\bar{c}(s_2)=\bar{c}(s_4)=0$. Hence we have $\bar{c}(s_1)=0$. q.e.d.

Corollary 2.8. *Let X and V be as in Theorem 2.1. Suppose that*

(i) $X^{Z_n} \neq \phi$,

(ii) *there exists a* Z_n*-map* $f: S(V) \to X$ *such that* $[f] = 0 \in [S^{2m-1}, X]$ $(\cong \pi_{2m-1}(X))$.

Let y_0 *be an arbitrary point of* X^{Z_n}. *Then there exists a* Z_n*-map*

$$F: D(V) \to X$$

such that $F|S(V) = f$ *and* $F(0) = y_0$. *Here* $D(V)$ *denotes the unit disk.*

3. Equivariant maps which are equivariantly homotopic to zero

Let n be a positive integer. Let V and W be real Z_n-representation spaces with $\dim_R V = \dim_R W = k > 0$. Let

$$\rho_V, \rho_W: Z_n \to GL(k, R)$$

be the Z_n-representations afforded by V, W respectively. Then a Z_n-action on $GL(k, R)$ is given by

$$s \circ A = \rho_W(s) A \rho_V(s)^{-1} \qquad \text{for } s \in Z_n,\ A \in GL(k, R),$$

and denote by $GL(V, W)$ this Z_n-space. Remark that $GL(k, R)$ has two connected components $GL^+(k, R)$ and $GL^-(k, R)$. If n is an odd integer, then we have

$$\rho_V(Z_n),\ \rho_W(Z_n) \subset GL^+(k, R).$$

Hence $GL^+(k, R)$ and $GL^-(k, R)$ are Z_n-subspaces of $GL(V, W)$ and are denoted by $GL^+(V, W)$ and $GL^-(V, W)$ respectively.

Let $F(S(V), S(W))$ denote the space of homotopy equivalent maps from $S(V)$ to $S(W)$ with the compact-open topology. A Z_n-action on $F(S(V), S(W))$ is given by

$$(s \circ f)(v) = sf(s^{-1}v) \qquad \text{for } s \in Z_n,\ f \in F(S(V), S(W)),\ v \in S(V).$$

It is well-known that $F(S(V), S(W))$ has two connected components $F^+(S(V), S(W))$ and $F^-(S(V), S(W))$ representing maps of degree $+1$ and -1 respectively. If n is an odd integer, then $F^+(S(V), S(W))$ and $F^-(S(V), S(W))$ are Z_n-subspaces of $F(S(V), S(W))$.

It is well-known that

(3.1) $GL^\varepsilon(V, W)$ *and* $F^\varepsilon(S(V), S(W))$ $(\varepsilon = \pm)$ *are path-connected and q-simple for* $q > 0$.

Moreover it is easy to see that

(3.2) *If* n *is an odd integer, then the maps of* $GL^\varepsilon(V, W)$ *and* $F^\varepsilon(S(V), S(W))$ $(\varepsilon = \pm)$ *into themselves given by the action of a generator of* Z_n *are homotopic to to the identity.*

Proposition 3.3. *Let n be a positive odd integer. Let V and W be real $\boldsymbol{Z}_n$-representation spaces with $\dim_{\boldsymbol{R}} V=\dim_{\boldsymbol{R}} W=k$. Let U be a complex $\boldsymbol{Z}_n$-representation space such that $\boldsymbol{Z}_n$ acts freely on $S(U)$ and $\dim_{\boldsymbol{R}} U=2m$. Assume that*

(i) $k\geqq 2m+1$,

(ii) *there exists a $\boldsymbol{Z}_n$-map $f\colon S(U)\to GL^{\varepsilon}(V, W)$ such that $[f]=0\in[S^{2m-1}, GL^{\varepsilon}(V, W)]$,*

(iii) $GL^{\varepsilon}(V, W)^{\boldsymbol{Z}_n}\neq\phi$,

where $\varepsilon=+$ or $-$. Then there exists a $\boldsymbol{Z}_n$-map $F\colon D(U)\to GL^{\varepsilon}(V, W)$ such that $F\,|\,S(U)=f$.

Proof. It is well-known that

$$\pi_i(GL^{\varepsilon}(V, W))\cong\begin{cases}\boldsymbol{Z}_2 & \text{if } i\equiv 0,\ 1 \bmod 8\,,\\ 0 & \text{if } i\equiv 2,\ 4,\ 5,\ 6 \bmod 8\,,\\ \boldsymbol{Z} & \text{if } i\equiv 3,\ 7 \bmod 8,\end{cases}$$

for $1\leqq i\leqq k-2$. Since n is odd, we have

$$\begin{cases}\operatorname{Hom}(\boldsymbol{Z}_n, \pi_{2i-1}(GL^{\varepsilon}(V, W)))=0 & \text{for } 1\leqq i\leqq m\,,\\ \operatorname{Ext}(\boldsymbol{Z}_n, \pi_{2i}(GL^{\varepsilon}(V, W)))=0 & \text{for } 1\leqq i\leqq m-1\,.\end{cases}$$

Therefore the result follows from Corollary 2.8. q.e.d.

Proposition 3.4. *Let n be a positive odd integer. Let V and W be real $\boldsymbol{Z}_n$-representation spaces with $\dim_{\boldsymbol{R}} V=\dim_{\boldsymbol{R}} W=k$. Let U be a complex $\boldsymbol{Z}_n$-representation space such that $\boldsymbol{Z}_n$ acts freely on $S(U)$ and $\dim_{\boldsymbol{R}} U=2m$. Assume that*

(i) $(n, s(2m-1))=1$,

(ii) $k\geqq 2m+2$,

(iii) *there exists a $\boldsymbol{Z}_n$-map $f\colon S(U)\to F^{\varepsilon}(S(V), S(W))$ such that $[f]=0\in[S^{2m-1}, F^{\varepsilon}(S(V), S(W))]$,*

(iv) $F^{\varepsilon}(S(V), S(W))^{\boldsymbol{Z}_n}\neq\phi$,

where $\varepsilon=+$ or $-$. Let φ be an arbitrary element of $F^{\varepsilon}(S(V), S(W))^{\boldsymbol{Z}_n}$. Then there exists a $\boldsymbol{Z}_n$-map $F\colon D(U)\to F^{\varepsilon}(S(V), S(W))$ such that $F|S(U)=f$ and $F(0)=\varphi$.

Proof. It follows from Atiyah [4; p. 294] that there exist isomorphisms

$$\pi_i(F^{\varepsilon}(S(V), S(W)))\cong G_i \qquad \text{for } 1\leqq i\leqq k-3\,.$$

By the assumptions (i) and (ii), we have

$$\begin{cases}\operatorname{Hom}(\boldsymbol{Z}_n, \pi_{2i-1}(F^{\varepsilon}(S(V), S(W))))=0 & \text{for } 1\leqq i\leqq m\,,\\ \operatorname{Ext}(\boldsymbol{Z}_n, \pi_{2i}(F^{\varepsilon}(S(V), S(W))))=0 & \text{for } 1\leqq i\leqq m-1\,.\end{cases}$$

Therefore the result follows from Corollary 2.8. q.e.d.

4. Equivariant *KO*-rings

In this section, we consider an isomorphism of S^1-vector bundles over $S^{2m-1}(p_1, p_2, \cdots, p_m)$ when the S^1-action is free or pseudofree.

Let V be a real S^1-representation space. Let X be a compact S^1-space. Denote by $\underline{\underline{V}}$ the trivial S^1-vector bundle

$$V \to X \times V \to X.$$

Let ξ and η be real S^1-vector bundles over X with $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta$. Let

$$p\colon \mathrm{Hom}(\xi, \eta) \to X$$

be the S^1-vector bundle defined by Atiyah [3; §1.2] and Segal [25; §1]. Let $\mathrm{Iso}(\xi, \eta) \subset \mathrm{Hom}(\xi, \eta)$ be the subspace of all isomorphisms from ξ_x to η_x for $x \in X$, where ξ_x (resp. η_x) denotes the fiber of ξ (resp. η) over x. Clearly, $\mathrm{Iso}(\xi, \eta)$ is an S^1-subspace of $\mathrm{Hom}(\xi, \eta)$ and

$$q = p \,|\, \mathrm{Iso}(\xi, \eta)\colon \mathrm{Iso}(\xi, \eta) \to X \tag{4.1}$$

is a surjective S^1-map. We remark that ξ and η are equivalent as S^1-vector bundles over X if and only if there exists an S^1-cross-section of q defined on X.

Let $p = (p_1, p_2, \cdots, p_m)$ be a sequence of positive integers. Denote by $D^{2m}(p_1, p_2, \cdots, p_m)$ the unit disk in $\boldsymbol{C}^m$ with the S^1-action φ_p (see §1).

Let $m > 1$ be an integer. We put

$$\begin{aligned}
M_k &= S^{2m-1}(p_1, p_2, \cdots, p_k, 1, \cdots, 1) && \text{for } 1 \leqq k \leqq m,\\
S_k &= S^{2m-3}(p_1, p_2, \cdots, p_{k-1}, 1, \cdots, 1) && \text{for } 2 \leqq k \leqq m,\\
D_k &= D^{2m-2}(p_1, p_2, \cdots, p_{k-1}, 1, \cdots, 1) && \text{for } 2 \leqq k \leqq m,\\
M_0 &= S^{2m-1}(1, 1, \cdots, 1),\\
S_1 &= S^{2m-3}(1, 1, \cdots, 1),\\
D_1 &= D^{2m-2}(1, 1, \cdots, 1).
\end{aligned}$$

Here we remark that $\partial D_k = S_k$ for $1 \leqq k \leqq m$.

In the following, for every positive integer n, we always regard the cyclic group $\boldsymbol{Z}_n$ as the subgroup of S^1 and regard an S^1-space as a $\boldsymbol{Z}_n$-space in respective context.

We define a $\boldsymbol{Z}_{p_k}$-map $j_k\colon D_k \to M_k$ by

$$j_k(z_1, \cdots, z_{k-1}, z_k, \cdots, z_{m-1}) = (z_1, \cdots, z_{k-1}, \sqrt{1 - |z_1|^2 - \cdots - |z_{m-1}|^2}, z_k, \cdots, z_{m-1}).$$

It is easy to see that j_k is a $\boldsymbol{Z}_{p_k}$-embedding and $j_k | S_k\colon S_k \to M_k$ is an S^1-embedding. In the following, D_k and S_k are regarded as a $\boldsymbol{Z}_{p_k}$-invariant subspace of M_k and an S^1-invariant subspace of M_k by j_k respectively. Let e_j $(1 \leqq j \leqq m)$ be the j-th unit vector of $\boldsymbol{C}^m$. Then we see that $e_1, e_2, \cdots, e_{k-1} \in S_k$ and $e_k \in D_k$ as the center of the disk.

We define a continuous map α: $S^1 \times D_k \to M_k$ by

$$\alpha(s, z) = sz \qquad \text{for } s \in S^1,\ z \in D_k .$$

Then we have

Lemma 4.2. *α is an identification map.*

The proof is easy.

Lemma 4.3. *Let X be an S^1-space and let p: $X \to M_k$ be a surjective S^1-map. If there exists a $\boldsymbol{Z}_{p_k}$-cross-section t_1: $D_k \to X$ of $p \mid p^{-1}(D_k)$ such that $t_1 \mid S_k$: $S_k \to X$ is an S^1-cross-section of $p \mid p^{-1}(S_k)$, then there exists an S^1-cross-section t: $M_k \to X$ of p such that $t \mid D_k = t_1$.*

Proof. By Lemma 4.2, α: $S^1 \times D \to M_k$ is surjective. Thus, given $z \in M_k$, there exists $s \in S^1$ such that $s^{-1}z \in D_k$. Define t: $M_k \to X$ by

$$t(z) = st(s^{-1}z) ,$$

where $s \in S^1$ is chosen as $s^{-1}z \in D_k$. Then it is easy to see that t is a well-defined S^1-cross-section of p such that $t \mid D_k = t_1$. q.e.d.

Define S^1-maps

$$h_k \colon M_k \to M_{k+1} \qquad \text{for } 0 \leqq k \leqq m-1$$

by

$$h_k(z_1, \cdots, z_k, z_{k+1}, z_{k+2}, \cdots, z_m) = \frac{(z_1, \cdots, z_k, z_{k+1}^{p_{k+1}}, z_{k+2}, \cdots, z_m)}{\|(z_1, \cdots, z_k, z_{k+1}^{p_{k+1}}, z_{k+2}, \cdots, z_m)\|}$$

and we put $h_m = id$: $M_m \to M_m$. Moreover we define

$$\tilde{h}_k = h_m \circ h_{m-1} \circ \cdots \circ h_k \colon M_k \to M_m \qquad \text{for } 0 \leqq k \leqq m .$$

Then it follows that

$$\tilde{h}_k(e_j) = e_j \qquad \text{for } 0 \leqq k \leqq m,\ 1 \leqq j \leqq m .$$

Let ξ and η be S^1-vector bundles over M_m with $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta = n$. We put

$$V_k = (\tilde{h}_k^* \xi)_{e_k} = \xi_{e_k},\ W_k = (\tilde{h}_k^* \eta)_{e_k} = \eta_{e_k} \qquad \text{for } 1 \leqq k \leqq m .$$

Here V_k, W_k $(1 \leqq k \leqq m)$ are regarded as $\boldsymbol{Z}_{p_k}$-representation spaces. Let q_k: $\mathrm{Iso}(\tilde{h}_k^* \xi, \tilde{h}_k^* \eta) \to M_k$ $(0 \leqq k \leqq m)$ be S^1-maps defined by (4.1). Then we have

Lemma 4.4. *There are Z_{p_k}-homeomorphisms*

$$\varphi_k\colon q_k^{-1}(D_k) \to D_k \times GL(V_k, W_k) \qquad \textit{for } 1 \leqq k \leqq m$$

such that the following diagram commutes:

$$\begin{tikzcd} q_k^{-1}(D_k) \arrow[rr, "\varphi_k"] \arrow[dr, "q_k \mid q_k^{-1}(D_k)"'] & & D_k \times GL(V_k, W_k) \arrow[dl, "\pi_1"] \\ & D_k, & \end{tikzcd}$$

where π_1 denotes the projection on the first factor.

Proof. Since D_k is Z_{p_k}-contractible, there exist isomorphisms of Z_{p_k}-vector bundles:

$$\begin{cases} \alpha\colon (\tilde{h}_k^* \xi) \mid D_k \to D_k \times V_k, \\ \beta\colon (\tilde{h}_k^* \eta) \mid D_k \to D_k \times W_k. \end{cases}$$

Let $\tilde{q}_k\colon \mathrm{Iso}(D_k \times V_k, D_k \times W_k) \to D_k$ be an S^1-map defined by (4.1). Then we can define Z_{p_k}-homeomorphisms

$$\begin{cases} \psi_1\colon \mathrm{Iso}((\tilde{h}_k^* \xi) \mid D_k, (\tilde{h}_k^* \eta) \mid D_k) \to \mathrm{Iso}(D_k \times V_k, D_k \times W_k), \\ \psi_2\colon \mathrm{Iso}(D_k \times V_k, D_k \times W_k) \to D_k \times GL(V_k, W_k), \end{cases}$$

by

$$\begin{cases} \psi_1(f_x) = \beta_x \circ f_x \circ \alpha_x^{-1} & \text{for } x \in D_k, f_x \in q_k^{-1}(x), \\ \psi_2(g_x) = (x, g_x) & \text{for } x \in D_k, g_x \in \tilde{q}_k^{-1}(x), \end{cases}$$

respectively. It is obvious that a Z_{p_k}-homeomorphism

$$\varphi_k = \psi_2 \circ \psi_1\colon q_k^{-1}(D_k) = \mathrm{Iso}((\tilde{h}_k^* \xi) \mid D_k, (\tilde{h}_k^* \eta) \mid D_k) \to D_k \times GL(V_k, W_k)$$

satisfies our condition. q.e.d.

Define an S^1-map $h\colon M_0 \to M_m$ by

$$h(z_1, z_2, \cdots, z_m) = \frac{(z_1^{p_1}, z_2^{p_2}, \cdots, z_m^{p_m})}{\|(z_1^{p_1}, z_2^{p_2}, \cdots, z_m^{p_m})\|}.$$

Lemma 4.5. *Let $m > 1$ be an integer and let p_i $(1 \leqq i \leqq m)$ be positive odd integers with $(p_i, p_j) = 1$ for $i \neq j$. Let ξ and η be real S^1-vector bundles over M_m such that $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta = n \geqq 2m-1$ and $\xi \supset \underline{\boldsymbol{R}}^1$ as an S^1-vector subbundle. Assume that*

(i) *$h^* \xi$ and $h^* \eta$ are equivalent as S^1-vector bundles over M_0,*

(ii) *ξ_{e_k} and η_{e_k} are equivalent as Z_{p_k}-representation spaces for $1 \leqq k \leqq m$. Then ξ and η are equivalent as S^1-vector bundles over M_m.*

Proof. Let q_k: $\mathrm{Iso}(\tilde{h}_k^*\xi, \tilde{h}_k^*\eta)\to M_k$ $(0\leqq k\leqq m)$ be S^1-maps defined by (4.1). We shall show that there exist S^1-cross-sections of q_k $(0\leqq k\leqq m)$:

$$t_k\colon M_k\to \mathrm{Iso}(\tilde{h}_k^*\xi, \tilde{h}_k^*\eta)\,,$$

by induction. Then the existence of the last S^1-cross-section t_m shows the result.

It follows from Iberkleid [11; Theorem 3.4] that the S^1-maps $\tilde{h}_0, h\colon M_0\to M_m$ are S^1-homotopic. Hence, by the assumption (i), we have

$$\tilde{h}_0^*\xi\cong h^*\xi\cong h^*\eta\cong\tilde{h}_0^*\eta\,,$$

where $\cong$ stands for *is equivalent to*. Therefore there exists an S^1-cross-section of q_0:

$$t_0\colon M_0\to \mathrm{Iso}(\tilde{h}_0^*\xi, \tilde{h}_0^*\eta)\,.$$

Let k be an integer greater than zero. We now assume that there exists an S^1-cross-section of q_{k-1}:

$$t_{k-1}\colon M_{k-1}\to \mathrm{Iso}(\tilde{h}_{k-1}^*\xi,\tilde{h}_{k-1}^*\eta)\,.$$

Remark that

$$\tilde{h}_{k-1}=\tilde{h}_k\circ h_{k-1}\colon M_{k-1}\to M_m\,.$$

It follows that there exist S^1-vector bundle maps

$$\begin{cases}\bar{h}_{k-1}\colon \tilde{h}_{k-1}^*\xi\to\tilde{h}_k^*\xi\,,\\ \bar{h}'_{k-1}\colon \tilde{h}_{k-1}^*\eta\to\tilde{h}_k^*\eta\,,\end{cases}$$

covering $h_{k-1}\colon M_{k-1}\to M_k$. We define an embedding $j'_k\colon D_k\to M_{k-1}$ by

$$j'_k(z_1,\cdots,z_{k-1},z_k,\cdots,z_{m-1})=(z_1,\cdots,z_{k-1},\sqrt{1-|z_1|^2-\cdots-|z_{m-1}|^2},z_k,\cdots,z_{m-1})\,.$$

Then the restriction $j'_k|S_k\colon S_k\to M_{k-1}$ is an S^1-embedding. Thus D_k and S_k are also regarded as a subspace of M_{k-1} and an S^1-invariant subspace of M_{k-1} by j'_k respectively. We put $D'_k=j'_k(D_k)$ and $S'_k=j'_k(S_k)$. It is easy to see that

$$\begin{cases}h_{k-1}|D'_k\colon D'_k\to D_k\subset M_k\,,\\ h_{k-1}|S'_k\colon S'_k\to S_k\subset M_k\,,\end{cases}$$

are a homeomorphism and an S^1-homeomorphism respectively. It follows that the restrictions

$$\begin{cases}\bar{h}_{k-1}|\{(\tilde{h}_{k-1}^*\xi)|D'_k\}\colon (\tilde{h}_{k-1}^*\xi)|D'_k\to(\tilde{h}_k^*\xi)|D_k\,,\\ \bar{h}'_{k-1}|\{(\tilde{h}_{k-1}^*\eta)|D'_k\}\colon (\tilde{h}_{k-1}^*\eta)|D'_k\to(\tilde{h}_k^*\eta)|D_k\,,\end{cases}$$

are isomorphisms of vector bundles. Moreover the restrictions

$$\begin{cases} \bar{h}_{k-1} \,|\, \{(\tilde{h}_{k-1}^{*}\xi) \,|\, S_k'\} : (\tilde{h}_{k-1}^{*}\xi) \,|\, S_k' \to (\tilde{h}_k^{*}\xi) \,|\, S_k\,, \\ \bar{h}_{k-1}' \,|\, \{(\tilde{h}_{k-1}^{*}\eta) \,|\, S_k'\} : (\tilde{h}_{k-1}^{*}\eta) \,|\, S_k' \to (\tilde{h}_k^{*}\eta) \,|\, S_k\,, \end{cases}$$

are isomorphisms of S^1-vector bundles. Using the S^1-cross-section $t_{k-1}\colon M_{k-1}\to \mathrm{Iso}(\tilde{h}_{k-1}^{*}\xi,\ \tilde{h}_{k-1}^{*}\eta)$, we can define a continuous cross-section of $q_k \,|\, q_k^{-1}(D_k)$:

$$u_k\colon D_k \to q_k^{-1}(D_k) \subset \mathrm{Iso}(\tilde{h}_k^{*}\xi, \tilde{h}_k^{*}\eta)$$

by putting $u_k(x) = \{\bar{h}_{k-1}' \,|\, (\tilde{h}_{k-1}^{*}\xi)_x\} \circ t_{k-1}((h_{k-1} \,|\, D_k')^{-1}(x)) \circ \{h_{k-1} \,|\, (\tilde{h}_{k-1}^{*}\xi)_x\}$ for $x \in D_k \subset M$. Then the restriction

$$v_k = u_k \,|\, S_k\colon S_k \to q_k^{-1}(S_k) \subset \mathrm{Iso}(\tilde{h}_k^{*}\xi,\ \tilde{h}_k^{*}\eta)$$

is an S^1-cross-section of $q_k \,|\, q_k^{-1}(S_k)$. Let $\pi_2\colon D_k \times GL^{\varepsilon}(V_k, W_k) \to GL^{\varepsilon}(V_k, W_k)$ be the projection on the second factor. It follows from Lemma 4.4 that v_k yields a $\boldsymbol{Z}_{p_k}$-map

$$\bar{v}_k\colon S_k \to GL^{\varepsilon}(V_k, W_k)$$

by $\bar{v}_k(x) = \pi_2(\varphi_k(v_k(x)))$ for $x \in S_k$, where $\varepsilon = +$ or $-$. Since $v_k = u_k \,|\, S_k$, we have

$$[\bar{v}_k] = 0 \in [S^{2m-3},\ GL^{\varepsilon}(V_k, W_k)]\,.$$

By the assumption (ii), V_k $(=(\tilde{h}_k^{*}\xi)_{e_k} = \xi_{e_k})$ and W_k $(=(\tilde{h}_k^{*}\eta)_{e_k} = \eta_{e_k})$ are equivalent as $\boldsymbol{Z}_{p_k}$-representation spaces and $V_k \supset \boldsymbol{R}^1$. This shows that

$$GL^{\varepsilon}(V_k,\ W_k)^{\boldsymbol{Z}_{p_k}} \neq \phi\,.$$

Moreover we remark that p_k is an odd integer and $\boldsymbol{Z}_{p_k}$ acts freely on S_k. Therefore it follows from Proposition 3.3 that there exists a $\boldsymbol{Z}_{p_k}$-map

$$\bar{w}_k\colon D_k \to GL^{\varepsilon}(V_k, W_k)$$

such that $\bar{w}_k \,|\, S_k = \bar{v}_k$. By Lemma 4.4, we can define a $\boldsymbol{Z}_{p_k}$-cross-section of $q_k \,|\, q_k^{-1}(D_k)$:

$$w_k\colon D_k \to q_k^{-1}(D_k) \subset \mathrm{Iso}(\tilde{h}_k^{*}\xi,\ \tilde{h}_k^{*}\eta)$$

by $w_k(x) = \varphi_k^{-1}(x, \bar{w}_k(x))$ for $x \in D_k$. Since $w_k \,|\, S_k = v_k$, it follows from Lemma 4.3 that there exists an S^1-cross-section of q_k:

$$t_k\colon M_k \to \mathrm{Iso}(\tilde{h}_k^{*}\xi,\ \tilde{h}_k^{*}\eta)\,.$$

In this way, we obtain S^1-cross-sections $t_0, t_1, \cdots, t_m$. q.e.d.

The following lemma is due to Segal (see [25; Proposition 2.1]).

Lemma 4.6. *Let G be a compact Lie group and let X be a compact Hausdorff G-space such that G acts freely on X. Then the projection $pr\colon X \to X/G$ induces* *an isomorphism of rings*

$$pr^*\colon KO(X/G) \to KO_G(X).$$

We put

$$\mu = (pr^*)^{-1}\colon KO_{S^1}(M_0) \xrightarrow{\cong} KO(CP^{m-1}).$$

Denote by $RO(G)$ the real representation ring of G. We define a homomorphism of rings

$$\Phi\colon KO_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m)) \to KO(CP^{m-1}) \oplus \bigoplus_{i=1}^{m} RO(\boldsymbol{Z}_{p_i})$$

by putting

$$\Phi(\xi-\eta) = \mu(h^*\xi - h^*\eta) \oplus \bigoplus_{i=1}^{m} (\xi_{e_i} - \eta_{e_i}).$$

Then we have

Theorem 4.7. *Let p_i $(1 \leqq i \leqq m)$ be positive odd integers such that $(p_i, p_j)=1$ for $i \neq j$. Then the homomorphism Φ is injective.*

Proof. If $m=1$, then $KO_{S^1}(S^1(p_1)) = KO_{S^1}(S^1/\boldsymbol{Z}_{p_1}) \cong RO(\boldsymbol{Z}_{p_1})$. Therefore we assume that $m>1$. If $\Phi(\xi-\eta)=0$, then $h^*\xi - h^*\eta = 0$ in $KO_{S^1}(M_0)$ and $\xi_{e_i} - \eta_{e_i} = 0$ in $RO(\boldsymbol{Z}_{p_i})$ for $1 \leqq i \leqq m$. Thus there exists an S^1-representation space U such that $h^*(\xi \oplus \underline{\underline{U}})$ is equivalent to $h^*(\eta \oplus \underline{\underline{U}})$. Then we put

$$\xi' = \xi \oplus \underline{\underline{\boldsymbol{R}}}^{2m} \oplus \underline{\underline{U}} \quad \text{and} \quad \eta' = \eta \oplus \underline{\underline{\boldsymbol{R}}}^{2m} \oplus \underline{\underline{U}}.$$

Since ξ' and η' satisfy the assumption of Lemma 4.5, ξ' is equivalent to η'. It follows that

$$\xi - \eta = \xi' - \eta' = 0 \qquad \text{in } KO_{S^1}(M_m).$$

Hence Φ is injective. q.e.d.

Next we consider the condition (i) of Lemma 4.5. Let ES^1 (resp. BS^1) be a universal S^1-space (resp. a classifying space for S^1). Let $\pi_k\colon ES^1 \underset{S^1}{\times} M_k \to BS^1$ $(0 \leqq k \leqq m)$ be the natural projection.

Lemma 4.8. *The homomorphism*

$$\pi_k^*\colon H^q(BS^1; \boldsymbol{Z}) \to H^q(ES^1 \underset{S^1}{\times} M_k; \boldsymbol{Z})$$

is an isomorphism for $0 \leqq q \leqq 2m-2$. Moreover the integral cohomology ring of $ES^1 \underset{S^1}{\times} M_k$ is

$$H^*(ES^1 \underset{S^1}{\times} M_k; \boldsymbol{Z}) = \boldsymbol{Z}[c]/(qc^m),$$

where $\deg c = 2$ *and* $q = \prod_{i=1}^{k} p_i$.

Proof. The map π_k is a projection of a sphere bundle associated with the complex m-plane bundle $\eta^{p_1}\oplus\cdots\oplus\eta^{p_k}\oplus\eta\oplus\cdots\oplus\eta$, where η is the canonical complex line bundle over BS^1. Then the result follows from the Thom-Gysin exact sequence. q.e.d.

Lemma 4.9. *Let $\tau\colon ES^1\times_{S^1} M_0\to M_0/S^1=CP^{m-1}$ be the natural projection. Then*

$$\tau^*\colon H^*(CP^{m-1};\boldsymbol{Z})\to H^*(ES^1\times_{S^1} M_0;\boldsymbol{Z})$$

is an isomorphism.

Proof. The result follows from the Vietoris-Begle Mapping Theorem (see Bredon [6; p. 371], Spanier [26; p. 344]).

Lemma 4.10. *The homomorphism*

$$(1\times_{S^1} h)^*\colon H^q(ES^1\times_{S^1} M_m;\boldsymbol{Z})\to H^q(ES^1\times_{S^1} M_0;\boldsymbol{Z})$$

is an isomorphism for $0\leqq q\leqq 2m-2$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} H^q(BS^1) & \xrightarrow{\quad id \quad} & H^q(BS^1) \\ \downarrow{\scriptstyle \pi_m^*} & & \downarrow{\scriptstyle \pi_0^*} \\ H^q(ES^1\times_{S^1} M_m) & \xrightarrow{(1\times_{S^1} h)^*} & H^q(ES^1\times_{S^1} M_0)\,. \end{array}$$

Since π_m^* and π_0^* are isomorphisms for $0\leqq q\leqq 2m-2$, $(1\times_{S^1} h)^*$ is an isomorphism for $0\leqq q\leqq 2m-2$. q.e.d.

Lemma 4.11. *Let ξ and η be real S^1-vector bundles over M_m with $\dim_{\boldsymbol{R}}\xi=\dim_{\boldsymbol{R}}\eta=k$. Assume that $m\not\equiv 2 \bmod 4$. Then the following two conditions are equivalent:*

(i) *$\mu(h^*\xi)=\mu(h^*\eta)$ in $KO(CP^{m-1})$,*

(ii) *$p_i(ES^1\times_{S^1}\xi)=p_i(ES^1\times_{S^1}\eta)$ in $H^{4i}(ES^1\times_{S^1} M_m;\boldsymbol{Z})$ for $1\leqq i\leqq\min([k/2],[(m-1)/2])$.*

Here $p_i(ES^1\times_{S^1}\xi)$ (resp. $p_i(ES^1\times_{S^1}\eta)$) denotes the i-th Pontrjagin class of the bundle $ES^1\times_{S^1}\xi\to ES^1\times_{S^1} M_m$ (resp. $ES^1\times_{S^1}\eta\to ES^1\times_{S^1} M_m$).

Proof. Remark that $\tau^*(\mu(h^*\xi))=ES^1\times_{S^1} h^*\xi$, where $\tau\colon ES^1\times_{S^1} M_0\to M_0/S^1=CP^{m-1}$ is the natural projection. Then we have

$$\tau^*(p_i(\mu(h^*\xi)))=p_i(ES^1\times_{S^1} h^*\xi)$$

and

$$(1 \underset{S^1}{\times} h)^*(p_i(ES^1 \underset{S^1}{\times} \xi)) = p_i(ES^1 \underset{S^1}{\times} h^*\xi)\,.$$

Hence it follows from Lemmas 4.9 and 4.10 that the condition (ii) is equivalent to the following:

$$p_i(\mu(h^*\xi)) = p_i(\mu(h^*\eta)) \text{ in } H^{4i}(CP^{m-1}; \boldsymbol{Z})$$

for $1 \leqq i \leqq \min([k/2], [(m-1)/2])$. Since $m \not\equiv 2 \bmod 4$, $KO(CP^{m-1})$ is a free abelian group (see Sanderson [24; Theorem 3.9]). It follows from Hsiang [10; §3] that

$$p_i(\mu(h^*\xi)) = p_i(\mu(h^*\eta)) \qquad \text{for } 1 \leqq i \leqq \min([k/2], [(m-1)/2])$$

if and only if

$$\mu(h^*\xi) = \mu(h^*\eta) \qquad \text{in } KO(CP^{m-1})\,. \qquad \text{q.e.d.}$$

By Theorem 4.7 and Lemma 4.11, we have

Theorem 4.12. *Let m be a positive integer such that $m \not\equiv 2 \bmod 4$. Let p_i $(1 \leqq i \leqq m)$ be positive odd integers with $(p_i, p_j)=1$ for $i \neq j$. Let ξ and η be real S^1-vector bundles over $S^{2m-1}(p_1, p_2, \cdots, p_m)$ with $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta = k$. Then $\xi = \eta$ in $KO_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m))$ if and only if the following two conditions are satisfied*:

(i) $\xi_{e_i} = \eta_{e_i}$ *in* $RO(\boldsymbol{Z}_{p_i})$ *for* $1 \leqq i \leqq m$,

(ii) $p_i(ES^1 \underset{S^1}{\times} \xi) = p_i(ES^1 \underset{S^1}{\times} \eta)$ *for* $1 \leqq i \leqq \min([k/2], [(m-1)/2])$.

REMARK 4.13. Let G be a compact Lie group and let X be a finite G-CW-complex in the sense of Matumoto [17]. Let ξ and η be G-vector bundles over X such that they are stably equivalent. But, in general, ξ and η are not equivalent even if $\dim \xi = \dim \eta > \dim X$ (cf. Sanderson [24; Lemma 1.2]). For example, for an arbitrary integer $n \geqq 0$, we put

$$\begin{cases} \xi = S^3(7, 11) \times t^2 \oplus t \oplus nt\,, \\ \eta = S^3(7, 11) \times t^9 \oplus t^{78} \oplus nt\,, \end{cases}$$

where t^d $(d \in \boldsymbol{Z})$ denotes the complex one-dimensional S^1-representation space defined by $t^d(s)z = s^d z$ for $s \in S^1$, $z \in \boldsymbol{C}^1$. It follows from Lemma 4.5 that

$$\xi \oplus \underline{\underline{\boldsymbol{R}}}^1 \cong \eta \oplus \underline{\underline{\boldsymbol{R}}}^1\,.$$

Now we assume that there exists an isomorphism of S^1-vector bundles:

$$\omega\colon \xi \to \eta\,.$$

Since ξ (resp. η) is a complex vector bundle, ξ (resp. η) has a canonical orientation. Then the isomorphism of $\boldsymbol{Z}_7$-representation spaces $\omega_{e_1}\colon \xi_{e_1} \to \eta_{e_1}$

is orientation-preserving, but the isomorphism of $\boldsymbol{Z}_{11}$-representation spaces ω_{e_2}: $\xi_{e_2} \to \eta_{e_2}$ is orientation-reversing. Since $S^3(7, 11)$ is connected, this is a contradiction. Therefore ξ and η are not equivalent.

5. Equivariant J-groups

In [12] and [14], Kawakubo has defined the notion of the equivariant J-group as follows:

Let G be a compact Lie group and let X be a compact G-space. Let ξ and η be real G-vector bundles over X. Denote by $S(\xi)$ (resp. $S(\eta)$) the unit sphere bundle associated with ξ (resp. η) with respect to some S^1-invariant metric. $S(\xi)$ and $S(\eta)$ are said to be *G-fiber homotopy equivalent* if $S(\xi)$ and $S(\eta)$ are homotopy equivalent by fiber-preserving G-maps and G-homotopies. Let $T_G(X)$ be the additive subgroup of $KO_G(X)$ generated by elements of the form $\xi-\eta$, where ξ and η are G-vector bundles over X whose associated sphere bundles are G-fiber homotopy equivalent. We define the equivariant J-group $J_G(X)$ by

$$J_G(X) = KO_G(X)/T_G(X)$$

and define the equivariant J-homomorphism J_G by the natural epimorphism

$$J_G\colon\ KO_G(X) \to J_G(X)\,.$$

When X is a point, $J_G(X)$ is denoted by $J_G(*)$.

In this section, we shall consider the equivariant J-group of $S^{2m-1}(p_1, p_2, \cdots, p_m)$ when the S^1-action is free or pseudofree. We shall use freely the notations in §§3 and 4.

Let X be a compact S^1-space. Let ξ and η be real S^1-vector bundles over X with $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta$. Let $E(S(\xi), S(\eta))$ denote the disjoint union of the function spaces $F(S(\xi_x), S(\eta_x))$ (see §3) and define

$$q'\colon\ E(S(\xi), S(\eta)) \to X \tag{5.1}$$

by

$$q'(F(S(\xi_x)), S(\eta_x))) = x\,.$$

Then there exists a canonical topology for $E(S(\xi), S(\eta))$ so that $E(S(\xi), S(\eta))$ is the total space of a fiber bundle with projection q' and with fibers $F(S(\xi_x), S(\eta_x))$. An S^1-action

$$\rho\colon\ S^1 \times E(S(\xi), S(\eta)) \to E(S(\xi), S(\eta))\,,$$

is given by $\rho(s, f)(v) = sf(s^{-1}v)$ for $s \in S^1$, $f \in F(S(\xi_x), S(\eta_x))$, $v \in S(\xi_{sx})$. Then q': $E(S(\xi), S(\eta)) \to X$ is an S^1-map.

Let p_i $(1 \leqq i \leqq m)$ be positive integers. Let ξ and η be real S^1-vector bundles over M_m $(=S^{2m-1}(p_1, p_2, \cdots, p_m))$ with $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta$. We choose and fix some S^1-invariant metrics on ξ and η. Then the S^1-vector bundles $h^*\xi$, $h^*\eta$, $\tilde{h}_k^*\xi$ and $\tilde{h}_k^*\eta$ $(0 \leqq k \leqq m)$ have canonical S^1-invariant metrics induced by the S^1-invariant metrics on ξ and η. We put

$$V_k = (\tilde{h}_k^*\xi)_{e_k} = \xi_{e_k}, \; W_k = (\tilde{h}_k^*\eta)_{e_k} = \eta_{e_k} \qquad \text{for } 1 \leqq k \leqq m\,.$$

Here V_k and W_k $(1 \leqq k \leqq m)$ are regarded as orthogonal $\boldsymbol{Z}_{p_k}$-representation spaces. Let $q'_k\colon E(S(\tilde{h}_k^*\xi), S(\tilde{h}_k^*\eta)) \to M_k$ $(0 \leqq k \leqq m)$ be S^1-maps defined by (5.1). Then we have

Lemma 5.2. *There are $\boldsymbol{Z}_{p_k}$-homeomorphisms*

$$\varphi'_k\colon q_k'^{-1}(D_k) \to D_k \times F(S(V_k), S(W_k)) \qquad \textit{for } 1 \leqq k \leqq m$$

such that the following diagram commutes:

$$\begin{tikzcd}
q_k'^{-1}(D_k) \arrow[rr, "\varphi'_k"] \arrow[dr, "q'_k | q_k'^{-1}(D_k)"'] & & D_k \times F_k(S(V_k), S(W_k)) \arrow[dl, "\pi_1"] \\
& D_k, &
\end{tikzcd}$$

where π_1 denotes the projection on the first factor and the restriction

$$\varphi'_k | q_k'^{-1}(e_k)\colon q_k'^{-1}(e_k) = F(S(V_k), S(W_k)) \to \{e_k\} \times F(S(V_k), S(W_k)) \subset D_k \times F(S(V_k), S(W_k))$$

is the identity.

The proof is parallel to that of Lemma 4.4, so we omit it.

Lemma 5.3. *Let $m > 1$ be an integer and let p_i $(1 \leqq i \leqq m)$ be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$ and $(p_i, s(2m-3)) = 1$ for $1 \leqq i \leqq m$. Let ξ and η be real S^1-vector bundles over M_m such tnht $\dim_{\boldsymbol{R}} \xi = \dim_{\boldsymbol{R}} \eta = n \geqq 2m$ and $\xi \supset \underline{\boldsymbol{R}}^1$ as an S^1-vector subbundle. Assume that*

(i) *$S(h^*\xi)$ and $S(h^*\eta)$ are S^1-fiber homotopy equivalent,*

(ii) *$S(\xi_{e_i})$ and $S(\eta_{e_i})$ are $\boldsymbol{Z}_{p_i}$-homotopy equivalent for $1 \leqq i \leqq m$.*

Then $S(\xi)$ and $S(\eta)$ are S^1-fiber homotopy equivalent.

Proof. We put

$$V_i = (\tilde{h}_i^*\xi)_{e_i} = \xi_{e_i} \quad \text{and} \quad W_i = (\tilde{h}_i^*\eta)_{e_i} = \eta_{e_i} \qquad \text{for } 1 \leqq i \leqq m\,.$$

By the assumption (ii), there exist $\boldsymbol{Z}_{p_i}$-homotopy equivalences

$$f_i\colon S(V_i) \to S(W_i) \qquad \text{for } 1 \leqq i \leqq m\,.$$

Since $\xi\supset\underline{\underline{\boldsymbol{R}}}^1$, there exist $\boldsymbol{Z}_{p_i}$-homeomorphisms

$$\tau_i\colon S(V_i)\to S(V_i)\qquad \text{for } 1\leqq i\leqq m$$

such that $\deg\tau_i=-1$. Remark that $f_i\circ\tau_i\colon S(V_i)\to S(W_i)$ is also a $\boldsymbol{Z}_{p_i}$-homotopy equivalence.

First we shall show that, for each $0\leqq k\leqq m$, there exists an S^1-cross-section of q'_k:

$$t'_k\colon M_k\to E(S(\tilde{h}_k^*\xi),\ S(\tilde{h}_k^*\eta))$$

such that $t'_k(e_j)=f_j$ or $f_j\circ\tau_j$ for $1\leqq j\leqq k$.

Since $\tilde{h}_0$, $h\colon M_0\to M_m$ are S^1-homotopic, it follows from the assumption (i) that

$$S(\tilde{h}_0^*\xi)\sim S(h^*\xi)\sim S(h^*\eta)\sim S(\tilde{h}_0^*\eta)\,,$$

where $\sim$ stands for *is S^1-fiber homotopy equivalent to.* Thus there exists an S^1-cross-section of q'_0:

$$t'_0\colon M_0\to E(S(\tilde{h}_0^*\xi),\ S(\tilde{h}_0^*\eta))\,.$$

Let k be an integer greater than zero. Suppose that we are given an S^1-cross-section of q'_{k-1}:

$$t'_{k-1}\colon M_{k-1}\to E(S(\tilde{h}_{k-1}^*\xi),\ S(\tilde{h}_{k-1}^*\eta))$$

such that $t'_{k-1}(e_j)=f_j$ or $f_j\circ\tau_j$ for $1\leqq j\leqq k-1$. Then there exist a continuous cross-section of $q'_k\,|\,q'^{-1}_k(D_k)$:

$$u'_k\colon D_k\to q'^{-1}_k(D_k)\subset E(S(\tilde{h}_k^*\xi),\ S(\tilde{h}_k^*\eta))$$

and an S^1-cross-section of $q'_k\,|\,q'^{-1}_k(S_k)$:

$$v'_k\colon S_k\to q'^{-1}_k(S_k)\subset E(S(\tilde{h}_k^*\xi),\ S(\tilde{h}_k^*\eta))$$

such that $v'_k=u'_k\,|\,S_k$ and $u'_k(e_j)=f_j$ or $f_j\circ\tau_j$ for $1\leqq l\leqq k-1$. This is proved similarly as Lemma 4.6, but we need give care to the condition $v'_k(e_j)=f_j$ or $f_j\circ\tau_j$ for $1\leqq j\leqq k-1$. Let $\pi_2\colon D_k\times F^\varepsilon(S(V_k),S(W_k))\to F^\varepsilon(S(V_k),S(W_k))$ denote the projection on the second factor. By Lemma 5.2, v'_k yields a $\boldsymbol{Z}_{p_k}$-map

$$\bar{v}'_k\colon S_k\to F^\varepsilon(S(V_k),\ S(W_k))$$

by putting $\bar{v}'_k(x)=\pi_2(\varphi'_k(v'_k(x)))$ for $x\in S_k$, where $\varepsilon=+$ or $-$. Since $v'_k=u'_k\,|\,S_k$, we have

$$[\bar{v}'_k]\ =\ 0\in[S^{2m-3},\ F^\varepsilon(S(V_k),\ S(W_k))]\,.$$

Moreover $f_k\in F^\varepsilon(S(V_k),\ S(W_k))^{\boldsymbol{Z}_{p_k}}$ or $f_k\circ\tau_k\in F^\varepsilon(S(V_k),\ S(W_k))^{\boldsymbol{Z}_{p_k}}$. It follows from Proposition 3.4 that there exists a $\boldsymbol{Z}_{p_k}$-map

$$\bar{w}'_k\colon D_k \to F^e(S(V_k),\ S(W_k))$$

such that $\bar{w}'_k \mid S_k = \bar{v}'_k$ and $\bar{w}'_k(e_k) = f_k$ or $f_k \circ \tau_k$. Using Lemma 5.2, we define a $\boldsymbol{Z}_{p_k}$-cross-section of $q'_k \mid q'^{-1}_k(D_k)$:

$$w'_k\colon D_k \to q'^{-1}_k(D_k) \subset E(S(\tilde{h}^*_k\xi),\ S(\tilde{h}^*_k\eta))$$

by putting $w'_k(x) = \varphi'^{-1}_k(x,\ \bar{w}'_k(x))$ for $x \in D_k$. Since $w'_k \mid S_k = v'_k$ and $w'_k(e_k) = f_k$ or $f_k \circ \tau_k$, it follows from Lemma 4.3 that there exists an S^1-cross-section of q'_k:

$$t'_k\colon M_k \to E(S(\tilde{h}^*_k\xi),\ S(\tilde{h}^*_k\eta))$$

such that $t'_k(e_j) = w'_k(e_j) = f_j$ or $f_j \circ \tau_j$ for $1 \leqq j \leqq k$.

By induction, we obtain S^1-cross-sections $t'_0, t'_1, \cdots, t'_m$. The last S^1-cross-section t'_m gives a fiber-preserving S^1-map

$$\omega\colon S(\xi) \to S(\eta)$$

such that $\omega_{e_j} = f_j$ or $f_j \circ \tau_j$ for $1 \leqq j \leqq m$. It is easy to see that, for every $x \in M_m$, $\omega_x\colon S(\xi_x) \to S(\eta_x)$ is an S^1_x-homotopy equivalence, where S^1_x denotes the isotropy group at $x \in M_m$. Therefore it follows from the equivariant Dold theorem that ω gives an S^1-fiber homotopy equivalence (cf. Kawakubo [12; Theorem 2.1] and [24; Theorem 2.1]). q.e.d.

By the same argument as in §2 of Segal [25], we obtain an isomorphism of groups:

$$pr^*\colon J(CP^{m-1}) \to J_{S^1}(M_0)$$

and the following diagram commutes:

$$\begin{array}{ccc} KO(CP^{m-1}) & \xrightarrow{pr^*} & KO_{S^1}(M_0) \\ J\big\downarrow & & \big\downarrow J_{S^1} \\ J(CP^{m-1}) & \xrightarrow{pr^*} & J_{S^1}(M_0) \end{array}$$

(cf. Lemma 4.6). We define

$$\tilde{\mu} = (pr^*)^{-1}\colon J_{S^1}(M_0) \xrightarrow{\cong} J(CP^{m-1})\,.$$

Now we define a homomorphism of groups

$$\tilde{\Phi}\colon J_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m)) \to J(CP^{m-1}) \oplus \bigoplus_{i=1}^{m} J_{Z_{p_i}}(*)$$

by putting

$$\tilde{\Phi}(J_{S^1}(\xi-\eta)) = \tilde{\mu}(J_{S^1}(h^*\xi - h^*\eta)) \oplus \bigoplus_{i=1}^{m} J_{Z_{p_i}}(\xi_{e_i} - \eta_{e_i})\,.$$

Then we have

Theorem 5.4. *Let p_i $(1 \leqq i \leqq m)$ be positive odd integers such that $(p_i, p_j) = 1$ for $i \neq j$ and $(p_i, s(2m-3)) = 1$ for $1 \leqq i \leqq m$. Then the homomorphism $\tilde{\Phi}$ is injective.*

Proof. We see easily that $J_{S^1}(S^1/\boldsymbol{Z}_{p_1}) \cong J_{\boldsymbol{Z}_{p_1}}(*)$. Hence Theorem 5.4 will follow from Lemma 5.3 by the same argument as in the proof of Theorem 4.7.

Let ψ^k denote the Adams operation on equivariant KO-theory.

Corollary 5.5. (cf. [18; Theorem 6.8].) *Let a and b be integers wiht $(a, b) = (ab, p_i) = 1$ for $1 \leqq i \leqq m$. For an arbitrary element α of $KO_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m))$, we have*

$$J_{S^1}((\psi^a-1)(\psi^b-1)(\alpha)) = 0 \text{ in } J_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m)).$$

Proof. By tom Dieck [7; Theorem 1] and tom Dieck-Petrie [8; Theorem 5], we have

$$J_{Z_{p_i}}((\psi^a-1)(\psi^b-1)(\alpha)_{e_i}) = 0 \text{ in } J_{Z_{p_i}}(*) \qquad \text{for } 1 \leqq i \leqq m.$$

On the other hand, by the solution of the Adams conjecture ([1], [22]), we see that

$$\tilde{\mu}(J_{S^1}(h^*(\psi^a-1)(\psi^b-1)(\alpha))) = J((\psi^a-1)(\psi^b-1)(\mu(h^*(\alpha))) = 0 \text{ in } J(CP^{m-1}).$$

Therefore the result follows from Theorem 5.4. q.e.d.

REMARK 5.7. i) The ring structure of $KO(CP^{m-1})$ and the group structure of $J(CP^{m-1})$ have been determined by Sanderson [24; Theorem 3.9] and Adams-Walker [2] (see also Suter [28]). ii) The group structure of $J_{Z_n}(*)$ has been determined by Kawakubo [13] and [15].

6. Quasi-equivalence

Let G be a compact Lie group and let X be a compact G-space. Let ξ and η be real G-vector bundles of the same dimension over X. In [18] and [21], a G-map $\omega: \xi \to \eta$ which is proper, fiber-preserving and degree one on fibers is called a *quasi-equivalence*. Let $\alpha = \eta - \xi \in KO_G(X)$ and define $\alpha \geqq 0$ to mean there exist a G-vector bundle θ over X and a quasi-equivalence $\omega: \xi \oplus \theta \to \eta \oplus \theta$.

Problem 6.1. ([18], [21].) Given $\alpha \in KO_G(X)$, given necessary and sufficient conditions for $\alpha \geqq 0$.

In this section, we consider the above problem when $G = S^1$ and $X = S^{2m-1}(p_1, p_2, \cdots, p_m)$ with a free or pseudofree S^1-action.

We have

Theorem 6.2. *Let p_i $(1\leqq i\leqq m)$ be positive odd integers such that $(p_i, p_j)=1$ for $i\neq j$ and $(p_i, s(2m-3))=1$ for $1\leqq i\leqq m$. Let ξ and η be real S^1-vector bundles of the same dimension over $S^{2m-1}(p_1, p_2, \cdots, p_m)$. Then $\alpha=\eta-\xi\geqq 0$ if and only if ξ and η satisfy the following two conditions:*

(i) *$J(\mu(h^*\xi))=J(\mu(h^*\eta))$ in $J(CP^{m-1})$,*

(ii) *$\alpha_{e_i}=\eta_{e_i}-\xi_{e_i}\geqq 0$ for $1\leqq i\leqq m$,*

where we regard α_{e_i} as an element of $KO_{Z_{p_i}}(*)\cong RO(Z_{p_i})$ for $1\leqq i\leqq m$.

Proof. It is obvious that $\alpha\geqq 0$ if and only if there exist an S^1-vector bundle θ over $S^{2m-1}(p_1, p_2, \cdots, p_m)$ and a fiber-preserving S^1-map $\zeta\colon S(\xi\oplus\theta)\to S(\eta\oplus\theta)$ such that $\deg\zeta_x=1$ for $x\in S^{2m-1}(p_1, p_2, \cdots, p_m)$. Then the proof is parallel to that of Lemma 5.3. q.e.d.

Corollary 6.3. (cf. [21; Corollary 1.13].) *Let α be an arbitrary element of $KO_{S^1}(S^{2m-1}(p_1, p_2, \cdots, p_m))$ such that $\alpha_{e_i}\geqq 0$ for $1\leqq i\leqq m$. Then there exists a non-negative integer n so that*

$$n\alpha\geqq 0\,.$$

Proof. Remark that $\mu(h^*\alpha)\in\widetilde{KO}(CP^{m-1})$. It is well-known that $\tilde{J}(CP^{m-1})$ is a finite abelian group. Hence there exists an integer n such that

$$J(\mu(h^*(n\alpha)))=nJ(\mu(h^*\alpha))=0 \qquad \text{in } J(CP^{m-1})\,.$$

Thus the result follows from Theorem 6.2. q.e.d.

Corollary 6.4. *Let k be an integer with $(k, p_i)=1$ for $1\leqq i\leqq m$. Let α be an arbitrary element of $KO_{S^1}(S^{2m-1}(p_1, \cdots, p_m))$. Then there exists a non-negative integer $e=e(k, \alpha)$ such that*

$$k^e(\psi^k-1)(\alpha)\geqq 0\,.$$

Proof. By the solution of the Adams conjecture (see [1], [22]), there exists a non-negative integer e such that

$$J(\mu(h^*(k^e(\psi^k-1)(\alpha))))=J(k^e(\psi^k-1)(\mu(h^*\alpha)))=0 \qquad \text{in } J(CP^{m-1})\,.$$

On the other hand, by Lee-Wasserman [16; Corollaries 3.3 and 4.8] and Atiyah-Tall [5; V. Theorem 2.8], we have

$$k^e(\psi^k-1)(\alpha_{e_i})\geqq 0 \qquad \text{for } 1\leqq i\leqq m\,.$$

Therefore the result follows from Theorem 6.2. q.e.d.

REMARK 6.5. When X is a point and $\alpha\in K_G(X)\cong R(G)$, Problem 6.1 is solved by the main theorem of [18; Theorem 5.1] (see also Atiyah-Tall [5] and Lee-Wasserman [16]).

References

[1] J.F. Adams: *On the groups $J(X)-I$*, Topology **2** (1963), 181–195.

[2] J.F. Adams and G. Walker: *On complex Stiefel manifolds*, Proc. Cambridge Philos. Soc. **61** (1965), 81–103.

[3] M.F. Atiyah: K-theory, Benjamin, 1967.

[4] M.F. Atiyah: *Thom complexes*, Proc. London Math. Soc. **11** (1961), 291–310.

[5] M.F. Atiyah and D.O. Tall: *Group representations, λ-rings and the J-homomorphism*, Topology **8** (1969), 253–297.

[6] G.E. Bredon: Introduction to compact transformation groups, Academic Press, 1972.

[7] T. tom Dieck: *Homotopy-equivalent group representations*, J. Reine Angew. Math. **298** (1978), 182–195.

[8] T. tom Dieck and T. Petrie: *Geometric modules over the Burnside ring*, Invent. Math. **47** (1978), 273–287.

[9] J. Folkman: Equivariant maps of spheres into the classical groups, Mem. Amer. Math. Soc. 95, 1971.

[10] W.C. Hsiang: *A note on free differentiable actions of S^1 and S^3 on homotopy spheres*, Ann. of Math. **83** (1966), 266–272.

[11] W. Iberkleid: *Pseudo-linear spheres*, Michigan Math. J. **25** (1978), 359–370.

[12] K. Kawakubo: *Compact Lie group actions and fiber homotopy type*, J. Math. Soc. Japan **33** (1981), 295–321.

[13] K. Kawakubo: *Equivariant homotopy equivalence of group representations*, J. Math. Soc. Japan **32** (1980), 105–118.

[14] K. Kawakubo: G-*homotopy equivalent manifolds and J_G-homomorphism*, Proc. Japan Acad. **54** (1978), 104–106.

[15] K. Kawakubo: *The groups $J_G(*)$ for compact abelian topological groups G*, Proc. Japan Acad. **54** (1978), 76–78.

[16] C.N. Lee and A.G. Wasserman: On the groups $JO(G)$, Mem. Amer. Math. Soc. 159, 1976.

[17] T. Matumoto: *On G-CW complexes and a theorem of J. H. C. Whitehead*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **18** (1971), 363–374.

[18] A. Meyerhoff and T. Petrie: *Quasi-equivalence of G modules*, Topology **15** (1976), 69–75.

[19] D. Montgomery and C.T. Yang: *Differentiable pseudo-free circle actions on homotopy seven spheres*, Proc. of the Second Conf. on Compact Transformation Groups, Amherst, Springer Lecture Note 298 (1971), 41–101.

[20] D. Montgomery and C.T. Yang: *Differentiable pseudo-free circle actions*, Proc. Nat. Acad. Sci. U.S.A. **68** (1971), 894–896.

[21] T. Petrie: *Pseudoequivalences of G-manifolds*, Proc. Sympos. Pure Math. **32** (1978), 169–210.

[22] D. Quillen: *The Adams conjecture*, Topology **10** (1971), 67–80.

[23] R.L. Rubinsztein: On the equivariant homotopy of spheres, Dissertations Math., Warszawa (1976).

[24] B.J. Sanderson: *Immersions and embeddings of projective spaces*, Proc. London Math. Soc. **14** (1964), 137–153.

[25] G. Segal: *Equivariant K-theory,* Inst. Hautes Etudes Sci. Publ. Math. Paris **34** (1968), 129–151.
[26] E.H. Spanier: Algebraic topology, McGraw-Hill Book Company, 1966.
[27] N. Steenrod: The topology of fibre bundles, Princeton University Press, 1951.
[28] U. Suter: *On the groups $J(CP^m)$ and $J(HP^n)$*, Bol. Soc. Brasil. Mat. **5** (1974), 221–231.

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