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ON A KRULL ORDER

Dedicated to Professor Gorô Azumaya on his 60th birthday

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Let R be a ring with $1(\neq 0)$, τ an automorphism of R, and D a τ-derivation of R (i.e. $D(ab)=D(a)\tau(b)+a\cdot D(b)$ for all $a, b\in R$). Then a skew polynomial ring $A=R[t;\tau, D]=R\oplus tR\oplus t^2R\oplus\cdots$ is well defined by $at=t\tau(a)+D(a)$ $(a\in R)$. Then if R is a two-sided simple ring, every ideal of A is invertible. On the other hand, as is well known, a (commutative) polynomial ring over a Krull domain is also a Krull domain. Furthermore, if R is a (non-commutative) Krull order in the sense of Marubayashi, then so is $R[t]$ ([11]). This is the case when $\tau=id$ and $D=0$. In this paper we define a new "Krull order", and prove the following. If R is a Krull order then A is also a Krull order. Further we obtain some results on the structure of the group of reflexive fractional ideals of A. Any two-sided simple ring is a Krull order in our sense. In the case when R is a prime Goldie ring, R is a Krull order if and only if R is a maximal order and the ascending chain condition on integral reflexive ideals holds.

As a matter of fact, we prove main results in a more general situation. Namely we take some "positively filtered ring" instead of $R[t;\tau, D]$. By virtue of this, for example, if M is an invertible R-bimodule over a Krull order R then the tensor ring $T(M)$ is a Krull order. We believe this generalizatiln is proper for this kind of study. However, if we assume R to be a prime Goldie ring, arguments may become more brief. But this exclude the case when R is a two-sided simple ring from our study. As is seen in §1, we take, as a starting point, the set of ideals which have trivial dual modules. This may be a feature of our study on Krull orders. Main results are analogous to those on a polynomial ring over a unique factorization domain.

For the completeness of this paper, we need some arguments on the construction of a positively filtered ring. But we postpone these until the forthcoming paper. However the case when $A=R[t;\tau, D]$ is treated in 4. Appendix. In all that follows, all rings are associative, but not necessarily commutative. Every ring has $1(\neq 0)$, which is preserved by homomorphisms, inherited by subrings and acts as the identity operator on modules.

1. Preliminary results

Let A, B be rings. If M is a left (resp. right) A-module, we write ${}_AM$

(resp. M_A). If N is a left A-, right B-bimodule we write ${}_AN_B$, and we briefly call N an A-B-module.

Let Q be a ring, and M an additive submodule of Q. We define the *left order* of M (in Q) as $O_l(M)=\{x\in Q: xM\subseteq M\}$. Similarly we define the *right order* of M as $O_r(M)=\{x\in Q: Mx\subseteq M\}$. Then, $\{x\in Q: MxM\subset M\}=\{x\in Q: Mx\subseteq O_l(M)\}=\{x\in Q: xM\subseteq O_r(M)\}$, which is denoted by M^{-1}. Evidently M^{-1} is an $O_r(M)$-$O_l(M)$-submodule, $M^{-1}M$ is an ideal of $O_r(M)$, and MM^{-1} is an ideal of $O_l(M)$. Let R be a subring of Q. By $T(Q; R)$ (abbr. $T(R)$) we denote the set of all ideals I satisfying the following conditions.

(i) I is faithful as a left R-module as well as a right R-module.

(ii) If $xI\subseteq R$ or $Ix\subseteq R$ $(x\in Q)$ then $x\in R$.

Evidently $T(R)$ satisfies the following.

(i) $R\in T(R)$.

(ii) If $I\in T(R)$, and I' is an ideal of R such that $I\subseteq I'$ then $I'\in T(R)$.

(iii) If $I_1, I_2\in T(R)$ then $I_1I_2\in T(R)$, and so $I_1\cap I_2\in T(R)$ (by (ii)).

(iv) If $I\in T(R)$ then $O_l(I)=R=O_r(I)$. Therefore if $xI=0$ or $Ix=0$ $(x\in Q)$ then $x=0$.

Proposition 1.1. *Let A, B be subrings of Q, and M an A-B-submodule of Q. Then the following conditions are equivalent.*

(1) *There are B-A-submodules M', M'' of Q such that $MM'\in T(A)$, $M''M\in T(B)$.*

(2) *$MM^{-1}\in T(A)$, and $M^{-1}M\in T(B)$.*

(3) *$O_l(M)=O_r(MM^{-1})=A$, and $O_r(M)=O_l(M^{-1}M)=B$. Further ${}_AM$, M_B, MM^{-1}_A, and ${}_BM^{-1}M$ are faithful modules.*

(4) *$O_l(M)=O_r(M^{-1})=A$, and $O_r(M)=O_l(M^{-1})=B$. Further ${}_AM$, M_B, M^{-1}_A, and ${}_BM^{-1}$ are faithful modules.*

Proof. The implication (2)$\Rightarrow$(1) is trivial, and it is easy to see that (2)$\Rightarrow$(3), (3)$\Rightarrow$(4). (1)$\Rightarrow$(2) Evidently $O_l(M)=A$, and $O_r(M)=B$. Therefore $M'\subseteq M^{-1}$, and $M''\subseteq M^{-1}$. Hence $MM'\subseteq MM^{-1}$, and $M''M\subseteq M^{-1}M$. Thus we obtain (2). (4)$\Rightarrow$(2) If $M^{-1}My\subseteq B$ then $M^{-1}MyM^{-1}\subseteq M^{-1}$, hence $MyM^{-1}\subseteq O_r(M^{-1})=A$. Therefore $yM^{-1}\subseteq M^{-1}$, so $y\in O_l(M^{-1})=B$. On the other hand, if $zM^{-1}M\subseteq B$ then $zM^{-1}\subseteq M^{-1}$, hence $z\in O_l(M^{-1})=B$. If $bM^{-1}M=0$ $(b\in B)$ then $bM^{-1}\subseteq O_l(M)=A$, and so $bM^{-1}=0$. Hence $b=0$. Thus ${}_BM^{-1}M$ is faithful. Similarly $M^{-1}M_B$ is faithful. Hence $M^{-1}M\in T(B)$. Symmetrically we have $MM^{-1}\in T(A)$. This completes the proof.

Let A, B be subrings of Q. By $F(Q; A, B)$ (abbr. $F(A, B)$) we denote the set of all A-B-submodules M satisfying the condition (1) of Proposition 1.1. We put $F(Q)=\bigcup_{A,B}F(Q; A, B)$, where A, B run through all subrings of Q. In the sequel, if $M\in F(Q; A, B)$ then we write ${}_AM_B\in F(Q)$, conveniently. Note that $T(Q; A)\subseteq F(Q; A, A)$, and that if ${}_AM_B\in F(Q)$ then $xM=0$ or $Mx=0(x\in$

Q) implies $x=0$.

Proposition 1.2. *Let* ${}_AM_B$, ${}_BN_C\in F(Q)$.
(i) ${}_BM_A^{-1}\in F(Q)$, *and* $MIM^{-1}\in T(A)$ *for any* $I\in T(B)$.
(ii) ${}_AMN_C\in F(Q)$.

Proof. (i) It follows from Proposition 1.1 that ${}_BM_A^{-1}\in F(Q)$. Let $I\in T(B)$. If $MIM^{-1}x\subseteq A\,(x\in Q)$ then $IM^{-1}x\subseteq M^{-1}$, and so $IM^{-1}xM\subseteq M^{-1}M\subseteq B$. Therefore $M^{-1}xM\subseteq B$, hence $M^{-1}x\subseteq M^{-1}$. Thus $x\in O_r(M^{-1})=A$. On the other hand, $yMIM^{-1}\subseteq A$ implies that $M^{-1}yMIM^{-1}M\subseteq B$, and so $M^{-1}yM\subseteq B$. Hence $M^{-1}y\subseteq M^{-1}$, and therefore $y\in A$. Thus $MIM^{-1}\in T(A)$. (ii) If $xMN\subseteq MN$ then $M^{-1}xMNN^{-1}\subseteq M^{-1}MNN^{-1}\subseteq B$, and so $M^{-1}xM\subseteq B$. Then $x\in A$ as in (i). Thus $O_l(MN)=A$, and similarly $O_r(MN)=C$. Now, $MNN^{-1}M^{-1}MN\subseteq MM^{-1}MN\subseteq MN$, and so $N^{-1}M^{-1}\subseteq (MM)^{-1}$. Therefore $MNN^{-1}M^{-1}\subseteq (MN)\,(MN)^{-1}$, and $N^{-1}M^{-1}MN\subseteq (MN)^{-1}(MN)$. Since $NN^{-1}\in T(B)$ and $M^{-1}M\in T(B)$, it follows from (i) that $(MN)\,(MN)^{-1}\in T(A)$ and $(MN)^{-1}(MN)\in T(C)$. By Proposition 1.1, we have ${}_AMN_C\in F(Q)$.

If ${}_AM_B\in F(Q)$ then ${}_BM_A^{-1}\in F(Q)$, and so ${}_A(M^{-1})_B^{-1}\in F(Q)$. Since $MM^{-1}\subseteq A$ we have $M\subseteq (M^{-1})^{-1}$. Then $M^{-1}\supseteq ((M^{-1})^{-1})^{-1}$. On the other hand, $M^{-1}\subseteq ((M^{-1})^{-1})^{-1}$. Hence $M^{-1}=((M^{-1})^{-1})^{-1}$. We put $M^*=(M^{-1})^{-1}$. Then $M\subseteq M^*=M^{**}$ for any $M\in F(Q)$.

Proposition 1.3. *For any* ${}_AM_B\in F(Q)$, $M^*=\{x\in Q$: $Ix\subseteq M$ *for some* $I\in T(A)\}=\{x\in Q$: $xJ\subseteq M$ *for some* $J\in T(B)\}$.

Proof. If $x\in M^*$ then $M^{-1}x\subseteq B$, and so $MM^{-1}x\subseteq M$, where $MM^{-1}\in T(A)$. Conversely if $Ix\subseteq M$ for some $I\in T(A)$, then $IxM^{-1}\subseteq MM^{-1}\subseteq A$, so $xM^{-1}\subseteq A$. Hence $x\in (M^{-1})^{-1}=M^*$. Symmetrically we obtain the latter half.

Evidently, for any subring A or Q, $T(Q;\,A)=\{I\in F(Q;\,A,\,A)$: $I^*=A\}$.

Proposition 1.4. *Let* ${}_AM_B$, ${}_BN_C\in F(Q)$. *Then* $(MN)^{-1}=(N^{-1}M^{-1})^*$, *and* $(M^*N)^*=(MN)^*=(MN^*)^*$.

Proof. Since $N^{-1}M^{-1}\subseteq (MN)^{-1}$, we have $(N^{-1}M^{-1})^*\subseteq ((MN)^{-1})^*=(MN)^{-1}$. On the other hand, $x\in (MN)^{-1}$ implies that $MNx\subseteq A$, and so $Nx\subseteq M^{-1}$. Then $N^{-1}Nx\subseteq N^{-1}M^{-1}$, hence $x\in (N^{-1}M^{-1})^*$, because of $N^{-1}N\in T(C)$. Thus $(MN)^{-1}=(N^{-1}M^{-1})^*$. Using this, $(MN)^*=((N^{-1}M^{-1})^*)^{-1}=(N^{-1}M^{-1})^{-1}$. As $(M^*)^{-1}=M^{-1}$, we have $(M^*N)^*=(N^{-1}M^{-1})^{-1}=(MN)^*$. Similarly $(MN^*)^*=(N^{-1}M^{-1})^{-1}=(MN)^*$.

If ${}_AM_B\in F(Q)$ and $M^*=M$, we call M a reflexive A-B-submodule of Q. By $F^*(Q;\,A,\,B)$ (abbr. $F^*(A,\,B)$) we denote the set of all reflexive A-B-submodules of Q, and we put $F^*(Q)=\bigcup_{A,B}F^*(Q;\,A,\,B)$, where A,B run through all subrings of Q. By $F_i(Q;\,A)$ (abbr. $F_i(A)$) we denote $\{M\in F(Q;\,A,\,A)$: $M\subseteq A\}$, and we denote $F_i(Q;\,A)\cap F^*(Q;\,A,\,A)$ by $F_i^*(Q;\,A)$ (abbr. $F_i^*(A)$). If $I\in$

$F_i(A)$ (resp. $I \in F_i^*(A)$) we call I an *integral ideal* (resp. *reflexive ideal*) of A.

Let ${}_AM_B$, ${}_BN_C \in F^*(Q)$. We define $M \circ N$ by $(MN)^*$. Then, from Proposition 1.2 and Proposition 1.4, we have the folllwing.

Theorem 1.5. *The set of all reflexive submodules of Q, $F^*(Q)$ is a Brandt groupoid. The set of identities of $F^*(Q)$ is the set of all subrings of Q.*

Let A, B be subrings of Q, and ${}_AM_B$ an A-B-submodule of Q. If there are B-A-submodules M', M'' of Q such that $MM'=A$ and $M''M=B$, we call M an invertible A-B-submodule of Q. Then it is easily seen that ${}_AM_B \in F^*(Q; A, B)$ and $M^{-1}=M'=M''$. Here we note the following

Proposition 1.6. *Let ${}_AM_B$, ${}_BN_C \in F^*(Q)$. If ${}_AM_B$ or ${}_BN_C$ is an invertible submodule then $M \circ N=MN$.*

Proof. We first assume that ${}_BN_C$ is invertible. If $xMN \subseteq C$ then $xM \subseteq N^{-1}$, so $NxM \subseteq NN^{-1}=B$. Therefore $Nx \subseteq M^{-1}$, and so $x \in N^{-1}M^{-1}$. Thus $(MN)^{-1}=N^{-1}M^{-1}$. Similarly $(MN)^{-1}=N^{-1}M^{-1}$, when ${}_AM_B$ is invertible. Hence $M \circ N=(N^{-1}M^{-1})^{-1}=M^*N^*=MN$, when ${}_AM_B$ or ${}_BN_C$ is invertible (cf. Proposition 1.4).

REMARK. Let ${}_AM_B$ be invertible in Q. Then $Q \otimes_A M \cong Q$, $q \otimes m \mapsto qm$ ($q \in Q$, $m \in M$) (, and symmetrically $M \otimes_B Q \cong Q$). In fact, if $1=\Sigma_i m_i' m_i$ ($m_i' \in M^{-1}$, $m_i \in M$) then the inverse of the homomorphism $Q \otimes_A M \to Q$ is given by the map $q \mapsto \sum_i qm_i' \otimes m_i$ ($q \in Q$). As is well known, M is an invertible A-B-bimodule, that is, M_B is finitely generated, projective, and a generator, and $A \cong \mathrm{End}_B(M)$ by the map induced by ${}_AM$ (cf. [3]).

Let A, B be subrings of Q. If there exists an A-B-submodule $M \in F^*(Q; A, B)$ we write $A \sim B$ (in Q). Then "$\sim$" is an equivalence relation on the subrings of Q.

If $O_l(I)=O_r(I)=A$ holds for any ideal I of A such that both ${}_AI$ and I_A are faithful, we say that A is *maxmial* in Q.

Proposition 1.7. *For any subring A of Q, the following conditions are equivalent*:

(1) *A is maximal in Q.*

(2) *${}_AI_A \in F(Q; A, A)$ for every ideal I of A such that both ${}_AI$ and I_A are faithful.*

Proof. The implication (2)$\Rightarrow$(1) is trivial, and (1)$\Rightarrow$(2) follows from Proposition 1.1 (3).

Proposition 1.8. *Let ${}_AU_B \in F^*(Q; A, B)$.*

(i) *If A is maximal in Q then so is B.*

(ii) *There is a group isomorphism $F^*(Q; A, A) \cong F^*(Q; B, B)$, $M \mapsto (U^{-1}MU)^*$*

$=U^{-1}\circ M\circ U\ (M\in F^*(Q;A,A))$.

(iii) *If A is a prime ring then so is B.*

Proof. (i) Let I' be an ideal of B such that ${}_BI'$, I'_B are faithful. Put $I=UI'U^{-1}$. It is easy to see that both ${}_AI$ and I_A are faithful. Therefore, by assumption, $O_l(I)=O_r(I)=A$. It $xI'\subseteq I'$ then $UxU^{-1}I=UxU^{-1}UI'U^{-1}\subseteq UxI'U^{-1}\subseteq UI'U^{-1}=I$, and so $UxU^{-1}\subseteq O_l(I)=A$. Then $xU^{-1}\subseteq U^{-1}$, so $xU^{-1}U\subseteq U^{-1}U$. Hence $x\in B$. Thus $O_l(I')=B$. Similarly $O_r(I')=B$. Hence B is maximal in Q. (ii) This follows from Theorem 1.5. (iii) Let I, J be ideals of B, and assume that $IJ=0$. Then $UIU^{-1}\cdot UJU^{-1}=0$, and so $UIU^{-1}=0$ or $UJU^{-1}=0$. If $UIU^{-1}=0$ then $UI=0$, so $I=0$. Hence B is a prime ring.

Proposition 1.9. *Let A, B be subrings of Q such that $A\sim B$ in Q, and assume that A is a prime ring and is maximal in Q. Let M be an A-B-submodule of Q. Assume that there are elements u, v of Q such that $0\neq uM\subseteq B$ and $0\neq Mv\subseteq A$. Then ${}_AM_B\in F(Q;A,B)$.*

Proof. By Proposition 1.8, B is a prime ring, and is maximal in Q. Since BuM and MvA are non-zero ideals of B and A respectively, we have $O_r(M)=B$ and $O_l(M)=A$. Since $M^{-1}\ni u, v, M^{-1}M$ and MM^{-1} are non-zero ideals of B and A, respectively. Then, by Proposition 1.1 (3), $M\in F(Q;A,B)$.

Now we define a Krull subring of Q. A subring A of Q is said to be a *Krull subring* of Q if A is maximal in Q and the ascending chain condition on reflexive ideals of A holds. The following proposition follows from Proposition 1.8.

Proposition 1.10. *Let A, B be subrings of Q such that $A\sim B$ in Q. If A is a Krull subring of Q then so is B.*

Let A be any subring of Q. Let $P\in F_i^*(Q;A)$, and let $P\neq A$. Then P is said to be *irreducible* if $P=I_1\circ I_2(I_1, I_2\in F_i^*(Q;A))$ implies that $P=I_1$ or $P=I_2$, and P is said to be *maximal* if $P\subsetneqq I'\in F_i^*(Q;A)$ implies that $I'=A$. Assume that P is maximal in $F_i^*(Q;A)$, and let $P=I_1\circ I_2$. Then $P=(I_1I_2)^*\subseteq I_i^*=I_i$ $(i=1,2)$, hence $P=I_i$ or $I_i=A$. Therefore P is irreducible. Conversely, if P is irreducible then P is maximal. Thus "maximal" and "irreducble" are equivalent.

Assume that A is maximal in Q, and let P be irreducible in $F_i^*(Q;A)$. If $IJ\subseteq P$ for some ideals I, J of A then $(I+P)(J+P)\subseteq P$. If $I\not\subseteq P$ and $J\not\subseteq P$ then $I+P, J+P\in T(Q;A)$ by Proposition 1.7, so that $(I+P)(J+P)\in T(Q;A)$. Then have a contradiction $P\in T(Q;A)$. Hence P is a prime ideal. Conversely if $P\in F_i^*(Q;A)$ is a (proper) prime ideal then P is irreducble. Therefore, as is well known, if P, P' are irreducible in $F_i^*(A)$ then $P\circ P'=P'\circ P$. Then, in the usual way, we have the following.

Proposition 1.11. *Let A be a Krull subring of Q. Then any irreducible re-*

flexive ideal of A is a prime ideal, and $F_i^*(Q; A)$ is commutative. Any element of $F_i^*(Q; A)$ is uniquely represented as a product of irreducible elements of $F_i^*(Q; A)$.

Proposition 1.12. *Let A be a Krull subring of Q, and let ${}_AM_B \in F(Q; A, B)$. Assume that A is a prime ring. Then any non-zero A-B-submodule of M belongs to $F(Q; A, B)$, and there are elements $x_1, \cdots, x_r$ of M such that $M^* = (\sum_{i=1,\cdots,r} Ax_iB)^*$.*

Proof. By Proposition 1.8, B is a prime ring and is maximal in Q. Let M_0 be a non-zero A-B-submodule of M. Then, since $M^{-1}M_0$ and M_0M^{-1} are non-zero ideals of B and A respectively, we have $M_0 \in F(Q; A, B)$, by virtue of Proposition 1.9. Now let $0 \neq x_1 \in M$. Then $Ax_1B \in F(Q; A, B)$, and $(Ax_1B)^* \subseteq M^*$. If $(Ax_1B)^* \subsetneqq M^*$ then there is an element $x_2 \in M$ with $x_2 \notin (Ax_1B)^*$. If $(Ax_1B + Ax_2B)^* \subsetneqq M^*$, then $(Ax_1B + Ax_2B)^* \subsetneqq (Ax_1B + Ax_2B + Ax_3B)^*$ for some $x_3 \in M$. Continueing this process we obtain $x_1, \cdots, x_r \in M$ such that $M^* = (\sum_i Ax_iB)^*$, because ACC holds on $\{N \in F^*(Q; A, B) : N \subseteq M^*\}$. (In fact, $N \subseteq M^*$ means $N \circ (M^*)^{-1} \subseteq A$, and conversely.)

Proposition 1.13. *Let Q' be any overring of Q, and A a prime subring of Q. Assume that, for any non-zero ideal I of A, $IQ = QI = Q$ holds. Then $T(Q; A) = T(Q'; A)$, and $F(Q; A, A) = \{M \in F(Q'; A, A) : M \subseteq Q, MQ = QM = Q\}$. Therefore $F_i(Q; A) = F_i(Q'; A)$, and $F_i^*(Q; A) = F_i^*(Q'; A)$.*

Proof. Evidently $T(Q; A) \supseteq T(Q'; A)$. Let $I \in T(Q; A)$, and let $Ix \subseteq A$ $(x \in Q')$. Then $Qx = QIx \subseteq QA = Q$, so $x \in Q$. Hence $x \in A$. Similarly $yI \subseteq A (y \in Q')$ implies that $y \in A$. Thus $I \in T(Q'; A)$. Let $M \in F(Q; A, A)$, and put $M' = \{x \in Q : MxM \subseteq M\}$. Then $MM', M'M \in T(Q; A) = T(Q'; A)$. Then, by Proposition 1.1 (1), we have $M \in F(Q'; A, A)$. Furthermore, $Q \supseteq MQ \supseteq MM'Q = Q$, and so $MQ = Q$. Similarly $QM = Q$. Conversely, let $N \in F(Q'; A, A)$, $N \subseteq Q$, and $NQ = QN = Q$. If $zN \subseteq A (z \in Q')$ then $zQ = zNQ \subseteq AQ = Q$, and so $z \in Q$. Hence $N \in F(Q; A, A)$. The remainder is obvious.

Corollary. *Assume the same assumptions as in Proposition 1.13. If A is maximal in Q (resp. a Krull subring of Q) then A is maximal in Q' (resp. a Krull subring of Q'), and conversely.*

Proof. This follows from Proposition 1.7 and Proposition 1.13.

Let A be a subring of Q. By $S(Q; A)$ (abbr. $S(A)$) we denote $\cup I^{-1}$, where I runs through reflexive ideals of A. Evidently $S(Q; A)$ is a subring containing A. We call $S(Q; A)$ the *Asano overring* of A in Q.

Proposition 1.14. *Let A be a prime Krull subring of Q. Assume that $I \cdot S(Q; A) = S(Q; A)I = S(Q; A)$ for any non-zero ideal I of A. Then any irreducible reflexive ideal of A is a (non-zero) minimal prime ideal of A, and con-*

versely (*cf.* [11]).

Proof. Let $P \in F_i^*(Q; A)$ be irreducible. Then P is a prime ideal. If there exists a non-zero prime ideal P' of A such that $P' \subsetneqq P$. Then $(P'P^{-1})P \subseteq P'$ implies that $P'P^{-1} \subseteq P'$. Then we have a contradiction $P^{-1} \subseteq A$. Hence P is minimal in the set of all non-zero prime ideals of A. Conversely, let P be a minimal prime ideal. Since $P \cdot S(Q; A) = S(Q; A) \ni 1$, there are reflexive ideals $I_1, \cdots, I_r$ of A such that $I_1 \cdots I_r \subseteq P$. Then $I_i \subseteq P$ for some i. Hence some irreducible component P'' of I_i is contained in P. Then, by the minimality of P, we have $P'' = P$. This completes the proof.

Note that, in the above case, A is a Krull subring of $S(Q; A)$, and $S(Q; A)$ is a left and right Utumi's quotient ring of A.

Proposition 1.15. *Let A be a prime subring of Q, and assume that A is maximal in Q. Let M be a non-zero left A-submodule of Q. Put $O_r(M) = B$ and $M' = \{x \in Q: Mx \subseteq A\}$.*

(α) *If $M'M \in T(B)$ then $M \in F(Q; A, B)$.*

(β) *Assume that M satisfies the following conditions:*

(i) *$xM' \neq 0$ for any non-zero $x \in M$.*

(ii) *M_B is faithful.*

(iii) *$\{y \in Q: yM' \subseteq A\} = M$.*

Then $M \in F^*(Q; A, B)$ (, and conversely). (*Cf.* [6].)

Proof. (α) As $M'M \in T(B)$, we have $MM'M \neq 0$, so $MM' \neq 0$. Hence $MM' \in F_i(Q; A)$, and so $O_l(M) = A$. Therefore $M' = M^{-1}$. If $MM'x \subseteq A$ then $M'x \subseteq M'$, so $MM'x \subseteq MM'$. Hence $x \in A$. If $yMM' \subseteq A$, then $MM'yMM' \subseteq MM'$, and so $MM'y \subseteq A$. Hence $y \in A$. Thus $MM' \in T(A)$. Hence $M \in F(Q; A, B)$. (β) Since MM' is a non-zero ideal of A, we have $MM' \in F_i(A)$, and $M' = M^{-1}$. If $xM' \subseteq M'$ then $MxM' \subseteq MM' \subseteq A$, hence $Mx \subseteq M$ by (iii). Therefore $x \in B$. If $xM' = 0$ then $x \in M$, hence $x = 0$ by (i). Thus $O_l(M') = B$, and ${}_BM'$ is faithful. Therefore (4) of Proposition 1.1 holds. Hence $M \in F^*(Q; A, B)$, by (iii).

2. A positively filtered ring over a Krull order

Let R be a subring of a ring Q. If R, Q satisfy the following conditions we call R a *Krull order* of Q.

(i) R is a Krull subring of Q.

(ii) Q is a left and right quotient ring of R.

(iii) $IQ = QI = Q$ for any non-zero ideal I of R.

REMARK. If R is a prime Goldie ring, and Q is the maximal quotient ring of R then (ii), (iii) hold. Evidently every two-sided simple ring is a Krull order of itself.

Let R be a Krull order of Q. Let M be a non-zero R-R-submodule of Q. Then $M\cap R\neq 0$, and so $Q(M\cap R)=Q=(M\cap R)Q$. Therefore $QM=Q=MQ$. Hence Q is a simple R-Q-module as well as a simple Q-R-module. In particular, Q is a two-sided simple ring. Let $M\in F(Q;\ R,\ R)$. Then $QM=Q\ni 1$, so that $I\subseteq M$ for some dense left ideal I of R. Then $IR\subseteq M$, and so $0\neq IR\cdot M^{-1}\subseteq R$. Put $IR\cdot M^{-1}=J$. Then $R\supseteq IR\cdot M^{-1}M=JM$, hence $M\subseteq J^{-1}$. Since $(IR)^*\circ M^{-1}=J^*$ we have $M^*=(IR)^*\circ J^{-1}$. Conversely, let N be a non-zero R-R-submodule of Q such that $N\subseteq J_1^{-1}$ for some non-zero ideal J_1 of R. Then, by Proposition 1.12, $N\in F(Q;\ R,\ R)$. Summing up, we have

Proposition 2.1. *Let R be a Krull order of Q.*

(i) *Both ${}_QQ_R$ and ${}_RQ_Q$ are simple.*

(ii) *Fcr a non-zero R-R-snbmodule N of Q, $N\in F(Q;\ R,\ R)$ if and only if $N\subseteq I^{-1}$ for some non-zero ideal I of R.*

(iii) $F^*(Q;\ R,\ R)=\{I\circ J^{-1}\colon I,\ J\in F_l^*(Q;\ R)\}$, *which is an abelian group.*

For any ring A we denote by $Q_l(A)$ (resp. $Q_r(A)$) the left (resp. right) maximal quotient ring of A. Further we put $Q(A)=Q_l(A)\cap Q_r(A)$, more precisely, $Q(A)=\{x\in Q_r(A);\ Iv\subseteq A$ for some dense left ideal $I\}$. By Corollary of Proposition 1.13, if R is a Krull order of Q, then R is a Krull order of $Q(R)$ $(\supseteq Q)$.

In the remainder of this paper we assume the followings: R is a Krull order of Q. X is Q-Q-module containing Q, as a Q-Q-submodule, and such that X/Q is an invertible Q-Q-module. Y is an R-R-submodule of X containing R, such that Y/R is an invertible R-R-module, and such that $X=Q\otimes_R Y=Y\otimes_R Q$. $Q\langle X\rangle$ is an overring of Q satisfying the following conditions:

(i) $Q\langle X\rangle\supseteq X$ as a Q-Q-submodule, and $Q\langle X\rangle=\bigcup_{i\geq 0}X^i$, where $X^0=Q$.

(ii) For any integer $i\geqq 1$, the canonical map

$$(X/Q)\otimes_Q\cdots\otimes_Q(X/Q)\ (i\text{-times})\to X^i/X^{i-1},$$

$(x_1+Q)\otimes\cdots\otimes(x_i+Q)\mapsto x_1\cdots x_i+X^{i-1}$ is an isomorphism (cf. [13]).

We put $R\langle Y\rangle=\bigcup_{i\geq 0}Y^i$, where $Y^0=R$. If $i<0$ then we put $X^i=Y^i=0$. Evidently $Q\otimes_R(Y/R)\xrightarrow{\sim}X/Q$, $q\otimes(y+R)\mapsto qy+Q$, and $(Y/R)\otimes_R Q\xrightarrow{\sim}X/Q$, $(y+R)\otimes q\mapsto yq+Q$. Therefore $Q\otimes_R(\overset{i}{\otimes}_R(Y/R))\xrightarrow{\sim}\overset{i}{\otimes}_Q(X/Q)$ as Q-R-modules, and $(\overset{i}{\otimes}_R(Y/R))\otimes_R Q\xrightarrow{\sim}\overset{i}{\otimes}_Q(X/Q)$ as R-Q-modules, where $\overset{i}{\otimes}_R(Y/R)=(Y/R)\otimes_R\cdots\otimes_R(Y/R)$ (i-times). For any $i\geqq 1$, the following diagram is commutative:

$$\begin{tikzcd}
\overset{i}{\otimes}_R(Y/R) \arrow[r, "\beta"] \arrow[d, "\alpha"'] & \overset{i}{\otimes}_Q(X/Q) \arrow[d, "\approx"] \\
Y^i/Y^{i-1} \arrow[r, "\delta"] & X^i/X^{i-1}
\end{tikzcd}$$

Since ${}_R\overset{i}{\otimes}_R(Y/R)$ is projective, the canonical map $\overset{i}{\otimes}_R(Y/R)\to Q\otimes_R(\overset{i}{\otimes}_R(Y/R))$ $(\cong \overset{i}{\otimes}_Q(X/Q))$ is a monomorphism, so that α is an isomorphism. Therefore δ is a monomorphism, that is, $Y^i\cap X^{i-1}=Y^{i-1}$. In particular, $Y\cap Q=R$. Using the diagram

$$\begin{tikzcd}
Q\otimes_R Y^i \arrow[r, "\varepsilon_i"] \arrow[d] & X^i \arrow[d] \\
Q\otimes_R(Y^i/Y^{i-1}) \arrow[r, "\approx"] & X^i/X^{i-1},
\end{tikzcd}$$

by induction on i, we can prove that each ε_i is an isomorphism. Therefore $Q\otimes_R R\langle Y\rangle=Q\langle X\rangle$, and symmetrically $R\langle Y\rangle\otimes_R Q=Q\langle X\rangle$. We put $\bar{Q}=Q\langle X\rangle$ and $\bar{R}=R\langle Y\rangle$.

REMARK. Let $Q=Q(R)$, and let Y be an R-R-module containing R, as an R-R-submodule, and such that Y/R is an invertible R-R-module. Then, X, $Q\langle X\rangle$, and $R\langle Y\rangle$ as above exist, and those are uniquely determined by $Y\supseteq R$. The proof is given in §4, in the case when $Y/R_R\cong R_R$.

First we prove the following

Theorem 2.2. *If R is a Krull order then $R\langle Y\rangle$ is also a Krull order.*

We need many lemmas.

Lemma 2.3. *For any integer $i\geqq 1$, there is a one to one correspondence from the set of all R-R-submodules of Q to the set of all R-R-submodules of X^i/X^{i-1}, such that $M\mapsto (MY^i+X^{i-1})/X^{i-1}$.*

Proof. This follows from [12; Proposition 3.3 and its proof].

Corollary 1. *For any integer $i\geqq 1$, X^i/X^{i-1} is a simple Q-R-module as well as a simple R-Q-module.*

Proof. This follows from the fact that ${}_QQ_R$, ${}_RQ_Q$ are simple.

Corollary 2. *For any integer $i\geqq 1$, there is a one to one correspondence $M\mapsto M'$ from the set of all R-R-submodules of Q to itself, which is defined by $M'Y^i+X^{i-1}=Y^iM+X^{i-1}$. (Note that this map is multiplicative.)*

Lemma 2.4. *Let M be an R-Q-submodule of $X^r (r\geqq 1)$ such that $X^{r-1}\oplus M=X^r$. Then $QM\subseteq M$.*

Proof. Any y in QM is written as a sum $y=y_1+y_2 (y_1\in X^{r-1}, y_2\in M)$, and $Iy\subseteq M$ for some dense left ideal I of R. Then, for any $a\in I$, $ay_1=ay-ay_2\in X^{r-1}\cap M=\{0\}$. Hence $Iy_1=0$. Since ${}_QX^{r-1}$ is projective, we have $y_1=0$. Thus

$y=y_2\in M$.

Lemma 2.5. *Let A be an R-$\bar{Q}$-submodule of $\bar{Q}$. Then $QA\subseteq A$.*

Proof. We may assume that $0\neq A\neq\bar{Q}$. Then, since ${}_RQ_Q$ is simple, we have $Q\cap A=0$. Therefore there exists an integer r such that $X^{r-1}\cap A=0$ and $X^r\cap A\neq 0$. Since X^r/X^{r-1} is a simple R-Q-module, we have $X^{r-1}\oplus(X^r\cap A)=X^r$, hence $\bar{Q}=X^{r-1}\oplus((X^r\cap A)\otimes_Q\bar{Q})$ by [13; Corollary 1 of Proposition 1]. Then $A=A\cap\bar{Q}=X^{r-1}\cap A+(X^r\cap A)\otimes_Q\bar{Q}=(X^r\cap A)\otimes_Q\bar{Q}$. By Lemma 2.4, $Q(X^r\cap A)\subseteq X^r\cap A$, and so $QA\subseteq A$.

Corollary. *If A is an ideal of $\bar{R}$ then $QA=AQ$ (, so that QA is an ideal of $\bar{Q}$).*

Proof. Noting that $\bar{Q}=Q\bar{R}=\bar{R}Q$, AQ is an R-$\bar{Q}$-submodule. Hence $QA\subseteq AQ$. Symmetrically we obtain $AQ\subseteq QA$.

The following is well known, but we give its proof for completeness.

Lemma 2.6. *Let B be a ring, and I an ideal of B. Then the following conditions are equivalent*:

(1) *I is an invertible B-B-module.*

(2) *I is invertible in $Q(B)$.*

Proof. The implication (2)$\Rightarrow$(1) is well known. (1)$\Rightarrow$(2) Put $\{a\in Q_r(B)\colon aI\subseteq B\}=I'$. Then, since I is a dense right ideal, $I'\cong\mathrm{Hom}\ (I_B, B_B)$ canonically (cf. [16]). Since I_B is a generator, we have $I'I=B$. Since I_B is finitely generated and projective, we have $II'=B$, Then, since I is a dense left ideal, $I'\subseteq Q_l(B)$, and so $I'\subseteq Q(B)$. Thus I is invertible in $Q(B)$.

Lemma 2.7. *Every non-zero ideal of $\bar{Q}$ is invertible.* (Cf. [14; Examples].)

Proof. Let A be any non-zero ideal of $\bar{Q}$. We may assume that $A\neq\bar{Q}$. Then there is an integer $r\geqq 1$ such that $X^{r-1}\cap A=0$ and $X^r\cap A\neq 0$. Put $M=X^r\cap A$. Then, as in the proof of Lemma 2.5, $X^{r-1}\oplus M=X^r$, and $A=M\otimes_Q\bar{Q}=\bar{Q}\otimes_Q M$. Since $M\cong X^r/X^{r-1}$, M is an invertible Q-Q-module. Then it is easily seen that $\bar{Q}\cong\mathrm{End}\ (A_{\bar{Q}})$ by the map induced by ${}_{\bar{Q}}A$, so that ${}_{\bar{Q}}A_{\bar{Q}}$ is invertible, because $A_{\bar{Q}}=M\otimes_Q\bar{Q}_{\bar{Q}}$ is finitely generated, projective, and a generator (cf. [12; Lemma 3.1]).

If every non-zero ideal of a ring B is invertible, B is said to be an Asano order. Noting Lemma 1.6, an Asano order is a Krull order. A Krull order R is an Asano order if and only if $T(Q(R); R)=\{R\}$.

Lemma 2.8. (i) $S(\bar{R})\subseteq S(\bar{Q})\subseteq Q(\bar{R})=Q(\bar{Q})$. (ii) *For any non-zero ideal A of $\bar{R}$, $A\cdot S(\bar{Q})=S(\bar{Q})A=S(\bar{Q})$. Therefore $\bar{R}$ is a prime ring.*

Proof. Since ${}_Q\bar{Q}$ is projective, $\{x\in\bar{Q}\colon Ix=0\}=0$ for any dense left ideal I of R. Then, as $Q\bar{R}=\bar{Q}$, we have $\bar{Q}\subseteq Q_l(\bar{R})$. Symmetrically $\bar{Q}\subseteq Q_r(\bar{R})$, and hence $\bar{Q}\subseteq Q(\bar{R})$. Thus $Q(\bar{R})=Q(\bar{Q})$. Since $AQ(=AQ)$ is a non-zero ideal of $\bar{Q}$, we have $S(\bar{Q})A=S(\bar{Q})QA=S(\bar{Q})$. Similarly $A\cdot S(\bar{Q})=S(\bar{Q})$. Therefore $\bar{R}$ is a prime ring, and $A^{-1}\subseteq S(\bar{Q})$. Hence $S(\bar{R})\subseteq S(\bar{Q})$.

In virtue of Propositions 1.13 and 2.8, the notations $T(R)$, $F_i(R)$, $F_i^*(R)$, $T(\bar{R})$, $F_i(\bar{R})$, and $F_i^*(\bar{R})$ do not produce ambiguity.

By ρ_i we denote the correspondence $M\mapsto M'$ given in Corollary 2 of Lemma 2.3. Then $\rho_i(M)Y^i+X^{i-1}=Y^iM+X^{i-1}$, and if $M\subseteq R$ then $\rho_i(M)Y^i+Y^{i-1}=Y^iM+Y^{i-1}$, because of $X^{i-1}\cap Y^i=Y^{i-1}$. Further, note that $\rho_i(M')\rho_i(M'')=\rho_i(M'M'')$ for any M', M''. Put $\rho_1=\rho$. Then it is easy to verify that $\rho_i=\rho^i$ for all $i\geqq 1$.

For any R-R-submodule M of $Q(\bar{R})$, we put $M^*=\{x\in Q(\bar{R})\colon xI\subseteq M$ for some $I\in T(R)\}$. Note that $R^*=R$ and $\bar{Q}^*=\bar{Q}$.

Lemma 2.9. (i) $\rho(T(R))=T(R)$. (ii) *For any R-R-submodule M of Q, $\rho(M^*)=(\rho(M))^*$ holds. Therefore $\rho(F_i^*(R))=F_i^*(R)$.*

Proof. (i) For any ideal I of R and any $x\in Q$, $I\cdot RxR\subseteq R$ (or $RxR\cdot I\subseteq R$) if and only if $\rho(I)\rho(RxR)\subseteq R$ (or $\rho(RxR)\rho(I)\subseteq R$), because of $\rho(R)=R$. Therefore we obtain (i). (ii) If $x\in M^*$ then $xI\subseteq M$ for some $I\in T(R)$. Then $\rho(RxR)\rho(I)\subseteq\rho(M)$, and so $\rho(RxR)\subseteq\rho(M)^*$ by (i). Thus $\rho(M^*)\subseteq(\rho(M))^*$. Similarly $\rho^{-1}(M^*)\subseteq(\rho^{-1}(M))^*$. Then $\rho^{-1}((\rho(M))^*)\subseteq M^*$, whence $(\rho(M))^*\subseteq\rho(M^*)$.

Lemma 2.10. *$\bar{R}$ is maximal in $Q(\bar{R})$.*

Proof. Let A be any non-zero ideal of $\bar{R}$, and let $yA\subseteq A(y\in Q(\bar{R}))$. Then $yAQ\in AQ$, and so $y\in\bar{Q}$, because AQ is an invertible ideal of $\bar{Q}$. Put $W=\{x\in\bar{Q}\colon xA\subseteq A\}$. Then W is an $\bar{R}$-$\bar{R}$-submodule containing $\bar{R}$. For any $i\geqq 0$, there exists a unique R-R-submodule W_i of Q such that $(W\cap X^i)+X^{i-1}=W_iY^i+X^{i-1}$, by Lemma 2.3. Similarly, for A, $(A\cap X^i)+X^{i-1}=A_iY^i+X^{i-1}$, where A_i is an R-R-submodule of Q. Since $W\supseteq\bar{R}$, we have $W\cap X^i\supseteq Y^i$, and so $W_i\supseteq R$. Since $A\subseteq\bar{R}$, we have $A\cap X^i=A\cap Y^i$, and so $A_i\subseteq R$. It is easy to verify that $W_j\cdot\rho^j(A_i)Y^{i+j}\subseteq A_{i+j}Y^{i+j}+X^{i+j-1}$ for all $i,j\geqq 0$. Therefore $W_j\cdot\rho^j(A_i)\subseteq A_{i+j}$ for all $i,j\geqq 0$. Noting that $A_0\subseteq A_1\subseteq A_2\subseteq\cdots$, we put $I=\bigcup_{i\geqq 0}A_i$. Then I is a non-zero ideal of R, and $W_j\rho^j(I)\subseteq I$ for all $j\geqq 0$. By Lemma 2.9, $\rho^j(I^*)=(\rho^j(I))^*$, and so $W_j\cdot\rho^j(I^*)\subseteq I^*$. By Lemma 2.9 (ii), the number of irreducible components of $\rho^j(I^*)$ is equal to the one of I^*. As $R\subseteq W_j$, we have $\rho^j(I^*)\subseteq I^*$, hence $\rho^j(I^*)=I^*$. Thereby $W_j\subseteq R$, and so $W\cap X^j\subseteq Y^j+X^{i-1}$. Noting that $W\supseteq\bar{R}$, we obtain $W\cap X^j=Y^j+W\cap X^{j-1}$ for all $j\geqq 0$. Now $W\cap Q=W_0\subseteq R$, hence $W\cap X^j\subseteq Y^j$ for all $j\geqq 0$. Thus $W\subseteq\bar{R}$, as requered. Similarly $Ax\subseteq A$ implies that $x\in\bar{R}$.

Lemma 2.11. *For any $i\geqq 1$, $(Y^i)^*=Y^i$, and $\bar{R}^*=\bar{R}$.*

Proof. Let f be any right R-homomorphism from $\bar{R}$ to R. Extend f to a right Q-homomorphism $\bar{f}$ from $\bar{Q}=\bar{R}\otimes_R Q$ to Q. If $y\in\bar{R}^*$ then $yI\subseteq\bar{R}$ for some $I\in T(R)$, and so $y\in\bar{Q}$. Then $\bar{f}(y)I\subseteq R$, and so $\bar{f}(y)\in R$ for any f. If (f_λ, u_λ) $(\lambda\in\Lambda)$ is a projective coordinate system for $\bar{R}_R$, then so is $(\bar{f}_\lambda, u_\lambda)$ $(\lambda\subset\Lambda)$ for $\bar{Q}_Q$. Therefore $y=\sum_\lambda u_\lambda\bar{f}_\lambda(y)\in\bar{R}$. Hence $\bar{R}^*=\bar{R}$. If $x\in(Y^i)^*$ then $xJ\subseteq Y^i$ for some $J\in T(R)$. Then, as $JQ=Q$, we have $x\in X^i$. Hence $x\in X^i\cap\bar{R}^* =X^i\cap\bar{R}=Y^i$.

Lemma 2.12. *Let A be any reflexive $\bar{R}$-$\bar{R}$-submodule of $Q(\bar{R})$. Then $A^*=A$.*

Proof. If $xI\subseteq A$ for some $I\in T(R)$, then $A^{-1}xI\subseteq A^{-1}A\subseteq\bar{R}$. Using Lemma 2.11, $A^{-1}x\subseteq\bar{R}$. Therefore $x\in(A^{-1})^{-1}=A$.

Lemma 2.13. *Let A be any non-zero $\bar{R}$-R-submodule of $\bar{R}$. Then there exists a finitely generated $\bar{R}$-R-submodule A_0 of A such that $A\subseteq\bigcup_{j\geq 0}\beta^j(A_0)$, where $\beta(M)=M^*$ for any R-R-submodule M of $Q(\bar{R})$.*

Proof. For any $i\geqq 0$, Y^i/Y^{i-1} is an invertible R-R-bimodule. Therefore there exists a unique ideal A_i of R such that $(A\cap Y^i)+Y^{i-1}=Y^iA_i+Y^{i-1}$. In particular, $A\cap R=A_0$. Since $Y(A\cap Y^i)\subseteq A\cap Y^{i+1}$, we have an ascending chain $A_0\subseteq A_1\subseteq A_2\subseteq\cdots$. If $A_i=0$ then $A\cap Y^i\subseteq Y^{i-1}$, and so $A_k\neq 0$ for some k. Then $A_k^*\subseteq A_{k+1}^*\subseteq\cdots$, which are reflexive ideals of R. Therefore, for some integer $m\geqq k$, $A_m^*=A_{m+1}{}^*=\cdots$. By Proposition 1.12, $A_m^*=(\sum_{j=1,\dots,t}Rz_jR)^*$ for some $z_1,\cdots,z_t\in A_m$. Noting that ${}_RY^m$ is finitely generated, we have that $\sum_j Y^mz_jR\subseteq\sum_i Rb_iR+Y^{m-1}$ for some $b_1,\cdots,b_s\in A\cap Y^m$. Let $n\geqq m$. Then $A\cap Y^n\subseteq Y^nA_n^*+Y^{n-1}=Y^nA_m^*+Y^{n-1}\subseteq Y^{n-m}(\sum_j Y^mz_jR)^*+Y^{n-1}$. Therefore, if $a\in A\cap Y^n$ then $aJ\subseteq Y^{n-m}(\sum_j Y^mz_jR)+Y^{n-1}$ for some $J\in T(R)$, and so $aJ\subseteq\sum_i Y^{n-m}b_iR+Y^{n-1}$. Then $aJ\subseteq\sum_i Y^{n-m}b_iR+A\cap Y^{n-1}$. Thus $A\cap Y^n\subseteq(\sum_i Y^{n-m}b_iR+A\cap Y^{n-1})^*$ for any $n\geqq m$. By induction we obtain $A\cap Y^n\subseteq\beta^{n-m+1}(\sum_i Y^{n-m}b_iR+A\cap Y^{m-1})$ $(n\geqq m)$. However, from the above proof, this holds whenever $A_m\neq 0$ and $A_m^*=\cdots=A_n^*$. Therefore, for any $n\geqq 0$ with $A_n\neq 0$, $A\cap Y^n\subseteq(\sum_i Rc_iR+A\cap Y^{n-1})^*$ for some $c_1,\cdots,c_v\in A\cap Y^n$. On the other hand, if $A_n=0$ then $0=A_0=\cdots=A_n$, and so $A\cap Y^n=0$. Hence there exists a finitely generated R-R-submodule W of $A\cap Y^{m-1}$ such that $A\cap Y^{m-1}\subseteq\beta^m(W)$. Then, for any $n\geqq m$, $A\cap Y^n\subseteq\beta^{n-m+1}(\sum_i Y^{n-m}b_iR+\beta^m(W))\subseteq\beta^{n+1}(\sum_i Y^{n-m}b_iR+W)$. This completes the proof.

Now we can complete the proof of Theorem 2.2 with the following

Lemma 2.14. *The ascending chain condition on reflexive ideals of $\bar{R}$ holds.*

Proof. Let $A_1\subseteq A_2\subseteq A_3\subseteq\cdots$ be an ascending chain of reflexive ideals of $\bar{R}$. Put $A=\bigcup_i A_i$. Then, by Lemma 2.13, $A\subseteq\bigcup_{j\geq 0}\beta^j(A')$ for some finitely generated $\bar{R}$-R-submodule A' of A. Then $A'\subseteq A_i$ for some i. By Lemma

2.12, $\beta(A_i)=A_i$, and so $\beta^j(A')\subseteq A_i$ for all j. Hence $A=A_i$.

Next we proceed to the proof of the following

Theorem 2.15. *For any non-zero ideal A of $Q\langle X\rangle$, $A\cap R\langle Y\rangle$ is a reflexive ideal of $R\langle Y\rangle$.*

Lemma 2.16. *The following conditions are equivalent.*

(1) *For any non-zero ideal A of $\bar{Q}$, $A\cap\bar{R}$ is a reflexive ideal of $\bar{R}$.*

(2) *For any $B\in T(\bar{R})$, $QB=\bar{Q}$ holds.*

(3) *For any non-zero ideal C of $\bar{R}$, $(QC)^{-1}=C^{-1}Q=QC^{-1}$ holds.*

Proof. (1)$\Rightarrow$(2) If $QB\subsetneqq\bar{Q}$ then $B\subseteq QB\cap\bar{R}\subsetneqq\bar{R}$, and $QB\cap\bar{R}$ is a reflexive ideal, a contradiction. (2)$\Rightarrow$(3) From $CQ=QC$, we have $C^{-1}CQC^{-1}=C^{-1}QCC^{-1}$. Then, by assumption, $QC^{-1}=C^{-1}Q$. Hence $(QC)^{-1}=C^{-1}Q=QC^{-1}$. (3)$\Rightarrow$(1) Let $C\in T(\bar{R})$. Then $QC=\bar{Q}$, because of $C^{-1}=\bar{R}$, Now, put $A\cap\bar{R}=A'$. If $Cx\subseteq A'(x\in\bar{R})$, then $\bar{Q}x=QCx\subseteq A$, and so $x\in A\cap\bar{R}=A'$. Similarly $yC\subseteq A'$ implies that $y\in A'$. Hence A' is a reflexive ideal, by Proposition 1.3.

REMARK 1. The condition (2) is equivalent to that $B\cap R\neq 0$ for any $B\in T(\bar{R})$.

REMARK 2. If C is an ideal of $\bar{R}$ such that $C\cap R\in T(R)$, then $C\in T(\bar{R})$. In fact, if $xC\in\bar{R}$ then $x(C\cap R)\subseteq\bar{R}$, and so $x\in\bar{R}$, by Lemma 2.11.

Lemma 2.17. *For any $I\in F(Q;R,R)$, $(\bar{R}I^{-1})^*=\bar{R}I^{-1}$ holds.*

Proof. The proof is similar to the one of Lemma 2.11.

Let M be a monic Q-Q-submodule of degree n (i.e. $X^{n-1}\oplus M=X^n$). Then, by [13; Corollary 1 of Proposition 1], $X^{n+m}=X^{n-1}\oplus(X^m\otimes_Q M)$ for any $m\geqq 0$. Therefore $X^m\otimes_Q M\cong X^{n+m}/X^{n-1}$ as Q-Q-bimodules, canonically. Since $Y^{n+m}\cap X^{n-1}=Y^{n-1}$, Y^{n+m}/Y^{n-1} is canonically embedded in X^{n+m}/X^{n-1}, and $Q\otimes_R(Y^{n+m}/Y^{n-1})\cong X^{n+m}/X^{n-1}$. Hence there exists a unique R-R-submodule V_m of $X^m\otimes_Q M$ such that the following diagram is commutative:

$$\begin{tikzcd} X^m\otimes_Q M \arrow[r, "\approx"] & X^{n+m}/X^{n-1} \\ V_m \arrow[u, hook] \arrow[r, "\approx"] & Y^{n+m}/Y^{n-1}. \arrow[u] \end{tikzcd}$$

Namely, $V_m+X^{n-1}=Y^{n+m}+X^{n-1}$. Then $Q\otimes_R V_m=X^m\otimes_Q M$, and $V_m=X^mM\cap(Y^{n+m}+X^{n-1})$. Therefore $V_0\subseteq V_1\subseteq V_2\subseteq\cdots$, where $V_0=M\cap(Y^n+X^{n-1})$. By [13; Corollary 1 of Proposition 1], $\bar{Q}=X^{n-1}\oplus(\bar{Q}\otimes_Q M)$. Put $A=\bar{Q}\otimes_Q M$. Then $A\cap(\bar{R}+X^{n-1})=\bigcup_i V_i$, and $A=\bigcup_{m\geqq 0}(X^m\otimes_Q M)=\bigcup_{m\geqq 0}(Q\otimes_R V_m)=Q\otimes_R V$, where $V=\bigcup_i V_i$. By Lemma 2.13, $A\cap\bar{R}\subseteq\bigcup_{j\geqq 0}\beta^j(A')$ for some finitely generated $\bar{R}$-R-submodule A' of $A\cap\bar{R}$. However, by virtue of Lemma 2.11, $\beta(A\cap\bar{R})=A\cap\bar{R}$, whence $A\cap\bar{R}=\bigcup_{j\geq 0}\beta^j(A')$. As ${}_{\bar{R}}A'_R$ is finitely generated, $A'\subseteq\bar{R}V_s$ for some s. Now we assume that M is invertible in $Q(\bar{R})$. Then, since V_0 is an invertible R-R-module and $Q\otimes_R V_0=M=V_0\otimes_R Q$, we know that ${}_RV_{0R}$ is invertible in $Q(\bar{R})$. In this situation, we need the following

Lemma 2.18. *For any R-R-submodule W of $\bar{Q}$, $W^*V_0^{-1}=(WV_0^{-1})^*$ holds.*

Proof. By virtue of Proposition 1.2, there is a one to one mapping $I\mapsto V_0^{-1}IV_0$ from $T(R)$ onto itself. Let x be in $W^*V_0^{-1}$. Then $xV_0\subseteq W^*$. Since V_{0R} is finitely generated, $xV_0I\subseteq W$ for some $I\in T(R)$. Then $xV_0IV_0^{-1}\subseteq WV_0^{-1}$, and so $x\in(WV_0^{-1})^*$. Similarly we can prove that $(WV_0^{-1})^*\subseteq W^*V_0^{-1}$,

We still assume that M is a monic Q-Q-submodule which is invertible in $Q(\bar{R})$, and notations are the same as before. Since $V_s\subseteq A=\bar{Q}M=\bar{Q}V_0$, we have $V_sV_0^{-1}\subseteq\bar{Q}$. Since both ${}_RV_s$ and ${}_RV_0^{-1}$ are finitely generated, ${}_RV_sV_0^{-1}$ is also finitely generated, and so $V_sV_0^{-1}I\subseteq\bar{R}$ for some non-zero ideal I of R, because of $\bar{Q}=\bar{R}Q$. Then, as $A'\subseteq\bar{R}V_s$, we have $A'V_0^{-1}I\subseteq\bar{R}$, and so $A'V_0^{-1}II^{-1}\subseteq\bar{R}I^{-1}$. Then, by Lemma 2.18 and 2.17, $\beta^j(A')V_0^{-1}=\beta^j(A'V_0^{-1})\subseteq\bar{R}I^{-1}$ for all $j\geq 0$. Hence, as $A\cap\bar{R}=\bigcup_{j\geq 0}\beta^j(A')$, we obtain $(A\cap\bar{R})V_0^{-1}I\subseteq\bar{R}$. Put $N=\{x\in Q(\bar{R})\colon (A\cap\bar{R})x\subseteq\bar{R}\}$ and $N'=\{y\in Q(\bar{R})\colon Ay\subseteq\bar{Q}\}$. Evidently $N'=V_0^{-1}\bar{Q}$, and $V_0^{-1}I\subseteq N$ implies that $N'\subseteq N\bar{Q}$. Next, let us prove that $N\bar{Q}\subseteq N'$. Since ${}_RV_0$ is finitely generated, there exists a non-zero ideal I' such that $V_0I'\subseteq\bar{R}$. Then $V_0I'=I''V_0$ for some non-zero ideal I'' of R, for ${}_RV_{0R}$ is invertible. Therefore $A=\bar{Q}V_0=\bar{Q}I''V_0=\bar{Q}V_0I'\subseteq\bar{Q}(A\cap\bar{R})$, whence $A=\bar{Q}(A\cap\bar{R})$. Hence $N\subseteq N'$. Thus $N'=N\bar{Q}$. Finally, $zN\subseteq\bar{R}$ implies $zN'=zN\bar{Q}\subseteq\bar{Q}$, and so $z\in\bar{Q}V_0=A$. Since $\bar{R}\subseteq N$, we have $z\in\bar{R}$. Hence $z\in A\cap\bar{R}$. Therefore a left $\bar{R}$-submodule $A\cap\bar{R}$ satisfies (β) of Proposition 1.15. Thus we have the following

Proposition 2.19. *Let M be a monic Q-Q-submodule which is invertible in $Q(\bar{R})$. Put $A=\bar{Q}M$, $A^{-1}=\{x\in Q(\bar{R})\colon Ax\subseteq\bar{Q}\}$, and $(A\cap\bar{R})^{-1}=\{x\in Q(\bar{R})\colon (A\cap\bar{R})x\subseteq\bar{R}\}$. Then $A=Q(A\cap\bar{R})$, and $A^{-1}=(A\cap\bar{R})^{-1}\bar{Q}$. Further, $A\cap\bar{R}\in F^*(Q(\bar{R});\bar{R},B)$, where $B=O_r(A\cap\bar{R})$.*

Evidently Theorem 2.15 follows from Proposition 2.16, Proposition 2.19 above and Lemma 20 below. (Cf. the proof of Lemma 2.7).

Lemma 2.20. *Let A be any non-zero ideal of $\bar{Q}$. Then $A=\bar{Q}M=M\bar{Q}$ for some monic Q-Q-submodule M. Such a M is uniquely determined by A, and is invertible in $S(\bar{Q})$.*

Proof. The first half follows from the proof of Lemma 2.7. Since $A=M\otimes_Q\bar{Q}$, any right Q-homomorphism from M to Q can be extended to a right $\bar{Q}$-homomorphism from A to $\bar{Q}$. Since A is invertible, this is given by a left multiplication of an element of A^{-1}. Therefore if we put $M'=\{x\in A^{-1};$

$xM\subseteq Q\}$, then $M'M=Q$, because M_Q is a generator. Symmetrically $MM''=Q$ for some Q-Q-submodule M'' of A^{-1}. Hence M is invertible in $S(\bar{Q})$. Let N be any monic Q-Q-submodule with $A=\bar{Q}N$. Let deg $N=r$. Then $\bar{Q}=X^{r-1}\oplus A$, and $N=A\cap X^r$, by [13; Corollary 1 of Proposition 1]. Hence N is uniquely determined by A.

In all that follows we denote $F^*(Q(\bar{Q});\bar{Q},\bar{Q})$, $F(Q(\bar{Q});\bar{Q},\bar{Q})$, $F^*(Q(\bar{R});\bar{R},\bar{R})$, and $F(Q(\bar{R}),\bar{R},\bar{R})$ by $F^*\{\bar{Q}\},F\{\bar{Q}\},F^*\{\bar{R}\}$, and $F\{\bar{R}\}$, respectively. Similarly we denote $F^*(Q;R,R)$ and $F(Q;R,R)$ by $F^*\{R\}$ and $F\{R\}$, respectively (cf. Proposition 1.13).

Let $M\in F\{\bar{R}\}$. Then $MI\subseteq\bar{R}$ for some $I\subset F_i(\bar{R})$, by Proposition 2.1. Using Corollary of Lemma 2.5, $QMI=MIQ=MQI$, and so $QMQ=MQ$, for QI is invertible. Symmetrically $QMQ=QM$, whence $MQ=QM$. Let $x\in Q(M^{-1})^{-1}$. Then $xC\subseteq QM$ for some $C\in T(\bar{R})$. Since $CQ=\bar{Q}$, we have $x\in QMQ=MQ$. Thus $QM=Q(M^{-1})^{-1}$. Therefore a group homomorphism ψ from $F^*\{\bar{R}\}$ to $F^*\{\bar{Q}\}$ is well defined by $\psi(M)=QM$. Let A, B be non-zero ideals of $\bar{Q}$. Then $AB\cap\bar{R}\supseteq(A\cap\bar{R})\circ(B\cap\bar{R})$. Since $AB\cap\bar{R}\subseteq B\cap\bar{R}$, we have $(AB\cap\bar{R})(B\cap\bar{R})^{-1}\subseteq\bar{R}$. By Proposition 2.19, $B^{-1}\supseteq(B\cap\bar{R})^{-1}$, and so $(AB\cap\bar{R})(B\cap\bar{R})^{-1}\subseteq A\cap\bar{R}$. Therefore $AB\cap\bar{R}\subseteq(A\cap\bar{R})\circ(B\cap\bar{R})$. Hence $AB\cap\bar{R}=(A\cap\bar{R})\circ(B\cap\bar{R})$. Then a group homomorphism ϕ from $F^*\{\bar{Q}\}$ to $F^*\{\bar{R}\}$ is well defined by $\phi(AB^{-1})=(A\cap\bar{R})\circ(B\cap\bar{R})^{-1}$. Because of Proposition 2.19, $\psi\phi=id$. Hence $F^*\{\bar{R}\}\cong\operatorname{Im}\phi\times\operatorname{Ker}\psi$, and $F^*\{\bar{Q}\}\cong\operatorname{Im}\phi$. Let I, J be in $F_i^*(\bar{R})$. If $IQ\subseteq JQ$ then $1\in\bar{Q}\subseteq I^{-1}JQ$, and so $G\subseteq I^{-1}J$ for some $G\in F_i(R)$. Then $(\bar{R}G\bar{R})^*\subseteq I^{-1}\circ J$. Therefore $I^{-1}\circ J\in\operatorname{Ker}\psi$ if and only if $(\bar{R}G\bar{R})^*\subseteq I^{-1}\circ J\subseteq((\bar{R}F\bar{R})^*)^{-1}$ for some $F,G\in F_i(R)$. In particular, $J\in\operatorname{Ker}\psi$ if and only if $J\cap R\neq 0$. In this case, $J\cap R\in F_i^*(R)$, by Lemma 2.12. Let $P'\in F_i^*(\bar{Q})$ be irreducible. Then, by Corollary of Lemma 2.5, $P'\cap\bar{R}$ is a prime ideal, so that $P'\cap\bar{R}$ is irreducible in $F_i^*(\bar{R})$, and $Q(P'\cap\bar{R})=P'$ by Proposition 2.19. Conversely, if $P\in F_i^*(\bar{R})$ is irreducible and $QP\neq\bar{Q}$ then, by the maximality of P in $F_i^*(\bar{R})$, we have $QP\cap\bar{R}=P$, and QP is maximal. Let $J\in F_i^*(\bar{R})$, and $J=P_1\circ\cdots\circ P_r$, where each P_i is irreducible in $F_i^*(\bar{R})$. Then $QJ\cap\bar{R}=(QP_1\cap\bar{R})\circ\cdots\circ(QP_r\cap\bar{R})$, and each $QP_i\cap\bar{R}$ is either P_i or $\bar{R}$. Let I', I'' be in $F_i^*(\bar{R})$. Then, $I'\circ I''^{-1}\in\operatorname{Ker}\psi\Leftrightarrow QI'=QI''\Leftrightarrow QI'\cap\bar{R}=QI''\cap\bar{R}$. Therefore $\operatorname{Ker}\psi=\coprod(P)$, where P ranges over all irreducible reflexive ideals P such that $P\cap R\neq 0$ (or equivalently, $QP=\bar{Q}$), and (P) denotes the infinite cyclic group generated by P.

Lemma 2.21. (i) *Let $I\in F\{R\}$, and assume that $I\bar{R}=\bar{R}I$. Then $\bar{R}I\in F\{\bar{R}\}$, $(\bar{R}I)^{-1}=\bar{R}I^{-1}=I^{-1}\bar{R}$, and $\bar{R}I\cap X^i=IY^i=Y^iI$ for all $i\geqq 0$. Therefore, $I\in F^*\{R\}$ then $\bar{R}I\in F^*\{\bar{R}\}$.*

(ii) *Let $J\in F_i^*(R)$ be irreducible, and asume that $JY=YJ$. Then, if $aRb\subseteq\bar{R}J$ $(a,b\in\bar{R})$ then $a\in\bar{R}J$ or $b\in\bar{R}J$. Therefore $\bar{R}J$ is irreducible in $F_i^*(\bar{R})$.*

Proof. (i) Since $0\neq\bar{R}I\cdot I^{-1}\bar{R}\subseteq\bar{R}$ and $0\neq\bar{R}I^{-1}\cdot I\bar{R}$, we have $\bar{R}I\in F\{\bar{R}\}$, by Proposition 1.9. Let $x\in(I\bar{R})^{-1}$. Then $xI\subseteq\bar{R}$, and so $xII^{-1}\subseteq\bar{R}I^{-1}$. By Lemma 2.17, $x\in\bar{R}I^{-1}$. Hence $(I\bar{R})^{-1}=\bar{R}I^{-1}$, and symmetrically $(\bar{R}I)^{-1}=I^{-1}\bar{R}$. Since Y^{i+1}/Y^i_R is projective, $Y^{i+1}=Y^i\oplus W$ for some right R-submodule W of Y^{i+1}. Then $\bar{R}=Y^i\oplus(W\otimes_R\bar{R})$, by [13; Proposition 1]. Then $\bar{Q}=\bar{R}\otimes_RQ=(Y^i\otimes_RQ)\oplus(W\otimes_R\bar{R}\otimes_RQ)=X^i\oplus(W\otimes_R\bar{Q})$, and $\bar{R}I=Y^iI\oplus W\bar{R}I$. Hence $X^i\cap\bar{R}I=Y^iI$, and symmetrically $X^i\cap I\bar{R}=IY^i$. (ii) By (i), $J\bar{R}\in F_i^*(\bar{R})$. Let B, C be R-R-submodules of $\bar{R}$ such that $BC\subseteq J\bar{R}$. Then, as $(B+J\bar{R})(C+J\bar{R})\subseteq J\bar{R}$, we may assume that $B, C\supseteq J\bar{R}$. For any integer $i\geqq 1$, there are ideals B_i, C_i of R such that $(B\cap Y^i)+Y^{i-1}=B_iY^i+Y^{i-1}$, $(C\cap Y^i)+Y^{i-1}=C_iY^i+Y^{i-1}$, because each Y^i/Y^{i-1} is an invertible R-R-bimodule. Then, as $J\bar{R}\cap Y^{i+j}=JY^{i+j}$, we have $B_j\cdot\rho^j(C_i)\subseteq J$ for all i, where ρ is the one as before. Now, assume that $B\supsetneqq J\bar{R}$. Then $B_j\not\subseteq J$ for some j, so that $\rho^j(C_i)\subseteq J$. Then $C_i\subseteq\rho^{-j}(J)=J$ for all i. Noting that $C_0=C\cap R\subseteq J$, this implies that $C\subseteq J\bar{R}$. This completes the proof.

Here we consider the following condition.

($\sharp$) For any $I\in F_i^*(Q;R)$, $IY=YI$.

Lemma 2.22. *Assume that the condition* ($\sharp$) *holds. Let $P\in F_i^*(\bar{R})$. Then P is an irreducible ideal such that $P\cap R\neq 0$ if and only if $P=I\bar{R}$ for some irreducible reflexive ideal I of R.*

Proof. The "if" part follows from Lemma 2.21. Conversely, let $P\in F_i^*(\bar{R})$ be irreducible, and let $P\cap R\neq 0$. Then $P\cap R\in F_i^*(R)$. If $IJ\subseteq P\cap R$ for some $I, J\in F_i(R)$, then $I^*J^*\subseteq P\cap R$, because of $P\cap R\in F_i^*(R)$. Then $I^*\bar{R}\cdot J^*\bar{R}\subseteq P$, whence $I^*\subseteq P$ or $J^*\subseteq P$, because P is a prime ideal. Hence $P\cap R$ is a prime ideal of R. Then, by Lemma 2.21, $(P\cap R)\bar{R}$ is irreducible. Hence $(P\cap R)\bar{R}=P$.

Assume that the condition ($\sharp$) holds. Let I, J be in $F^*\{R\}$. Then $(\bar{R}I)\circ(\bar{R}J)=((\bar{R}I\cdot\bar{R}J)^{-1})^{-1}=((\bar{R}IJ)^{-1})^{-1}=\bar{R}(I\circ J)$, by Lemma 2.21. Therefore the mapping $\theta\colon I\mapsto\bar{R}I$ is a homomorphism from $F^*\{R\}$ to Ker ψ. Evidently $I\subseteq\bar{R}I\cap Q$. Let $I=F\circ G^{-1}(F, G\in F_i^*(R))$. Then $(\bar{R}I\cap Q)G\subseteq\bar{R}F\cap Q=\bar{R}F\cap R=F$, because R_R is a direct summand of $\bar{R}_R$. Therefore $(\bar{R}I\cap Q)GG^{-1}\subseteq FG^{-1}$, and so $\bar{R}I\cap Q\subseteq F\circ G^{-1}=I$. Hence $I=\bar{R}I\cap Q$. On the other hand, all irreducible $P\in F_i^*(\bar{R})$ with $P\cap R\neq 0$ generate Ker ψ. Therefore, by Lemma 2.22, θ is an isomorphism from $F^*\{R\}$ to Ker ψ. Thus we obtain the following

Theorem 2.23. *Assume that the condition* ($\sharp$) *holds. Then* θ: $F^*\{R\}\cong$ Ker ψ, $I\mapsto\bar{R}I$, *as groups. Further,* $\bar{R}I\cap Q=I$ *for all* $I\in F^*\{R\}$.

Proposition 2.24. *Assume that the condition* ($\sharp$) *holds. If* $I\cdot S(R)=S(R)I=S(R)$ *for all* $I\in F_i(R)$, *then* $A\cdot S(\bar{R})=S(\bar{R})A=S(\bar{R})$ *for all* $A\in F_i(\bar{R})$. (Cf.

Proposition 1.14.)

Proof. From (♯), it follows that $S(R)\subseteq S(\bar{R})$. Let $A\in F_i(\bar{R})$. Then $AA^{-1}\cap R\neq 0$, because of Lemma 2.16. Therefore $S(R)\subseteq AA^{-1}S(R)\subseteq A\cdot S(\bar{R})$, hence $A\cdot S(\bar{R})=S(\bar{R})$. Symmetrically $S(\bar{R})A=S(\bar{R})$.

3. In this section, we study further on reflexive R-R-submodules of $Q(\bar{R})$. For any additive submodules V, W of $Q(\bar{R})$, we put $(V.\cdot W)=\{x\in Q(\bar{R})\colon xW\subseteq V\}$, and $(W\cdot .V)=\{x\in Q(\bar{R})\colon Wx\subseteq V\}$.

Proposition 3.1. (i) *If $N\in F(Q(\bar{R}); R, R)$ and $N\subseteq\bar{R}$, then $QN=NQ$.*

(ii) *Let $N\in F(Q(\bar{R}); R, R)$, and assume that $QN=NQ$. Then QN is an invertible Q-Q-submodule of $Q(\bar{R})$, $(QN)^{-1}=QN^{-1}=N^{-1}Q$, $QN^*=N^*Q$, and $\bar{R}N^*=(\bar{R}N)^*$. Furthermore, $(\bar{R}N\cdot .R)=N^{-1}\bar{R}$, and $(\bar{R}.\cdot N^{-1}\bar{R})=\bar{R}N^*$.*

(iii) *Let M be a Q-Q-submodule of Q, and assume that M is invertible in $Q(\bar{R})$. Then $M\cap\bar{R}\in F^*(Q(\bar{R}); R, R)$, and $Q(M\cap\bar{R})=M=(M\cap\bar{R})Q$. Further there is an invertible R-R-submodule V_0 of $Q(\bar{R})$ such that $V_0^{-1}(M\cap\bar{R})$, $(M\cap\bar{R})V_0^{-1}\in F^*\{R\}$ and $QV_0=M=V_0Q$.*

Proof. (i) First we prove that ${}_QQN_R$ is simple. Let U be any non-zero Q-R-submodule of QN. Then $Q=QNN^{-1}\supseteq UN^{-1}\neq 0$, and so $Q=UN^{-1}$, because ${}_QQ_R$ is simple. Then $QN=UN^{-1}N\subseteq U$, whence $U=QN$. Thus ${}_QQN_R$ is simple. Then there is an integer $n\geqq 0$ such that $QN\cap X^{n-1}=0$ and $QN\cap X^n\neq 0$. By making use of Corollary 1 of Lemma 2.3, we have $X^{n-1}\oplus QN=X^n$. Then, by Lemma 2.4, $QN\supseteq NQ$. Symmetrically $QN\subseteq NQ$, whence $QN=NQ$, as desired. (ii) $QN=NQ$ yields $N^{-1}Q=N^{-1}QNN^{-1}=N^{-1}NQN^{-1}=QN^{-1}$, and so $N^{-1}Q=QN^{-1}$. Therefore ${}_QQN_Q$ is invertible in $Q(\bar{R})$, and $(QN)^{-1}=N^{-1}Q=QN^{-1}$. Hence $QN=((QN)^{-1})^{-1}=N^*Q=QN^*$. Now, $\bar{R}\otimes_R QN=\bar{R}\otimes_R Q\otimes_Q QN=\bar{Q}\otimes_Q QN\simeq\bar{Q}\cdot QN=\bar{R}QN$ (cf. Remark to Proposition 1.6), and therefore any right R-homomorphism f from $\bar{R}$ to R can be extended to a right Q-homomorphism $\bar{f}$ from $\bar{R}QN$ to QN. Then, for any $x\in(\bar{R}N)^*$, we can see that $\bar{f}(x)\in N^*$, whence it follows that $x\in\bar{R}N^*$, because $\bar{R}_R$ is projective. (Cf. the proof of Lemma 2.11.) Since $\bar{R}N^*\subseteq(\bar{R}N)^*$ is evident we have $\bar{R}N^*=(\bar{R}N)^*$. Symmetrically, $Jy\subseteq N\bar{R}$ ($J\in T(R)$) implies that $y\in N^*\bar{R}$. Let $\bar{R}Nz\subseteq\bar{R}$. Then $N^{-1}Nz\subseteq N^{-1}\bar{R}$, and so $z\in N^{-1}\bar{R}$, because of $N^{-1}=(N^{-1})^*$. If $uN^{-1}\bar{R}\subseteq\bar{R}$ then $uN^{-1}N\subseteq\bar{R}N$, whence $u\in(\bar{R}N)^*=\bar{R}N^*$. This completes the proof of (ii). (iii) Since ${}_QQ_Q$ is simple, an invertible Q-Q-module M is also simple. Then, as in the proof of (i), $X^{n-1}\oplus M=X^n$ for some $n\geqq 0$. Then $M\simeq X^n/X^{n-1}$, canonically. Let V_0 be as in Lemma 2.18. Then $M=Q\otimes_R V_0=V_0\otimes_R Q$, and ${}_RV_{0R}$ is invertible in $Q(\bar{R})$. Put $N=M\cap\bar{R}$. Then $N\neq 0$, for $\bar{R}_R$ is essentialin $\bar{Q}_R$. Put $I=\{x\in Q\colon V_0x\subseteq N\}$. Then $N=V_0\otimes_R I$, because V_0 is invertible. Since V_{0R} is finitely generated, $JV_0\subseteq\bar{R}$ for some non-zero ideal J of R. Put Hom $(\bar{R}_R, R_R)(JV_0)=J'$. Then, since $\bar{R}_R$ is projective, J' is a non-zero ideal of R. Noting that $\bar{R}\otimes_R Q=\bar{Q}$, we have $J'I\subseteq R$. Therefore, by Proposition 2.1, $I\in F\{R\}$. If $zI'\subseteq I(z\in Q, I'\in T(R))$ then $V_0zI'\subseteq V_0I=N$, and so $V_0z\subseteq M\cap\bar{R}=N$, that is, $z\in I$. Thus $I\in F^*\{R\}$, and hence $N=V_0I=V_0\circ I\in F^*(Q(\bar{R}); R, R)$. Further, $NQ=V_0IQ=V_0Q=M$. Likewise $QN=M$. It is evident that $I=V_0^{-1}N$. Symmetrically $NV_0^{-1}\in F^*\{R\}$.

Let $N'\in F^*(Q(\bar{R}); R, R)$, and assume that $N'\subseteq\bar{R}$. Put $QN'\cap\bar{R}=N$. Then $N\in F^*(Q(\bar{R}); R, R)$. Therefore if we put $J=N'\circ N^{-1}$, then $J\in F_l^*(R)$, and $N'=J\circ N$. Evidently $QN\cap\bar{R}=N$. Further, as in (iii) above, $N=IV_0$, where $I\in F^*\{R\}$. Therefore $N'=(J\circ I)V_0$, where $J\circ I\in F^*\{R\}$, and V_0 is an invertible R-R-submodule of $Q(\bar{R})$ with $V_0Q=QV_0=QN'$.

Proposition 3.2. *Let $U\in F(Q(\bar{R}); R, R)$, and suppose that $\bar{R}U=U\bar{R}$ and $QU=UQ$.*

(i) *$\bar{R}U\in F(Q(\bar{R}); \bar{R}, \bar{R})$, $(\bar{R}U)^{-1}=\bar{R}U^{-1}=U^{-1}\bar{R}$, and $QU^{-1}=U^{-1}Q$. Therefore $((\bar{R}U)^{-1})^{-1}=\bar{R}U^*=U^*\bar{R}$, and $QU^*=U^*Q$.*

(ii) *QU is written as a product $QU=M_2M_1^{-1}$ with monic Q-Q-submodules M_i such that $\bar{Q}M_i=M_i\bar{Q}$ $(i=1,2)$.*

(iii) *$U^*Y=YU^*$.*

Proof. (i), (ii) Put $M=QU$. Then, by assumption, $\bar{Q}M=M\bar{Q}$. By Proposition 3.1, $U^{-1}Q=QU^{-1}=M^{-1}$, and hence $\bar{Q}M\in F^*\{\bar{Q}\}$, because of $\bar{Q}M^{-1}=M^{-1}\bar{Q}=(\bar{Q}M)^{-1}$. Therefore $\bar{Q}M=(\bar{Q}M_2)^{-1}(\bar{Q}M_1)$ for some monic Q-Q-submodules M_i such that $\bar{Q}M_i=M_i\bar{Q}$ $(i=1,2)$, by Lemma 2.20. Since $(\bar{Q}M_2)^{-1}=\bar{Q}M_2^{-1}=M_2^{-1}\bar{Q}$, we have $\bar{Q}M=\bar{Q}M_2^{-1}M_1$ and so $\bar{Q}M_2^{-1}M_1M^{-1}=\bar{Q}$. Then $M_2^{-1}M_1M^{-1}$ is a monic Q-Q-submodule, and so $M_2^{-1}M_1M^{-1}=Q$, by [13; Corollary 1 of Proposition 1]. Hence $M=M_2^{-1}M_1$. As $\bar{R}U=U\bar{R}$, we have $U^{-1}\bar{R}UU^{-1}=U^{-1}U\bar{R}U^{-1}$, whence $U^{-1}\bar{R}=\bar{R}U^{-1}$ by Proposition 3.1 (ii). Since $UU^{-1}\in T(R)$, it follows from Remark 2 of Lemma 2.16 that $RU\cdot U^{-1}\bar{R}\in T(\bar{R})$. Similarly $\bar{R}U^{-1}\cdot U\bar{R}\in T(\bar{R})$. Hence $\bar{R}U\in F\{\bar{R}\}$. The remainder follows from Proposition 3.1 (ii). (iii) By (i), we may assume that $U=U^*$. Since $\bar{Q}M_i=M_i\bar{Q}$, it follows from [13; Corollary 1 of Proposition 1] that $XM_i=X^{n_i+1}\cap\bar{Q}M_i=X^{n_i+1}\cap M_i\bar{Q}=M_iX$, where $n_i=\deg M_i$ $(i=1,2)$. Then, as $M=M_2^{-1}M_1$, we have $XM=MX$. Since $U^{-1}\subseteq M^{-1}$, $UYU^{-1}\subseteq MXM^{-1}=X$, and so $UYU^{-1}\subseteq X\cap\bar{R}=Y$. Then $UYU^{-1}U\subseteq YU$. Now, $XM=X\otimes_Q M=Y\otimes_R M$, so that any right R-homomorphism from Y to R can be extended to a right Q-homomorphism form XM to M. Then, since Y_R is projective, we have $(YU)^*=YU$. Therefore $UY\subseteq YU$, and symmetrically $YU\subseteq UY$. Thus $YU=UY$. (Cf. the proof of Lemma 2.11.)

Theorem 3.3. *Assume that the condition (♯) holds. Let M be a monic Q-Q-submodule of $Q\langle X\rangle$ such that $Q\langle X\rangle M=MQ\langle X\rangle$, and let $N=M\cap R\langle Y\rangle$. Then M is invertible in $S(Q\langle X\rangle)$, $N\in F^*(Q(\bar{R}); R, R)$, $M=QN=NQ$, and $Q\langle X\rangle M\cap R\langle Y\rangle=R\langle Y\rangle N=NR\langle Y\rangle$.*

Proof. By Lemma 2.20 and Proposition 3.1, M is invertible in $S(\bar{Q})$, $M=QN=NQ$, and $N\in F^*(Q(\bar{R});\ R,\ R)$. Put $A=\bar{Q}M\cap\bar{R}$ and $\bar{R}N=B$. Then $A\supseteq B$, and $QB=BQ=\bar{Q}M=QA=AQ$. By Proposition 2.16 and Theorem 2.15, $(QA)^{-1}=QA^{-1}=A^{-1}Q$. Therefore $QA^{-1}B=QA^{-1}\cdot QB=(QA)^{-1}QA=\bar{Q}$, hence $I\subseteq A^{-1}B$ for some $I\in F_i(R)$. Then $AI\subseteq B$, so $AI^*\subseteq B^*=B$ by Proposition 3.1. Therefore if we put $I=\{x\in R\colon Ax\subseteq B\}$ then $I=I^*$. Assume that $I\neq R$. Then $I\subseteq P$ for some irreducible $P\in F_i^*(R)$. Put $B'=(B^{\cdot}\cdot\bar{R})$. Then, by Proposition 3.1 (ii), $B'=N^{-1}\bar{R}$, and $BB'AI\subseteq AI\subseteq\bar{R}P$. Now $AI\cdot\bar{R}B'=AIB'\subseteq BB'\subseteq\bar{R}$, and so $\bar{R}B'\subseteq(AI)^{-1}$. Then, by Proposition 1.11, $\bar{R}B'\in F\{\bar{R}\}$, and so $\bar{R}B'\cdot AI\subseteq\bar{R}$ by virtue of the commutativity of $F^*\{\bar{R}\}$. Then, by Lemma 2.21 (ii), $B\subseteq\bar{R}P$ or $B'AI\subseteq\bar{R}P$. However, if $B\subseteq\bar{R}P$ then $NP^{-1}\subseteq\bar{R}\cap M=N$, so $P^{-1}\subseteq R$, a contradiction. On the other hand, if $B'AI\subseteq\bar{R}P$ then $\bar{R}B'AIP^{-1}\subseteq\bar{R}$, and so $\bar{R}B'\cdot A\cdot\bar{R}(I\circ P^{-1})\subseteq\bar{R}$. Therefore $A\cdot\bar{R}(I\circ P^{-1})\cdot\bar{R}B'\subseteq\bar{R}$, hence $A(I\circ P^{-1})\subseteq(\bar{R}.\cdot B')=B$ by Propositions 3.1 and 3.2. This is a contradiction. Thus $I=R$. Hence $A=B$, that is, $\bar{Q}M\cap\bar{R}=\bar{R}(\boldsymbol{M}\cap\bar{R})$. Symmetrically $M\bar{Q}\cap\bar{R}=(M\cap\bar{R})\bar{R}$. This complete the proof.

Theorem 3.4. *Assume that the condition (♯) holds. If every reflexive ideal of R is invertible then so is* $R\langle Y\rangle$.

Proof. Let A be any reflexive ideal of $\bar{R}$. Then A can be written as $A=(I\bar{R})\circ(B\cap\bar{R})$, where $I\in F_i^*(R)$, and $B=QA=AQ$ (cf. Theorem 2.23). By assumption, $I\bar{R}$ is invertible. On the other hand, $B=\bar{Q}M=M\bar{Q}$ for some monic Q-Q-submodule M, by Lemma 2.20. Put $M\cap\bar{R}=N$. Then $B\cap\bar{R}=\bar{R}N=N\bar{R}$ by Theorem 3.3. By Proposition 3.1 (iii), N is written as a product $N=JV_0$, where $J\in F^*\{R\}$, and V_0 is an invertible R-R-submodule of $Q(\bar{R})$. By Propositions 2.1 and 1.6, J is invertible, hence so is N. Then $B\cap\bar{R}$ is invertible. In fact, $(B\cap\bar{R})^{-1}=N^{-1}\bar{R}=\bar{R}N^{-1}$. Thus A is invertible.

Theorem 3.5. *Assume that the condition (♯) holds. Put* $\bar{S}=\{N\in F^*(Q(\bar{R});\ R,\ R)\colon QN=NQ,\ \bar{R}N=N\bar{R}\}$. *Then* $\lambda\colon\bar{S}\cong F^*\{\bar{R}\}$ *as group, where* $\lambda(N)=\bar{R}N$.

Proof. By Proposition 3.2, λ is well defined, and is a group homomorphism. If $\bar{R}N=\bar{R}$ then $N, N^{-1}\subseteq\bar{R}$. On the other hand, $\bar{Q}\cdot QN=\bar{Q}$, and so $QN=Q$, as in the proof of Proposition 3.2. Hence N, $N^{-1}\subseteq Q$. Therefore $N, N^{-1}\subseteq\bar{R}\cap Q=R$. Thus $N=R$. Let $A\in F_i^*(\bar{R})$. Then $A=(\bar{R}I)\circ(\bar{R}N)$, where $I\in F_i^*(R)$, and N is as in Theorem 3.3. Therefore Im $\lambda\supseteq F_i^*(\bar{R})$, and so Im $\lambda=F^*\{\bar{R}\}$, because of Proposition 2.1.

Assume that the condition (♯) holds. Evidently $\lambda(N)\subseteq\bar{R}$ if and only if $N\subseteq\bar{R}$, so that λ induces a semi-group isomorphism from $S=\{N\in\bar{S}\colon N\subseteq\bar{R}\}$ to $F_i^*(\bar{R})$. Further, by Theorem 3.3, $S_p=\{N\in S\colon QN\cap\bar{R}=N\}$ is isomorphic to $\{A\in F_i^*(\bar{R})\colon QA\cap\bar{R}=A\}$. Therefore $\bar{S}_p=\{N_1\circ N_2^{-1}\colon N_1,\ N_2\in$

$S_p\}\simeq \mathrm{Im}\ \phi(\simeq F^*\{\bar{Q}\})$ as group. Hence the direct product $F^*\{\bar{R}\}=\mathrm{Im}\ \phi\times$ Ker ψ induces the direct product $\bar{S}=\bar{S}_p\times F^*\{R\}$. Let $N\in S_p$. Then N is written as a product $N=V_0I$, where $I\in F^*\{R\}$, and V_0 is an invertible R-R-submodule of $Q(\bar{R})$ such that $QV_0=V_0Q$. Then $\bar{R}N=N\bar{R}=V_0I\bar{R}=V_0\bar{R}I$, and so $\bar{R}V_0II^{-1}=V_0\bar{R}II^{-1}$. Hence $V_0\bar{R}\subseteq(\bar{R}V_0)^*=\bar{R}V_0$ by Proposition 3.1. Symmetrically $\bar{R}V_0\subseteq V_0\bar{R}$, whence $V_0\bar{R}=\bar{R}V_0$. Therefore $\bar{S}$ is generated by $F^*\{R\}$ and the subgroup of all invertible R-R-submodules V of $Q(\bar{R})$ with $QV=VQ$, $\bar{R}V=V\bar{R}$.

Finally we note the following

Lemma 3.6. *If R is a prime Goldie ring and $Q=Q(R)$, then any monic Q-Q-submodule is invertible in $Q(\bar{R})$.*

Proof. Let M be a monic Q-Q-submodule of degree n. We may assume that $n\geqq 1$. Then, since $M\bar{Q}=M\otimes_Q\bar{Q}$, any right Q-homomorphism f from M to Q can be extended to a right $\bar{Q}$-homomorphism $\bar{f}$ from $M\bar{Q}$ to $\bar{Q}$. Since $Q(\bar{R})_{\bar{Q}}$ is injective (cf. §4. Appendix), $\bar{f}$ is given by a left multiplication of an element of $Q(\bar{R})$. Since M_Q is a generator, if we put $M'=\{x\in Q(\bar{R})\colon xM\subseteq Q\}$ then $M'M=Q$. Symmetrically $MM''=Q$ for some Q-Q-submodule M'' of $Q(\bar{R})$. Hence ${}_QM_Q$ is invertible in $Q(\bar{R})$.

4. Appendix

Lemma 4.1 *If ${}_RR$ is Noetherian then so is ${}_{\bar{R}}\bar{R}$.*

Proof. It suffices to prove that any left ideal of $\bar{R}$ is finitely generated. Let I be any left ideal of $\bar{R}$. For any integer $n\geqq 0$, Y^n/Y^{n-1} is an invertible R-R-bimodule, and hence there exists a unique left ideal I_n of R such that $I\cap Y^n+Y^{n-1}=Y^nI_n+Y^{n-1}$. Then $I_0=I\cap R\subseteq I_1\subseteq I_2\subseteq\cdots$. Therefore, $I_m=I_{m+1}=\cdots$ for some m. Put $J=I_m$. Since ${}_RJ$ and ${}_RY^m$ are finitely generated, ${}_RY^mJ$ is also finitely generated, so that $Y^mJ\subseteq\sum_i Ra_i+Y^{m-1}$ for some $a_1,\cdots,a_r$ of $I\cap Y^m$. Then, for any $n\geqq m$, $I\cap Y^n\subseteq Y^nJ+Y^{n-1}\subseteq\sum_i Y^{n-m}a_i+Y^{n-1}$, and so $I\cap Y^n=\sum_i Y^{n-m}a_i+I\cap Y^{n-1}$. Therefore $I\cap Y^n\subseteq\sum_i \bar{R}a_i+I\cap Y^{m-1}$ for all $n\geqq m$. Hence $I=\sum_i \bar{R}a_i+I\cap Y^{m-1}$. Since ${}_RI\cap Y^{m-1}$ is finitely generated, ${}_{\bar{R}}I$ is finitely generated.

If R is a prime Goldie ring and $Q=Q(R)$, then $\bar{Q}$ is a prime Goldie ring, by Lemma 4.1. Hence, as is well known, $Q(\bar{Q})_{\bar{Q}}$ is injective.

In the sequel, R is any ring. Let σ, τ be automorphisms of R, and D an endomorphism of R as an additive group. If $D(xy)=\sigma(x)D(y)+D(x)\tau(y)$ for all $x, y\in R$, then D is said to be a (σ, τ)-derivation of R ([5]). If $\sigma=id_R$, D is called a τ-derivation. Let I be a dense right ideal of R, and f a right R-homomorphism form I to $Q_r(R)$. Then, as is well known, there exists a unique element b of $Q_r(R)$ such that $f(x)=bx$ for all $x\in I$ (cf. [16]). Let ν

be any automorphism of R. Then ν is uniquely extended to an automorphism of $Q_r(R)$, and symmetrically of $Q_l(R)$. And these induce the same automorphism of $Q(R)$. Therefore we denote these automorphisms by ν, too.

Lemma 4.2. *Let τ be an automorphism of R, and g an additive homomorphism from a dense right ideal I to $Q_r(R)$ such that $g(xa)=g(x)\tau(a)$ for all $x\in I$, $a\in R$. Then there exists a unique element b of $Q_r(R)$ such that $g(x)=b\cdot\tau(x)$ for all $x\in I$.*

Proof. Put $h=g\tau^{-1}$. Then h is a right R-homomorphism from a dense right ideal $\tau(I)$ to $Q_r(R)$. Hence there exists a unique element b of $Q_r(R)$ such that $h(\tau(x))=b\cdot\tau(x)$ for all $x\in I$.

Lemma 4.3. *Let D be a (σ, τ)-derivation of R. Then D is uniquely extended to a (σ, τ)-derivation of $Q_r(R)$, and symmetrically of $Q_l(R)$. And these induce the same (σ, τ)-derivation of $Q(R)$.*

Proof. Let $b\in Q_r(R)$, and let I be a dense right ideal of R such that $bI\subseteq R$. A map g from I to $Q_r(R)$ is defined by $g(x)=D(bx)-\sigma(b)D(x)$ $(x\in I)$. Then g is as in Lemma 4.2, whence there exists a unique $b'\in Q_r(R)$ such that $g(x)=b'\cdot\tau(x)$ for all $x\in I$. Note that b' does not depend on the choice of I. Put $D'(b)=b'$. Then D' is a unique (σ, τ)-derivation of $Q_r(R)$ such that $D'|R=D$. Similarly D is uniquely extended to a (σ, τ)-derivation D'' of $Q_l(R)$, and it is easy to verify that $D'|Q(R)=D''|Q(R)$.

We denote D', D'', and $D'|Q(R)$ by D, too.

Let D be a τ-derivation of R, and put $Q=Q(R)$. By Lemma 4.2, the skew polynomial ring $R[t; \tau, D]$ defined by $at=t\tau(a)+D(a)$ $(a\in R)$ is a subring of the skew polynomial ring $Q[t; \tau, D]$. Put $Y=R+tR$ and $X=Q+tQ$. Then, for any $i\geqq 1$, $Y^i=R+tR+\cdots+t^iR$, and $X^i=Q+tQ+\cdots+t^iQ$. It is easy to see that these satisfy the conditions in §2.

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