Hashimoto, S.
Osaka J. Math.
18 (1981), 501-507

THE TRANSFER MAP IN THE KR_G-THEORY

SHIN HASHIMOTO

(Received November 27, 1979)

In his work [10] Nishida defined the equivariant transfer maps and studied some properties of the transfer maps in the equivariant K-theory. And making use of them, he gave a new proof of the Adams conjecture in complex case. Following his work and introducing the transfer maps in the Real equivariant K-theory, we give here a proof of the Adams conjecture in real case.

In §1 we introduce the transfer maps in the KR_G-theory, and in §2 we discuss induced representations of Real representations and real representations. Nishida [10] used the monomiality of complex representations [11]. Instead of this fact, we prove in §3 that the identity representation of any odd dimensional orthogonal group is a linear combination of representations which are induced from one or two dimensional representations of appropriate subgroups. Then, by a parallel argument to Nishida [10], the proof of the Adams conjecture in real case is given in §4.

The author expresses his hearty gratitude to Professor S. Araki, Professor H. Minami and Dr. A. Kono for their valuable suggestions.

1. The transfer map

Let G be a compact Lie group and X a compact G-space. For an admissible G-bundle $\xi=(p: E\rightarrow X)$ [10, 8], Nishida [10] defined a G-equivariant trace $t: X_+\wedge V^c\rightarrow E_+\wedge V^c$ of ξ for a suitable real representation space V of G, and proved that it is unique up to stable G-homotopies.

Let G be a compact Real Lie group with involution τ [5]. We denote by $G\times_\tau \boldsymbol{Z}_2$ the semidirect product of G with $\boldsymbol{Z}_2$, the group generated by τ. Atiyah [4] introduced KR_G, the Real equivariant K-theory, which is a contravariant functor from the category of Real G-spaces (that is, $G\times_\tau \boldsymbol{Z}_2$-spaces) to the category of abelian groups. When the involution acts trivially on G and a Real G-space X, then $KR_G(X)$ is naturally isomorphic to $KO_G(X)$.

Let V be a Real representation space of G and X a Real G-space. Atiyah [4] proved the Thom isomorphism $\Phi: KR_G(X)\cong KR_G(X\times V)$. Let $\xi=(p: E\rightarrow X)$ be an admissible $G\times_\tau \boldsymbol{Z}_2$-bundle. We can choose a $G\times_\tau \boldsymbol{Z}_2$-equivariant trace $t: X_+\wedge V^c\rightarrow E_+\wedge V^c$ of ξ [10] in such a way that V is a Real representation space of G. Then we define

$$p_!: KR_G(E) \to KR_G(X)$$

the transfer map for ξ in the KR_G-theory as the composite of the following sequence

$$KR_G(E) \xrightarrow{\Phi} KR_G(E\times V) \xrightarrow{t^*} KR_G(X\times V) \xrightarrow{\Phi^{-1}} KR_G(X).$$

This definition is well defined since the trace is unique. Similarly we define the transfer for an admissible G-bundle in the K_G-theory.

Let X be a Real G-space. If we forget the involution on X, then we may regard X as a G-space, which is denoted by ψX. We define the forgetful map

$$\psi: KR_G(X) \to K_G(\psi X)$$

by foregtting conjugate linear involutions on vector bundles. The following lemma is obtained straightforward from the definitions of the Thom elements.

Lemma 1. *The Thom isomorphisms commute with the forgetful maps, i.e., the diagram*

$$\begin{tikzcd} KR_G(X) \arrow[r, "\Phi"] \arrow[d, "p_!"] & KR_G(X\times V) \arrow[d, "p_!"] \\ K_G(\psi X) \arrow[r, "\Phi"] & K_G(\psi X\times \psi V) \end{tikzcd}$$

commutes, where V is a Real representation space of G.

Forgetting the involutions, an admissible $G\times_\tau \boldsymbol{Z}_2$-bundle becomes an admissible G-bundle and a $G\times_\tau \boldsymbol{Z}_2$-equivariant trace becomes a G-equivariant trace. So we have

Proposition 2. *The transfer maps commute with the forgetful maps, i.e., the following diagram commutes*

$$\begin{tikzcd} KR_G(E) \arrow[r, "\psi"] \arrow[d, "p_!"] & K_G(\psi E) \arrow[d, "p_!"] \\ KR_G(X) \arrow[r, "\psi"] & K_G(\psi X). \end{tikzcd}$$

2. The induced representation

Let G be a compact Lie group, H a closed subgroup and $i: H\subset G$ the inclusion map. Segal [11] defined the induction homomorphism $i_!: R(H)\to R(G)$ and Nishida [10] showed that the transfer map for a G-bundle ($p: G/H\to point$)

in the K_G-theory coincides with the induction homomorphism through the natural isomorphism $K_G(G/H)\cong R(H)$.

Let G be a compact Real Lie group, H a closed Real subgroup and $i: H\subset G$ the inclusion map. $R_R(G)$ denotes the Real representation ring of G [5]. The forgetful map

$$\psi: R_R(G) \to R(G)$$

is defined by forgetting conjugate linear involutions. It is well known that this forgetful map is injective. When the involution acts trivially on G, then $R_R(G)$ is naturally isomorphic to $RO(G)$ and the forgetful map coincides with the complexification map $c: RO(G)\to R(G)$. The diagram

$$\begin{tikzcd}
R_R(H) \arrow[ddd, "\psi"'] & & & R_R(G) \arrow[ddd, "\psi"] \\
& KR_G(G/H) \arrow[ul, "\cong"'] \arrow[r, "p_!"] \arrow[d, "\psi"] & KR_G(\text{point}) \arrow[ur, "\cong"] \arrow[d, "\psi"] & \\
& K_G(\psi(G/H)) \arrow[dl, "\cong"] \arrow[r, "p_!"] & K_G(\text{point}) \arrow[dr, "\cong"'] & \\
R(H) \arrow[rrr, "i_!"] & & & R(G)
\end{tikzcd}$$

commutes by the definition of the natural isomorphism $KR_G(G/H)\cong R_R(H)$, Proposition 2 and [10], Theorem 5.2. We define an induction homomorphism

$$i_!: R_R(H) \to R_R(G)$$

as the composite of the upper horizontal map and two isomorphisms of this diagram. In case the involution is trivial, we have an induction homomorphism

$$i_!: RO(H) \to RO(G)\,.$$

Since the forgetful map and the complexification map preserve the characters, these induction homomorphisms satisfy the character formula [11], p. 119–120.

Let E be a compact Real G-space such that ψE is a free G-space. For a Real representation space M of G, we define $\alpha(M)$ as a Real G-vector bundle $(E\times_G M\to E/G)$. The correspondence $M\to\alpha(M)$ induces a homomorphism

$$\alpha: R_R(G) \to KR(E/G)\,.$$

When the involution acts trivially on G and E, we have a homomorphism

$$\alpha: RO(G) \to KO(E/G)\,.$$

Proposition 3. *The diagram*

$$\begin{array}{ccc} R_R(H) & \xrightarrow{\alpha} & KR(E/H) \\ \downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle p_!} \\ R_R(G) & \xrightarrow{\alpha} & KR(E/G) \end{array}$$

commutes, where $p_!$ *is the transfer for an admissible* $\boldsymbol{Z}_2$*-bundle* $(p\colon E/H \to E/G)$.

This proof is parallel to [10] Proposition 5.4, so we omit it. In case the involution is trivial, we have

Corollary 4. *The following diagram commutes*

$$\begin{array}{ccc} RO(H) & \xrightarrow{\alpha} & KO(E/H) \\ \downarrow{\scriptstyle i_!} & & \downarrow{\scriptstyle p_!} \\ RO(G) & \xrightarrow{\alpha} & KO(E/G)\,. \end{array}$$

3. Real representations of the orthogonal group

In this section we put $G=O(2m+1)$ and $H=O(2)\times O(2m-1)$. Let $i\colon H\subset G$ be the standard inclusion, i.e., $i(B,C)=\begin{pmatrix} B & 0 \\ 0 & C\end{pmatrix}$. Let ι and ν be representations of G, whose actions are $\iota(A)x=Ax$ and $\nu(A)y=\det A\cdot y$ for $A\in G$, $x\in\boldsymbol{R}^{2m+1}$ and $y\in\boldsymbol{R}$. And let μ be a representation of H, whose action is $\mu(B,C)z=Bz$ for $(B,C)\in H$ and $z\in\boldsymbol{R}^2$.

Proposition 5. $\iota=i_!\mu+\nu$

Proof. We take the characters of both representations and we shall see that they are equal as class functions. Since G consists of exactly two connected components, we have two conjugacy classes of Cartan subgroups of G in the sense of Segal [11], and we may choose T^m and $T^m\times\boldsymbol{Z}_2$ as representatives of them, where T^m is the standard maximal torus of $SO(2m+1)$ and $\boldsymbol{Z}_2$ is generated by $-I_{2m+1}$, the diagonal matrix with -1 as diagonal entries. Let $g(\theta_1,\theta_2,\cdots,\theta_m;\varepsilon)$ be a matrix

$$\begin{pmatrix} D(\theta_1) & & & & \\ & D(\theta_2) & & & \\ & & \ddots & & \\ & & & D(\theta_m) & \\ & & & & \varepsilon \end{pmatrix}$$

where $D(\theta)=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}$, $0\leqq\theta_k<2\pi$, $\varepsilon=\pm 1$. Every topological generators of T^m (resp. $T^m\times\boldsymbol{Z}_2$) can be expressed as $g=g(\theta_1,\theta_2,\cdots,\theta_m;1)$ (resp. $g'=g(\theta_1,$

$\theta_1, \cdots, \theta_m; -1)$) such that $\theta_1, \theta_2, \cdots, \theta_m$ and π are linearly independent over the rational field. Since the topological generators of Cartan subgroups are dense in G, it is sufficient to show that those characters coincide on g and g'. It is easy to see that $\chi_\iota(g) = \sum_{k=1}^m 2\cos\theta_k + 1$, $\chi_\iota(g') = \sum_{k=1}^m 2\cos\theta_k - 1$, $\chi_\mu(g) = \chi_\mu(g') = 2\cos\theta_1$, $\chi_\nu(g) = 1$, $\chi_\nu(g') = -1$. By the character formula, the character of $i_!\mu$ is written as

$$\chi_{i_!\mu}(g) = \sum_{x\in F}\chi_\mu(x^{-1}gx)$$
$$\chi_{i_!\mu}(g') = \sum_{y\in F'}\chi_\mu(y^{-1}g'y)$$

where F (resp. F') is the set of representatives of fixed points of the action of g (resp. g') on G/H. We shall describe F' explicitly. $y \in F'$ means $y^{-1}gy \in H$, and $y^{-1}gy$ generates a Cartan subgroup T of H which is isomorphic to $T^m \times \boldsymbol{Z}_2$. Put $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and let U_0 be the subgroup of H generated by $\begin{pmatrix} A & \\ & I_{2m-1} \end{pmatrix}$ and U_1 the subgroup of H generated by $\begin{pmatrix} A & \\ & -I_{2m-1} \end{pmatrix}$. T^{m-1} denotes the maximal torus of $SO(2m-1)$ which we regard as a subgroup of H. U_0 and U_1 are subgroups of $Z_H(T^{m-1})$, the centralizer of T^{m-1} in H. We define $S_0 = U_0 \times T^{m-1}$ and $S_1 = U_1 \times T^{m-1}$. S_0 and S_1 are isomorphic to $T^{m-1} \times \boldsymbol{Z}_2$ and they are Cartan subgroups of H which are not conjugate. According to Segal [11], there are just four conjugacy classes of Cartan subgroups of H since H/H^0, the group of components is isomorphic to $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. And we may take T^m, $T^m \times \boldsymbol{Z}_2$, S_0 and S_1 as representatives of those conjugacy classes. Thus T, the group generated by $y^{-1}g'y$, is conjugate to $T^m \times \boldsymbol{Z}_2$ in H, i.e., there exists an element h of H such that $T^m \times \boldsymbol{Z}_2 = h^{-1}Th$. Then $yh \in N_G(T^m \times \boldsymbol{Z}_2)$, the normalizer, and y and yh are in the same coset in G/H. So we can take F' as a subset of $N_G(T^m \times \boldsymbol{Z}_2)$. The natural projection $G \to G/H$ sends $N_G(T^m \times \boldsymbol{Z}_2)$ to $N_G(T^m \times \boldsymbol{Z}_2)/(N_G(T^m \times \boldsymbol{Z}_2) \cap H)$ and evidently $N_G(T^m \times \boldsymbol{Z}_2) \cap H = N_H(T^m \times \boldsymbol{Z}_2)$. So we identify F' with $N_G(T^m \times \boldsymbol{Z}_2)/N_H(T^m \times \boldsymbol{Z}_2)$. It is easy to see that

$$N_G(T^m \times \boldsymbol{Z}_2)/(T^m \times \boldsymbol{Z}_2) \cong \textstyle\sum_m \int \boldsymbol{Z}_2$$

$$N_H(T^m \times \boldsymbol{Z}_2)/(T^m \times \boldsymbol{Z}_2) \cong \boldsymbol{Z}_2 \times \textstyle\sum_{m-1} \int \boldsymbol{Z}_2$$

So we shall identify them. Let $y = (\sigma; \varepsilon_1, \cdots, \varepsilon_m) \in \sum_m \int \boldsymbol{Z}_2$ and $y' = (\delta, \rho; \delta_1, \cdots, \delta_{m-1}) \in \boldsymbol{Z}_2 \times \sum_{m-1} \int \boldsymbol{Z}_2$. Then

$$y^{-1}g(\theta_1, \cdots, \theta_m; -1)y = g(\varepsilon_1\theta_{\sigma^{-1}(1)}, \cdots, \varepsilon_m\theta_{\sigma^{-1}(m)}; -1)$$
$$y'^{-1}g(\theta_1, \cdots, \theta_m; -1)y' = g(\delta\theta_1, \delta_1\theta_{1+\rho^{-1}(1)}, \cdots, \delta_{m-1}\theta_{1+\rho^{-1}(m-1)}; -1)\,.$$

Since $\# F' = m$, $\chi_{i_!\mu}(g') = \sum_{k=1}^m 2\cos\theta_k$. Similarly $\chi_{i_!\mu}(g) = \sum_{k=1}^m 2\cos\theta_k$. This completes the proof.

4. The Adams conjecture

We state the Adams conjecture in real case and prove it. Let F_n be the monoid of based homotopy equivalences of S^n. Let BF_n be the classifying space of F_n and $BF=\varinjlim BF_n$. The homotopy set $[X_+, BF]$ is isomorphic to the group of stable fibre homotopy equivalence classes of spherical fibre spaces [14]. For a finite CW-complex X, an abelian group $Sph(X)$ is defined as $[X_+, BF\times \boldsymbol{Z}]$, and the J-homomorphism $J\colon KO(X)\to Sph(X)$ is defined by $J(\xi)=([\xi], \dim \xi)$ for a real vector bundle ξ where $[\xi]$ denotes the class of the associated sphere bundle. By Segal [13], $\{O(n)\}$ and $\{F_n\}$ are Γ-spaces and the map $j=\{j_n\colon O(n)\to F_n\}$ is a map of Γ-spaces. So $BO\times \boldsymbol{Z}$ and $BF\times \boldsymbol{Z}$ become infinite loop spaces and $j\colon BO\times \boldsymbol{Z}\to BF\times \boldsymbol{Z}$ becomes an infinite loop map. Remark that this infinite loop space structure of $BO\times \boldsymbol{Z}$ coincides with the infinite loop space structure induced from the Thom isomorphism [15]. So $Sph(X)$ is a 0-th term of a generalised cohomology theory and $J=j^*$ is a stable natural transformation. By [10], Proposition 4.3, we have

Lemma 6. *The transfer commutes with the J-homomorphism.*

Let q be a prime number. For an abelian group A, $A\otimes \boldsymbol{Z}\left[\frac{1}{q}\right]$ is denoted by $A\left[\frac{1}{q}\right]$. Let ψ^q be the q-th Adams operation. Since $\alpha\colon RO(G)\to KO(E/G)$ is a λ-ring homomorphism, α commutes with ψ^q. It is well known that ψ^q is a stable operation on $KO(X)\left[\frac{1}{q}\right]$. So we have

Lemma 7. *The transfer commutes with ψ^q in the $KO(\ )\left[\frac{1}{q}\right]$-theory.*

Now we prove

Theorem 8 (Adams conjecture).

$$J(\psi^q-1)=0;\, KO(X)\left[\frac{1}{q}\right]\to Sph(X)\left[\frac{1}{q}\right].$$

Proof. Adams [2] proved this theorem for one and two dimensional vector bundles. Since odd dimensional vector bundles generate $KO(X)$ as an abelian group, it is sufficient to prove the theorem for odd dimensional vector bundles. Let ξ be a $(2m+1)$-dimensional real vector bundle over X and $(E\to X)$ the associated principal $O(2m+1)$-bundle. Let G and H be the same groups as in §3. Consider the following commutative diagram

$$\begin{array}{ccccc}
RO(H)\left[\frac{1}{q}\right] & \xrightarrow{\alpha} & KO(E/H)\left[\frac{1}{q}\right] & \xrightarrow{J} & Sph(E/H)\left[\frac{1}{q}\right] \\
\big\downarrow i_! & & \big\downarrow p_! & & \big\downarrow p_! \\
RO(G)\left[\frac{1}{q}\right] & \xrightarrow{\alpha} & KO(E/G)\left[\frac{1}{q}\right] & \xrightarrow{J} & Sph(E/G)\left[\frac{1}{q}\right]
\end{array}$$

where $p_!$ is the transfer for the bundle (p: $E/H \to E/G$). Clearly $\xi = \alpha(\iota)$. Since ν is a one dimensional representation and μ is a two dimensional representation, we have

$$\begin{aligned} J(\psi^q - 1)(\xi) &= J(\psi^q - 1)\alpha(\iota) \\ &= J(\psi^q - 1)\alpha i_!(\mu) + J(\psi^q - 1)\alpha(\nu) \\ &= J(\psi^q - 1)p_!\alpha(\mu) + J(\psi^q - 1)\alpha(\nu) \\ &= p_! J(\psi^q - 1)\alpha(\mu) + J(\psi^q - 1)\alpha(\nu) \\ &= 0 . \end{aligned}$$

This completes the proof.

References

[1] J.F. Adams: *Vector fields on spheres,* Ann. of Math. **75** (1962), 603–632.

[2] J.F. Adams: *On the group $J(X)$-I.* Topology **2** (1963), 181–195.

[3] M.F. Atiyah: K-theory, Benjamin 1967.

[4] M.F. Atiyah: *Bott periodicity and the index of elliptic operators,* Quart. J. Math. Oxford (2), **19** (1968), 113–140.

[5] M.F. Atiyah and G.B. Segal: *Equivariant K-theory and completions,* J. Differential Geom. **3** (1969), 1–18.

[6] M.F. Atiyah and I.M. Singer: *The index of elliptic operators* IV, Ann. of Math. **87** (1968), 484–530.

[7] J.C. Becker and D.H. Gottlieb: *The transfer map and fibre bundles,* Topology **14** (1975), 1–12.

[8] T. tom Dieck: *Faserbündel mit Gruppenoperation* Arch. Math. **20** (1969), 136–143.

[9] C. Kosniowski: *Equivariant cohomology and stable cohomotopy,* Math. Ann. **210** (1974), 83–104.

[10] G. Nishida: *The transfer homomorphism in equivariant generalised cohomology,* J. Math. Kyoto Univ. **18** (1978), 435–451.

[11] G.B. Segal: *The representation ring of a compact Lie group,* Inst. Hautes Études Sci. Publ. Math. **34** (1968), 113–128.

[12] G.B. Segal: *Equivariant K-theory,* Inst. Hautes Études Sci. Publ. Math. **34** (1968), 129–151.

[13] G.B. Segal: *Categories and cohomology theories,* Topology **13** (1974), 293–312.

[14] D. Stasheff: *A classification theorem for fibre spaces,* Topology **2** (1963), 239–246.

[15] J.P. May: E_∞ ring spaces and E_∞ ring spectra, Lecture Notes in Math. 577, Springer.

Department of Mathematics
Osaka City University
Osaka 558
Japan