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REMARKS ON PROOF OF A THEOREM OF KATO AND KOBAYASI ON LINEAR EVOLUTION EQUATIONS

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1. Introduction

Let

$$du/dt+A(t)u=f(t)\,,\quad 0\leqq t\leqq T\,, \tag{1.1}$$

be an evolution equation of "hyperbolic" type in a Banach space E with $A(t)$ having a domain containing a fixed dense linear subspace F. T. Kato [1], [2], J.R. Dorroh [3], S. Ishii [4],[5], K. Kobayasi [7] etc. have developed methods of constructing an evolution operator for (1.1). The main theorem due to T. Kato and K. Kobayasi is stated as follows:

Theorem. *Let E and F be Banach spaces such that F is densely and continuously embedded in E, and $\{A(t)\}_{0\leqq t\leqq T}$ be a family of closed linear operators in E with the domains*

$$D(A(t))\supset F\,.$$

Assume that

(I) $\{A(t)\}_{0\leqq t\leqq T}$ *is stable on E,*

(II) $A\in C([0, T]; \mathcal{L}_s(F; E))$,

(III) *There is family $\{S(t)\}_{0\leqq t\leqq T}$ of isomorphisms from F onto E such that*

$$S\in C^1([0, T]; \mathcal{L}_s(F; E))\,,$$

and

$$S(t)A(t)S(t)^{-1}=A(t)+B(t)$$

for each $t\in[0, T]$ with some

$$B\in C([0, T]; \mathcal{L}_s(E))\,.$$

Then we can construct an unique evolution operator $\{U(t, s)\}_{0\leqq s\leqq t\leqq T}$ with the following properties

a) $U\in C(\{(t, s);\ 0\leqq s\leqq t\leqq T\}\,;\ \mathcal{L}_s(E))$,

b) $U\in C(\{(t, s);\ 0\leqq s\leqq t\leqq T\}\,;\ \mathcal{L}_s(F))$,

c) $U(t, s)U(s, r)=U(t, r),\ 0\leqq r\leqq s\leqq t\leqq T;\ U(s, s)=I,\ 0\leqq s\leqq T,$

d) $U(\cdot, s)\in C^1([s, T]; \mathcal{L}_s(F; E))$, $0\leqq s<T$; $(\partial/\partial t)U(t, s)=-A(t)U(t, s)$,
e) $U(t, \cdot)\in C^1([0, t]; \mathcal{L}_s(F; E))$, $0<t\leqq T$; $(\partial/\partial s)U(t, s)=U(t, s)A(s)$.

T. Kato [1] first proved this theorem under stronger condition that $A(t)$ is norm continuous in t: $A\in C([0, T]; \mathcal{L}(F; E))$. J.R. Dorroh [3] then simplified the proof of the differentiability of $U(t, s)$. The author [6] noticed that if E and F are reflexive Banach spaces, then the norm continuity of $A(t)$ is weakened to the strong continuity (II). K. Kobayasi [7] recently eliminated this restriction and proved the theorem for general Banach spaces. He showed that a way of parting intervals used in the case of non-linear evolution equations (e.g. [8]) is available also for this linear problem. In this paper we will notice that though in [7] he used the partition of each $[s, T]$ depending on s, it can be replaced by an appropriate partition of the whole interval $[0, T]$. We need more detailed consideration than [7] to obtain the partition independent of s. But it makes it possible to utilize the Yoside approximation $A_n(t)$ of $A(t)$ in proof of the theorem. We give in section 3 the proof in this method. Once it is established that the evolution operator $U_n(t, s)$ for $A_n(t)$ is strongly convergent, we can verify more immediately that the limit $U(t, s)$ is really an evolution operator for $A(t)$.

Throughout this paper, we use the same notation and terminology as in [6]. $||\cdot||_E$ is the norm of a normed space E. For two normed spaces E and F, $\mathcal{L}(E; F)$ is the normed space of all bounded linear operators from E to F with the operator norm $||\cdot||_{E,F}$, and $\mathcal{L}_s(E; F)$ is the locally convex space $\mathcal{L}(E; F)$ equipped with the strong topology. $\mathcal{L}_s(E; E)$ is abbreviated as $\mathcal{L}_s(E)$, and $||\cdot||_{E,E}$ as $||\cdot||_E$, if there is no fear of confusion. For a locally convex space E, $E\text{-}\lim_{\lambda\to\lambda_0} x_\lambda$ is the limit in E of a convergent family $\{x_\lambda\}_{\lambda\in\Lambda}$ of E, $C(D; E)$ is the set of all continuous mappings from a metric space D to E, and $C^1([a, b]; E)$ is the set of all continuously differentiable functions in the interval $[a, b]$. $C_1, C_2, \cdots$ denote constants determined by $\sup_t ||A(t)||_{F,E}$, $\sup_t ||S(t)||_{F,E}$, $\sup_t ||S(t)^{-1}||_{E,F}$, $\sup_t ||dS/dt||_{F,E}$, $\sup_t ||B(t)||_E$, T, c_0 and $\{M, \beta\}$ alone; where c_0 is a constant such that $||\cdot||_E\leqq c_0||\cdot||_F$, and $\{M, \beta\}$ are the constants of stability of $\{A(t)\}$ on E. It is known that the part of $\{A(t)\}$ in F is stable with the constants of stability $\{\tilde{M}, \tilde{\beta}\}$ given by

$$\tilde{M} = M\sup_t ||S(t)||\sup_t ||S(t)^{-1}||\exp\{TM\sup_t ||S(t)^{-1}||\sup_t ||dS/dt||\}$$
$$\tilde{\beta} = \beta+M\sup_t ||B(t)||$$

(see [1], [9]).

2. Existence of the appropriate partition of [0, T]

For a finite partition Δ: $0=T_0<T_1<\cdots<T_N=T$ of $[0, T]$, A_Δ denotes a

step function of A

$$A_{\Delta}(t)=\begin{cases} A(T_j)\,, & T_j \leqq t < T_{j+1}\,, \\ A(T_N)\,, & t=T\,, \end{cases}$$

and $\{U_{\Delta}(t,s)\}_{0\leqq s\leqq t\leqq T}$ is the evolution operator for A_{Δ}

$$U_{\Delta}(t,s)=\begin{cases} \exp(-(t-s)A(T_j))\,, & T_j \leqq s \leqq t \leqq T_{j+1}\,, \\ \exp(-(t-T_j)A(T_j))\cdots\exp(-(T_{i+1}-s)A(T_i))\,, & \\ & T_i \leqq s \leqq T_{i+1}\cdots T_j \leqq t \leqq T_{j+1}\,. \end{cases}$$

Proposition 2.1. *For any $\varepsilon>0$ and any $y\in F$, there exists a finite partition Δ of $[0,T]$ such that*

$$\sup_{0\leqq s\leqq t\leqq T} \|\{A(t)-A_{\Delta}(t)\}U_{\Delta}(t,s)y\|_E \leqq \varepsilon\,.$$

Proof. We define inductively an increasing sequence $\{T_k\}_{k=0,1,2,\cdots}$ of $[0,T]$ in the following way. $T_0=0$. Assume that $\{T_j\}_{0\leqq j\leqq k}$ is defined so that the estimate

$$\sup_{0\leqq s\leqq t\leqq T_k} \|\{A(t)-A_{\Delta_k}(t)\}U_{\Delta_k}(t,s)y\|_E \leqq \varepsilon \tag{2.1}$$

holds for the partition Δ_k: $0=T_0<\cdots<T_k=T_k$ of $[0,T_k]$. If $T_k<T$, we consider a set J_k of all elements $h\in(0,T-T_k]$ such that

$$\sup_{\substack{T_k\leqq t<T_k+h \\ 0\leqq \tau<h}} \|\{A(t)-A(T_k)\}\exp(-\tau A(T_k))z\|_E \leqq \varepsilon$$

holds for every

$$z\in L_k=\{U_{\Delta_k}(t,s)y;\ 0\leqq s\leqq t\leqq T_k\}\,.$$

Since L_k is compact in F, J_k is non-empty and has the maximum. Putting $h_k=\mathrm{Max}\,J_k$, we define $T_{k+1}=T_k+h_k$. Then the estimate

$$\sup_{0\leqq s\leqq t\leqq T_{k+1}} \|\{A(t)-A_{\Delta_{k+1}}(t)\}U_{\Delta_{k+1}}(t,s)y\|_E \leqq \varepsilon \tag{2.2}$$

is valid. In fact, (2.2) is trivial if $t=T_{k+1}$. If $T_{k+1}>t\geqq s\geqq T_k$, $A_{\Delta_{k+1}}(t)=A(T_k)$ and $U_{\Delta_{k+1}}(t,s)y=\exp(-(t-s)A(T_k))y$. Therefore it follows that

$$\begin{aligned} &\|\{A(t)-A_{\Delta_{k+1}}(t)\}U_{\Delta_{k+1}}(t,s)y\|_E \\ &\quad\leqq \sup_{\substack{T_k\leqq t<T_k+h_k \\ 0\leqq \tau<h_k}} \|\{A(t)-A(T_k)\}\exp(-\tau A(T_k))y\|_E \leqq \varepsilon\,. \end{aligned}$$

If $T_{k+1}>t\geqq T_k>s$, $U_{\Delta_{k+1}}(t,s)y=\exp(-(t-T_k)A(T_k))U_{\Delta_k}(T_k,s)y$. Similarly $U_{\Delta_k}(T_k,s)y$ is an element of L_k. Finally if $T_k>t\geqq s$, $U_{\Delta_{k+1}}(t,s)y=U_{\Delta_k}(t,s)y$. (2.2) is nothing but the assumption (2.1). Untill T_k reaches T, we continue the inductive procedure. In order to complete the proof, it remains now to prove that such a procedure finishes within finite times. Suppose the contrary. Then we would have an infinite sequence $\{T_k\}_{k=0,1,2,\dots}$ of $[0, T)$ satisfying (2.1) for each k. To reach a contradiction we will prove that

$$L=\bigcup_{k=0}^{\infty} L_k$$

is relatively compact in F by using the next lemma essentially due to K. Kobayasi [7].

Lemma 2.2. *There exists a constant C_1 such that the estimation*

$$\|\prod_{k=1}^{p}\exp(-\tau_k A(t_k))z-\prod_{k=1}^{q}\exp(-\tau_k A(t_k))z\|_F$$

$$\leqq C_1\{\sum_{i=q+1}^{p}\tau_i\exp(\tilde{\beta}\sum_{k=r+1}^{i}\tau_k)\}\|x\|_F \tag{2.3}$$

$$+C_1\exp(\tilde{\beta}\sum_{k=r+1}^{p}\tau_k)\|S(t_r)\prod_{k=1}^{r}\exp(-\tau_k A(t_k))z-x\|_E \tag{2.4}$$

$$+C_1\{(t_p-t_r)+\sum_{k=r+1}^{p}\tau_k\}\exp(\tilde{\beta}\sum_{k=r+1}^{p}\tau_k)\|x\|_E \tag{2.5}$$

holds for any $x, z\in F$, $\tau_k\geqq 0$ $(1\leqq k\leqq p)$, $0\leqq t_1\leqq\cdots\leqq t_p\leqq T$, and integers $p\geqq q\geqq r\geqq 1$.

Proof.

$$\prod_{k=1}^{p}\exp(-\tau_k A(t_k))z-\prod_{k=1}^{q}\exp(-\tau_k A(t_k))z$$

$$=\{\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))-\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))\}S(t_r)^{-1}\times$$

$$\times\{S(t_r)\prod_{k=1}^{r}\exp(-\tau_k A(t_k))z-x\}$$

$$+\{\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))-\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))\}S(t_r)^{-1}x$$

$$=R_1+R_2\,.$$

R_1 is estimated by (2.4).

$$R_2=S(t_p)^{-1}\{S(t_p)\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))S(t_r)^{-1}$$

$$-S(t_q)\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))S(t_r)^{-1}\}x$$

$$+S(t_p)^{-1}\{S(t_q)-S(t_p)\}\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))S(t_r)^{-1}x$$

$$=R_3+R_4\,.$$

R_4 is estimated by (2.5).

$$\begin{aligned} R_3 &= S(t_p)^{-1}\{S(t_p)\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))S(t_r)^{-1}-\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))\}x \\ &\quad -S(t_p)^{-1}\{S(t_q)\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))S(t_r)^{-1}-\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))\}x \\ &\quad +S(t_p)^{-1}\{\prod_{k=r+1}^{p}\exp(-\tau_k A(t_k))-\prod_{k=r+1}^{q}\exp(-\tau_k A(t_k))\}x \\ &= R_5+R_6+R_7 . \end{aligned}$$

$$\begin{aligned} S(t_p)R_5 &= \sum_{i=r+1}^{p}[\prod_{k=i+1}^{p}\exp(-\tau_k A(t_k))\{S(t_i)\exp(-\tau_i A(t_i)) \\ &\quad -\exp(-\tau_i A(t_i))S(t_i)\}\prod_{k=r+1}^{i-1}\exp(-\tau_k A(t_k))]S(t_r)^{-1}x \\ &\quad +\sum_{i=r+1}^{p}[\prod_{k=i}^{p}\exp(-\tau_k A(t_k))\{S(t_i)-S(t_{i-1})\}\prod_{k=r+1}^{i-1}\exp(-\tau_k A(t_k))]S(t_r)^{-1}x . \end{aligned}$$

From this we obtain the estimate of R_5 by (2.5), and similarly that of R_6. From

$$S(t_p)R_7 = \sum_{i=q+1}^{p}\{\exp(-\tau_i A(t_i))-I\}\prod_{k=r+1}^{i-1}\exp(-\tau_k A(t_k))x$$

it follows that R_7 is estimated by (2.3).

Let $T_\infty=\lim_{k\to\infty} T_k$. Noting that $U_{\Delta_k}(t,s)y$ coinsides for all k such that $t\leqq T_k$, we define

$$U_{\Delta_\infty}(t,s)y = \lim_{k\to\infty} U_{\Delta_k}(t,s)y$$

for $0\leqq s\leqq t<T_\infty$. By the preceding lemma we have the following:

Lemma 2.3. *For each* $0\leqq s\leqq T_\infty$ *there exists a limit*

$$F\text{-}\lim_{\substack{(t',s')\to(T_\infty,s)\\ T_\infty>t'\geqq s'\geqq 0}} U_{\Delta_\infty}(t',s')y . \tag{2.6}$$

Proof. If $s<T_\infty$, $s<T_j<T_\infty$ with some j. In this case the limit (2.6) is easily reduced to

$$F\text{-}\lim_{t'\to T_\infty} U_{\Delta_\infty}(t',T_j)z \tag{2.7}$$

with $z=U_{\Delta_j}(T_j,s)y\in F$. Let $t''>t'>T_j$ be such that

$$T_j<\cdots<T_{j+r-1}<\cdots<T_{j+q-1}\leqq t'<T_{j+q}\cdots T_{j+p-2}\leqq t''<T_{j+p-1}$$

with some $p>q>r$, and apply Lemma 2.2 with

$$t_k = \begin{cases} T_{j+k-1}, & 1\leqq k\leqq q , \\ T_{j+k-2}, & q+1\leqq k\leqq p , \end{cases}$$

$$\tau_k = \begin{cases} T_{j+k} - T_{j+k-1}\,, & 1 \leqq k \leqq q-1 \\ t' - T_{j+q-1}\,, & k = q \\ T_{j+q} - t'\,, & k = q+1 \\ T_{j+k-1} - T_{j+k-2}\,, & q+2 \leqq k \leqq p-1 \\ t'' - T_{j+p-2}\,, & k = p\,. \end{cases}$$

Then we get

$$\begin{aligned} &\|U_{\Delta_\infty}(t'', T_j)z - U_{\Delta_\infty}(t', T_j)z\|_F \\ &\quad \leqq C_1 e^{\tilde{\beta} T}\{(T_\infty - T_{j+q-1})\|x\|_F + \|S(t_r)\prod_{k=1}^{r} \exp(-\tau_k A(t_k))z - x\|_E \\ &\quad + 2(T_\infty - T_{j+r-1})\|x\|_E\}\,. \end{aligned} \tag{2.8}$$

For any $\eta > 0$, $T_\infty - T_{j+r_0-1} \leqq \eta$ with some r_0, and

$$\|S(t_{r_0})\prod_{k=1}^{r_0} \exp(-\tau_k A(t_k))z - x_0\|_E \leqq \eta$$

with some $x_0 \in F$. $\|x_0\|_E$ is dominated by

$$\|x_0\|_E \leqq \eta + \|S(t_{r_0})\prod_{k=1}^{r_0} \exp(-\tau_k A(t_k))z\|_E \leqq \eta + \tilde{M} e^{\tilde{\beta} T} \sup_t \|S(t)\|\ \|z\|_F\,.$$

Therefore if $t'' > t' > T_{j+r_0-1}$, (2.8) is smaller than

$$C_2\{(T_\infty - T_{j+q-1})\|x_0\|_F + (1 + \|z\|_F)\eta\}\,.$$

If q_0 is large enough for $(T_\infty - T_{j+q_0-1})\|x_0\|_F \leqq \eta$, then $t'' > t' > T_{j+q_0-1}$ implies

$$\|U_{\Delta_\infty}(t'', T_j)z - U_{\Delta_\infty}(t', T_j)z\|_F \leqq C_3(1 + \|z\|_F)\eta\,,$$

which shows the existence of (2.7). If $s = T_\infty$, we can prove

$$y = F\text{-}\lim_{(t', s') \to (T_\infty, T_\infty)} U_{\Delta_\infty}(t', s')y\,. \tag{2.9}$$

Let $t' > s'$ be such that

$$T_j \leqq s' < T_{j+1} \cdots T_{j+p-2} \leqq t' < T_{j+p-1}$$

with some j and $p \geqq 2$, and apply Lemma 2.2 with

$$t_k = \begin{cases} T_j\,, & k = 1, 2 \\ T_{j+k-2}\,, & 3 \leqq k \leqq p\,, \end{cases}$$

$$\tau_k = \begin{cases} 0\,, & k = 1 \\ T_{j+1} - s'\,, & k = 2 \\ T_{j+k-1} - T_{j+k-2}\,, & 3 \leqq k \leqq p-1 \\ t' - T_{j+p-2}\,, & k = p\,, \end{cases}$$

and $q=r=1$. Then we get

$$\begin{aligned}||U_{\Delta_\infty}(t', s')y-y||_F &\leqq C_4 e^{\tilde{\beta}T}\{(T_\infty-T_j)||x||_F+||S(T_j)y-x||_E\}\\ &\leqq C_5\{(T_\infty-T_j)(||x||_F+||y||_F)+||S(T_\infty)y-x||_E\}\ .\end{aligned}$$

For any $\eta>0$, $||S(T_\infty)y-x_0||_E\leqq\eta$ with some $x_0\in F$, and $(T_\infty-T_{j_0})(||x_0||_F+||y||_F)\leqq\eta$ with some j_0. Therefore $t'\geqq s'>T_{j_0}$ implies

$$||U_{\Delta_\infty}(t', s')y-y||_F\leqq C_6\eta\ ,$$

which shows (2.9)

We have known that $U_{\Delta_\infty}(t, s)y$ can be extended on $0\leqq s\leqq t\leqq T_\infty$ continuously. Hence L contained in $\{U_{\Delta_\infty}(t, s)y;\ 0\leqq s\leqq t\leqq T_\infty\}$ is a relatively compact set in F.

Lemma 2.4. *For any $\eta>0$ there exists $\delta_1>0$ such that*

$$\sup_{\substack{|t-T_\infty|<\delta_1\\ 0\leqq\tau<\delta_1}} ||\exp(-\tau A(t))z-z||_F\leqq\eta$$

for every $z\in L$.

Proof. Let $p=2$, $q=r=1$, $t_1=t_2=t$ and $\tau_1=0$, $\tau_2=\tau$ in Lemma 2.2. Then

$$\begin{aligned}||\exp(-\tau A(t))z-z||_F &\leqq C_7 e^{\tilde{\beta}\tau}\{\tau||x||_F+||S(t)z-x||_E\}\\ &\leqq C_8 e^{\tilde{\beta}\tau}\{\tau||x||_F+||S(T_\infty)z-x||_E+|t-T_\infty|\,||z||_F\}\ .\end{aligned}$$

Since $S(T_\infty)(L)$ is precompact in E and F is dense in E, $S(T_\infty)(L)$ can be covered with a finite number of open balls $\{B(y_i;\eta/3C_8e^{\tilde{\beta}})\}_{1\leqq i\leqq l}$ with centers $y_i\in F$. Hence for any $z\in L$

$$||\exp(-\tau A(t))z-z||_F\leqq C_8 e^{\tilde{\beta}\tau}\{\tau \max_{1\leqq i\leqq l}||y_i||_F+|t-T_\infty|\,||z||_F\}+e^{\tilde{\beta}(\tau-1)}\eta/3\ .$$

Similarly for any $\eta>0$ there exists $\delta_2>0$ such that

$$\sup_{|t-T_\infty|<\delta_2} ||\{A(t)-A(T_\infty)\}z||_E\leqq\eta$$

for every $z\in L$. Put $h=\min\{\delta_1, \delta_2\}$. Then for $T_k>T_\infty-h$ the estimation

$$\begin{aligned}&\sup_{\substack{T_k\leqq t<T_k+h\\ 0\leqq\tau<h}} ||\{A(t)-A(T_k)\}\exp(-\tau A(T_k))z||_E\\ &\leqq \sup_{\substack{T_k\leqq t<T_k+h\\ 0\leqq\tau<h}} [||\{A(t)-A(T_k)\}z||_E+C_9||\exp(-\tau A(T_k))z-z||_F]\\ &\leqq C_{10}\eta\end{aligned}$$

holds for every $z\in L\supset L_k$. This shows $h\in J_k$, if we take $\eta=C_{10}^{-1}\varepsilon$. But $h\in J_k$ contradicts

$$h > T_\infty - T_k > T_{k+1} - T_k = h_k .$$

3. Proof of the theorem

For each integer $n > \tilde{\beta}$, $A_n(t)$ is the Yosida approximation of $A(t)$

$$A_n(t) = n - n(I + n^{-1}A(t))^{-1} . \tag{3.1}$$

Lemma 3.1. $A_n \in \mathcal{C}([0, T]; \mathcal{L}_s(E)) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F))$.

Proof. In view of (3.1) it suffices to prove

$$(\lambda + A(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E)) \cap \mathcal{C}([0, T]; \mathcal{L}_s(F)) \tag{3.2}$$

for $\lambda > \tilde{\beta}$. For $x \in F$ we can write

$$\begin{aligned} &\{(\lambda + A(t+h))^{-1} - (\lambda + A(t))^{-1}\} x \\ &\quad = -(\lambda + A(t+h))^{-1}\{A(t+h) - A(t)\}(\lambda + A(t))^{-1} x . \end{aligned}$$

Together with the uniform boundness of $||(\lambda + A(\cdot))^{-1}||_E$, this shows that $(\lambda + A(\cdot))^{-1}x$ is continuous in $||\cdot||_E$. For general $x \in E$ it follows from the density of F in E. To see the strong continuity in F of (3.2) we have only to show

$$(\lambda + A(\cdot) + B(\cdot))^{-1} \in \mathcal{C}([0, T]; \mathcal{L}_s(E)) , \tag{3.3}$$

since

$$(\lambda + A(t))^{-1} = S(t)^{-1}(\lambda + A(t) + B(t))^{-1}S(t) \tag{3.4}$$

on F. But (3.3) follows from

$$(\lambda + A(t) + B(t))^{-1} = (\lambda + A(t))^{-1}\{I + B(t)(\lambda + A(t))^{-1}\}^{-1} \tag{3.5}$$

and the strong continuity of $(\lambda + A(\cdot))^{-1}$ in E proved above.

Lemma 3.2. *$\{A_n(t)\}_{0 \leq t \leq T}$ is stable on E (resp. F) with constants of stability $\{M, \beta n(n-\beta)^{-1}\}$ (resp. $\{\tilde{M}, \tilde{\beta} n(n-\tilde{\beta})^{-1}\}$).*

Proof. The stability of $\{A_n(t)\}$ is observed directly by

$$(\lambda + A_n(t))^{-1} = \frac{1}{\lambda + n} + \left(\frac{n}{\lambda + n}\right)^2 \left(\frac{\lambda n}{\lambda + n} + A(t)\right)^{-1} .$$

For $n > \tilde{\beta}$ let $\{U_n(t, s)\}_{0 \leq s \leq t \leq T}$ be the evolution operator for $\{A_n(t)\}_{0 \leq t \leq T}$. From Lemma 3.1 and 3.2 we conclude

$$U_n \in \mathcal{C}(\{(t, s);\ 0 \leqq s \leqq t \leqq T\};\ \mathcal{L}_s(E)),\ ||U_n(t, s)||_E \leqq M e^{\beta_n(t-s)} \tag{3.6}$$

with $\beta_n = \beta n(n-\beta)^{-1}$ and

$$U_n \in \mathcal{C}(\{(t, s);\ 0 \leqq s \leqq t \leqq T\};\ \mathcal{L}_s(F)),\ ||U_n(t, s)||_F \leqq \tilde{M} e^{\tilde{\beta}_n(t-s)} \tag{3.7}$$

with $\tilde{\beta}_n=\tilde{\beta}n(n-\tilde{\beta})^{-1}$.

Let $y\in F$ be arbitrarily fixed, ε be any positive number and Δ be the partition of $[0, T]$ satisfying

$$\sup_{0\leqq s\leqq t\leqq T} \|\{A(t)-A_\Delta(t)\}U_\Delta(t, s)y\|_E\leqq\varepsilon \tag{3.8}$$

whose existence is guaranteed by Proposition 2.1. We can estimate the difference between $U_n(t, s)y$ and $U_\Delta(t, s)y$ by the following:

Proposition 3.3. *There exists an integer N such that for any $n\geqq N$*

$$\sup_{0\leqq s\leqq t\leqq T} \|U_n(t, s)y-U_\Delta(t, s)y\|_E\leqq 2MT\varepsilon e^{\beta_n T}.$$

Proof.

$$\begin{aligned}\{U_\Delta(t, s)-U_n(t, s)\}y &= \int_s^t U_n(t, \tau)\{A_n(\tau)-A_\Delta(\tau)\}U_\Delta(\tau, s)yd\tau \\ &= \int_s^t U_n(t, \tau)\{A_n(\tau)-A(\tau)\}U_\Delta(\tau, s)yd\tau \\ &+ \int_s^t U_n(t, \tau)\{A(\tau)-A_\Delta(\tau)\}U_\Delta(\tau, s)yd\tau .\end{aligned}$$

The second term is evaluated by (3.8). Hence our proposition follows from the next lemma.

Lemma 3.4. *For any compact set K of E, there exists an integer N such that for any $n\geqq N$*

$$\sup_{0\leqq t\leqq T} \|(I+n^{-1}A(t))^{-1}x-x\|_E\leqq\varepsilon$$

holds for every $x\in K$.

Proof. K is covered with a finite number of open balls $\{B(y_i; \varepsilon/2(M+1)\}_{1\leqq i\leqq l}$ in E with centers $y_i\in F$. Hence for any $x\in K$, taking some y_i,

$$\begin{aligned}&\|(I+n^{-1}A(t))^{-1}x-x\|_E \\ &\leqq\|\{(I+n^{-1}A(t))^{-1}-I\}(x-y_i)\|_E+\|\{(I+n^{-1}A(t))^{-1}-I\}y_i\|_E \\ &\leqq(\varepsilon/2)n(n-\beta)^{-1}+M(n-\beta)^{-1}\operatorname*{Max}_{1\leqq i\leqq l}\|A(t)y_i\|_E .\end{aligned}$$

We can now prove that $\{U_n(t, s)\}_{n>\tilde{\beta}}$ is convergent in $\mathcal{L}_s(E)$ uniformly in (t, s). In fact we have

$$\sup_{0\leqq s\leqq t\leqq T} \|U_m(t, s)y-U_n(t, s)y\|_E\leqq 2MT\varepsilon(e^{\beta_m T}+e^{\beta_n T})$$

for any $m, n\geqq N$ by the mediation of $U_\Delta(t, s)y$. $\{U_n(t, s)y\}_{n>\tilde{\beta}}$ is convergent in E uniformly in (t, s). Since $y\in F$ was arbitrary and $\|U_n(t, s)\|_E$ is uniformly bounded by (3.6), $\{U_n(t, s)x\}_{n>\tilde{\beta}}$ is uniformly convergent in E for any $x \in E$. Thus the operator $U(t, s)$ is defined by

$$U(t, s) = \mathcal{L}_s(E)\text{-}\lim_{n\to\infty} U_n(t, s). \tag{3.9}$$

Obviously $U(t, s)$ satisfies a) and c). To see the remaining properties we introduce bounded operators on E

$$W_n(t, s) = S(t)U_n(t, s)S(s)^{-1}, \quad 0 \leqq s \leqq t \leqq T,$$

for each $n > \tilde{\beta}$ analogously to [1]. By (3.7)

$$W_n \in \mathcal{C}(\{(t, s);\ 0 \leqq s \leqq t \leqq T\};\ \mathcal{L}_s(E)).$$

$W_n(t, s)$ is connected with $U_n(t, s)$ by

$$\begin{aligned} W_n(t, s) - U_n(t, s) &= \int_s^t \frac{\partial}{\partial\tau} U_n(t, \tau)S(\tau)U_n(\tau, s)S(s)^{-1} d\tau \\ &= \int_s^t U_n(t, \tau)C_n(\tau)W_n(\tau, s) d\tau \end{aligned}$$

with

$$C_n(t) = A_n(t) - S(t)A_n(t)S(t)^{-1} + \frac{dS}{dt}(t)S(t)^{-1}, \quad 0 \leqq t \leqq T.$$

Lemma 3.5.

$$\mathcal{L}_s(E)\text{-}\lim_{n\to\infty} C_n(t) = -B(t) + \frac{dS}{dt}(t)S(t)^{-1} \tag{3.10}$$

uniformly in t.

Proof. Clearly (3.10) is equivalent to

$$\mathcal{L}_s(E)\text{-}\lim_{n\to\infty} \{S(t)A_n(t)S(t)^{-1} - A_n(t)\} = B(t) \tag{3.11}$$

uniformly in t. By (3.1), (3.4) and (3.5)

$$\begin{aligned} &S(t)A_n(t)S(t)^{-1} \\ &\quad = (A(t)+B(t))\{I+n^{-1}(A(t)+B(t))\}^{-1} \\ &\quad = (A(t)+B(t))(I+n^{-1}A(t))^{-1}\{I+n^{-1}B(t)(I+n^{-1}A(t))^{-1}\}^{-1} \\ &\quad = \{A_n(t)+B(t)(I+n^{-1}A(t))^{-1}\}\{I+n^{-1}B(t)(I+n^{-1}A(t))^{-1}\}^{-1} \\ &\quad = \{A_n(t)+(n^{-1}A_n(t)+(I+n^{-1}A(t))^{-1})B(t)(I+n^{-1}A(t))^{-1}\}\times \\ &\qquad\qquad \times\{I+n^{-1}B(t)(I+n^{-1}A(t))^{-1}\}^{-1} \\ &\quad = A_n(t)+(I+n^{-1}A(t))^{-1}B(t)(I+n^{-1}A(t))^{-1}\times \\ &\qquad\qquad \times\{I+n^{-1}B(t)(I+n^{-1}A(t))^{-1}\}^{-1}. \end{aligned}$$

(3.11) is reduced to

$$\mathcal{L}_s(E)\text{-}\lim_{n\to\infty}(I+n^{-1}A(t))^{-1}=I\,,$$

but this has already been established (Lemma 3.4).

Let $\{W(t,s)\}_{0\leqq s\leqq t\leqq T}$ be a solution of the integral equation

$$W(t,s)=U(t,s)+\int_s^t U(t,\tau)C(\tau)W(\tau,s)d\tau$$

in $\mathcal{L}_s(E)$ with the kernel (3.10)

$$C(t)=-B(t)+\frac{dS}{dt}(t)S(t)^{-1}\,.$$

Obviously

$$W\in C(\{(t,s);\,0\leqq s\leqq t\leqq T\}\,;\,\mathcal{L}_s(E))\,.$$

We can deduce from (3.9) and (3.10)

$$W(t,s)=\mathcal{L}_s(E)\text{-}\lim_{n\to\infty}W_n(t,s)$$

uniformly in (t,s). In other words

$$S(t)^{-1}W(t,s)S(s)=\mathcal{L}_s(F)\text{-}\lim_{n\to\infty}U_n(t,s) \tag{3.12}$$

uniformly in (t,s). We know that all other properties are immediate consequences of (3.12) ([9]).

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