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UNIRATIONAL QUASI-ELLIPTIC SURFACES IN CHARACTERISTIC 3

Dedicated to the memory of Taira Honda

MASAYOSHI MIYANISHI

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0. A non-singular projective surface X is called a *quasi-elliptic* surface if there exists a morphism $f: X \to C$, a curve, with almost all fibres irreducible singular rational curves E with $p_a(E)=1$ (cf. [4]). According to Tate [5], such surfaces can occur only in the case where the characteristic p of the ground field k is either 2 or 3, and almost all fibres E have single ordinary cusps. Let $\mathfrak{k}$ be the function field of C. Then the generic fibre of f with the unique singular point taken off is an elliptic $\mathfrak{k}$-form of the affine line A^1 (cf. [2], [3]); if this form has a $\mathfrak{k}$-rational point(*) it is birational over $\mathfrak{k}$ to one of the following affine plane curves:

(i) If $p=3$, $t^2=x^3+\gamma$ with $\gamma \in \mathfrak{k}-\mathfrak{k}^3$.

(ii) If $p=2$, $t^2=x^3+\beta x+\gamma$ with $\beta, \gamma \in \mathfrak{k}$ and $\beta \notin \mathfrak{k}^2$ or $\gamma \notin \mathfrak{k}^2$.

On the other hand, if X is unirational C must be a rational curve. Conversely if C is a rational curve X is unirational. Indeed, $k(X)\otimes_{\mathfrak{k}}\mathfrak{k}^{1/3}$ is rational over k in the first case, and $k(X)\otimes_{\mathfrak{k}}\mathfrak{k}^{1/2}$ is rational over k in the second case. In this article we consider a unirational quasi-elliptic surface with a rational cross-section only in characteristic 3. Thus X is birational to a hypersurface $t^2=x^3+\phi(y)$ in the affine 3-space A^3, where $\phi(y)\in\mathfrak{k}=k(y)$. If $\phi(y)$ is not a polynominal, write $\phi(y)=a(y)/b(y)$ with $a(y), b(y)\in k[y]$. Substituting t, x by $b(y)^3t$, $b(y)^2x$ respectively and replacing $\phi(y)$ with $b(y)^5a(y)$ we may assume that $\phi(y)\in k[y]$. Moreover, after making suitable birational transformations we may assume that $\phi(y)$ has no monomial terms whose degree are congruent to 0 modulo 3; especially that $d=\deg_y \phi$ is prime to 3. It is easy to see that under this assumption $f(x, y)=x^3+\phi(y)$ is irreducible.

A main result of this article is:

Theorem. *Let k be an algebraically closed field of characteristic* 3. *Then*

(*) This is equivalent to saying that f has a rational cross-section which is different from the section formed by the (movable) singular points of the fibres.

any unirational quasi-elliptic surface with a rational cross-section defined over k is birational to a hypersurface in $\boldsymbol{A}^3$: $t^2=x^3+\phi(y)$ with $\phi(y)\in k[y]$. Let $K=k(t, x, y)$ be an algebraic function field of dimension 2 generated by t, x, y over k such that $t^2=x^3+\phi(y)$ with $\phi(y)\in k[y]$ and $d=\deg_y \phi$ prime to 3. Let m be the quotient of d divided by 6, and let H_0 be the (non-singular) minimal model of K when K is not rational over k(*). *Moreover if $d\geqq 7$ assume that the following conditions hold*(**):

(1) *For* every *root α of $\phi'(y)=0$, $v_\alpha(\phi(y)-\phi(\alpha))\leqq 5$, where v_α is the $(y-\alpha)$-adic valuation of $k[y]$ with $v_\alpha(y-\alpha)=1$.*

(2) *If, moreover,* $\phi(y)-\phi(\alpha)=a(y-\alpha)^3+$(terms of higher degree in $y-\alpha$) *for* some *root α of $\phi'(y)=0$ and $a\in k-(0)$ then $v_\alpha(\phi(y)-\phi(\alpha)-a(y-\alpha)^3)\leqq 5$.*

Then we have the following:

(i) *If* $m=0$, i.e., $d\leqq 5$, *then K is rational over k. If $d\geqq 7$, K is not rational over k, and the minimal model H_0 exists.*

(ii) *If* $m=1$, i.e., $7\leqq d\leqq 11$, *then H_0 is a $K3$-surface.*

(iii) *If* $m>1$, i.e., $d\geqq 13$, *then* $p_a(H_0)=p_g(H_0)=m$, $q=\dim H^1(H_0, \mathcal{O}_{H_0})=0$, *the r-genus* $P_r(H_0)=r(m-1)+1$ *for every positive integer r, and* $\kappa(H_0)=1$.

We use the following notations: Let X be a non-singular projective surface. Then $K_X=$the canonical divisor class on X, $p_g(X)=\dim H^0(X, K_X)=$ the geometric genus, $q=\dim H^1(X, \mathcal{O}_X)=$the irregularity, $p_a(X)=p_g(X)-q=$the arithmetic genus, $\kappa(X)=$the Kodaira dimension of X, and $P_r(X)=\dim H^0(X, K_X^{\otimes r})$ $=$the r-genus for a positive integer r. For divisors D, D' etc. on X, $(D\cdot D')$ or (D^2) is the intersection number. We use sometimes the notation $D\cdot D'$ or D^2 to indicate the intersection number if there is no fear of confusion.

1. Let k be an algebraically closed field of characteristic $p=3$, let $\phi(y)$ be a polynomial in y with coefficients in k of degree $d>0$ and let $f(x, y)=x^3+\phi(y)$. Consider a hypersurface $t^2=x^3+\phi(y)$ in the projective 3-space $\boldsymbol{P}^3$, which is birational to a double covering(***) of $F_0=\boldsymbol{P}^1\times\boldsymbol{P}^1$. After a birational transfromation of type $(x, y, t)\mapsto(x+\rho(y), y, t)$ with $\rho(y)\in k[y]$ we may assume that $(d, 3)=1$ and moreover that $\phi(y)$ does not contain monomial terms whose degrees are congruent to zero modulo 3. Since K is apparently rational if $d=1$ or 2 we may assume that $d>3$.

The equation $x^3+\phi(y)=0$ defines a closed irreducible curve C in F_0. First of all, we shall look into singular points of C and the normalization $\bar{C}$ of C. Let $P : (x, y)=(\beta, \alpha)$ be a singular point of C lying on the affine part $\boldsymbol{A}^2=F_0-(x=$

(*) Note that if K is ruled and unirational then K is rational. Hence if K is not rational K has the minimal model.

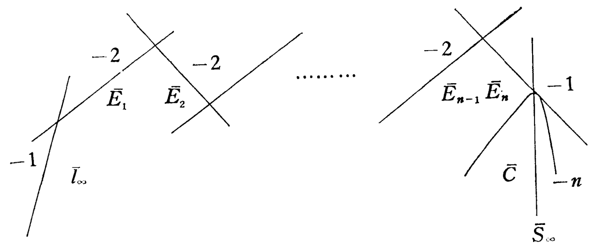
(**) If either one of these conditions is violated we can drop the degree d by 6 by a suitable birational transformation.

(***) A morphism $f: X'\to X$ of complete integral algebraic surfaces is called a double covering if f induces a separable quadratic extension of function fields $k(X')/k(X)$.

$\infty) \cup (y=\infty)$. Then $\phi'(\alpha)=0$ and $\beta^3+\phi(\alpha)=0$. Conversely every root of $\phi'(y)=0$ gives rise to a singular point of C lying on A^2. Since $\phi'(y)=0$ has at least one root, C has at least one singular point on $A^2 \subset F_0$. The point Q of C, which is situated outside of A^2, is given by $(\xi, u)=(0, 0)$, where $x=1/\xi$, $y=1/u$ and $u^d+\xi^3\psi(u)=0$ with $\psi(u)=u^d\phi(1/u)$ and $\psi(0)\neq 0$. Hence Q is a cuspidal singular point with multiplicity $(\underbrace{3, 3, \cdots, 3}_{n}, 1, \cdots)$(*) if $d=3n+1$; and $(\underbrace{3, 3, \cdots, 3}_{n}, 2, 1, \cdots)$ if $d=3n+2$.

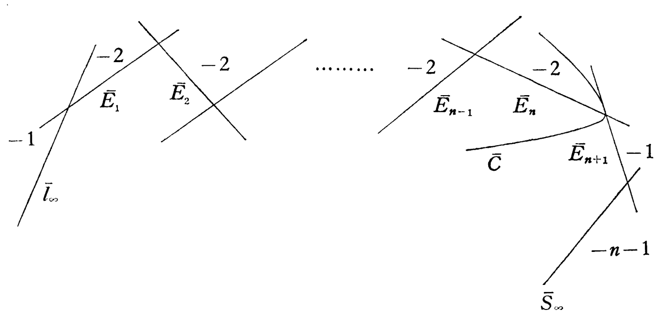
Here we introduce the following notations: Consider a fibration $\mathcal{F}=\{l_\alpha: l_\alpha \text{ is defined by } y=\alpha\}$ on F_0. We denote by l_∞ the fibre $y=\infty$, and by S_∞ the cross-section $x=\infty$. We denote by l a general fibre of $\mathcal{F}$.

Let $\sigma: F\to F_0$ be the smallest blowings-up of F_0 with centers at all singular points of C and their infinitely near singular points, by which the proper transform $\bar{C}=\sigma' C$ of C on F becomes non-singular. Let $\bar{S}_\infty=\sigma' S_\infty$, and let $\bar{l}_\infty=\sigma' l_\infty$. The following figures will indicate the configuration of F in a neighbourhood of $\sigma^{-1}(l_\infty \cup C \cup S_\infty)$.



(Fig. 1)

where $d=3n+1$ and $(\bar{C}\cdot\bar{E}_n)=3$;



(Fig. 2)

where $d=3n+2$ and $(\bar{C}\cdot\bar{E}_{n+1})=2$.

Since $(f)_\infty|_{F_0}=3S_\infty+dl_\infty$, we have

$$(f)|_F = \bar{C}+(3\bar{E}_1+6\bar{E}_2+\cdots+3n\bar{E}_n)+D-3(\bar{S}_\infty+\bar{E}_1+2\bar{E}_2+\cdots +n\bar{E}_n)-d(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n) = \bar{C}-3\bar{S}_\infty+D-d(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)$$

(*) By this notation we mean that Q is a point with multiplicity 3, the infinitely near point of C in the first neighborhood (which is a single point in this case) has multiplicity 3, etc.

if $d=3n+1$, where D is a positive divisor with support in the union $\mathcal{E}$ of exceptional curves which arise from the blowings-up with centers at the singular points and their infinitely near singular points of C in the affine part $A^2\subset F_0$; and also

$$\begin{aligned}(f)|_F &= \bar{C}+(3\bar{E}_1+6\bar{E}_2+\cdots+3n\bar{E}_n+(3n+2)\bar{E}_{n+1})\\ &+D-3(\bar{S}_\infty+\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n+(n+1)\bar{E}_{n+1})-d(\bar{l}_\infty+\bar{E}_1+\cdots\\ &+\bar{E}_{n+1}) = \bar{C}-3\bar{S}_\infty-\bar{E}_{n+1}-d(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n+1})+D\end{aligned}$$

if $d=3n+2$.

On the other hand since $K_{F_0}\sim -2S_\infty-2l_\infty$, we have

$$\begin{aligned}K_F\sim &-2(\bar{S}_\infty+\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n)-2(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)\\ &+\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n+D_3 \qquad \text{if } d=3n+1;\end{aligned}$$

and

$$\begin{aligned}K_F\sim &-2(\bar{S}_\infty+\bar{E}_1+\cdots+(n+1)\bar{E}_{n+1})-2(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n+1})\\ &+\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n+(n+1)\bar{E}_{n+1}+D_3 \qquad \text{if } d=3n+2,\end{aligned}$$

where D_3 is a positive divisor with support in $\mathcal{E}$.

We are now going to consider four cases separately.

(I) If $d=6m+1$ then $d=3n+1$ with $n=2m$. Let $B=\bar{C}+\bar{S}_\infty+(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)+D_1$ and let $Z=2\bar{S}_\infty+(3m+1)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)-D_2$, where D_1 and D_2 are the divisors uniquely determined by the conditions that $D_1\geqq 0$, every irreducible component of D_1 has multiplicity 1, $D_2\geqq 0$, $D_1+2D_2=D$, and $\operatorname{Supp}(D_1)\cup\operatorname{Supp}(D_2)\subset\mathcal{E}$. Then $(f)=B-2Z$, and $K_F+Z\sim(3m-1)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)-(\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n)+(D_3-D_2)\sim(3m-1)\sigma^{-1}(l)-(\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n)+(D_3-D_2)$. Hence $Z\cdot(K_F+Z)=2(3m-1)-2n+D_2\cdot(D_2-D_3)=2m-2+D_2\cdot(D_2-D_3)$, and $p_a(Z)=Z\cdot(K_F+Z)/2+1=m+D_2\cdot(D_2-D_3)/2$.

(II) If $d=6m+2$ then $d=3n+2$ with $n=2m$. Let $B=\bar{C}+\bar{S}_\infty+\bar{E}_{n+1}+D_1$, and let $Z=2\bar{S}_\infty+\bar{E}_{n+1}+(3m+1)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n+1})-D_2$, where D_1 and D_2 are divisors chosen as in the case (I). Then $(f)=B-2Z$, and $K_F+Z\sim(3m-1)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n+1})-(\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n+n\bar{E}_{n+1})+(D_3-D_2)\sim(3m-1)\sigma^{-1}(l)-(\bar{E}_1+2\bar{E}_2+\cdots+n\bar{E}_n+n\bar{E}_{n+1})+(D_3-D_2)$. Hence $Z\cdot(K_F+Z)=2(3m-1)-2n-n+n+D_2\cdot(D_2-D_3)=2m-2+D_2\cdot(D_2-D_3)$, and $p_a(Z)=m+D_2\cdot(D_2-D_3)/2$.

(III) If $d=6m+4$ then $d=3n+1$ with $n=2m+1$. Let $B=\bar{C}+\bar{S}_\infty+D_1$ and let $Z=2\bar{S}_\infty+(3m+2)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)-D_2$, where D_1 and D_2 are divisors chosen as above. Then $(f)=B-2Z$, and $K_F+Z\sim 3m\sigma^{-1}(l)-(\bar{E}_1+\cdots+n\bar{E}_n)+(D_3-D_2)$. Hence $Z\cdot(K_F+Z)=6m-2n+D_2\cdot(D_2-D_3)=2m-2+D_2\cdot(D_2-D_3)$, and $p_a(Z)=m+D_2\cdot(D_2-D_3)/2$.

(IV) If $d=6m+5$ then $d=3n+2$ with $n=2m+1$. Let $B=\bar{C}+\bar{S}_\infty+(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)+D_1$ and let $Z=2\bar{S}_\infty+(3m+3)(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n+1})-D_2$, where D_1 and D_2 are divisors chosen as above. Then $(f)=B-2Z$, and $K_F+Z\sim(3m+1)\sigma^{-1}(l)-(\bar{E}_1+\cdots+n\bar{E}_n+(n+1)\bar{E}_{n+1})+(D_3-D_2)$. Hence $Z\cdot(K_F+Z)=2(3m+1)$

$-2(n+1)+D_2\cdot(D_2-D_3)=2m-2+D_2\cdot(D_2-D_3)$, and $p_a(Z)=m+D_2\cdot(D_2-D_3)/2$.

In each case, $p_a(Z)=m+D_2\cdot(D_2-D_3)/2$. Let $\bar{F}\to F$ be the smallest blowings-up which make the branch locus of the double covering on $\bar{F}$ non-singular, let H be the normalization of $\bar{F}$ in the function field $K=k(t, x, y)$ and let $\pi: H \to F$ be the canonical morphism. Then H is a non-singular projective surface called the *canonical model* of K, which is a double covering of F with branch locus B(*) in each of the above four cases (cf. Artin [1]). Let K_H be the canonical divisor of H. By Artin [1], we know that $K_H\sim\pi^{-1}(K_F+Z)$ and $p_a(H)=2p_a(F)+p_a(Z)$, since the singular points on the branch locus B on F are all negligible singularities(**) and since $p_a(F)=0$.

Thus we proved:

Lemma 1. *Let m be the quotient of d divided by 6. Then $p_a(H)=m+D_2\cdot(D_2-D_3)/2$.*

Now we show:

Lemma 2. *With the notations and assumptions as above, H is a rational surface if $d\leqq 5$.*

Proof. First of all, we may assume that $d\leqq 4$. In effect, if $d=5$ we may assume that $\phi(y)$ has no constant and degree 1 terms after a suitable change of variables x and y. Then by a change of variables : $t'=t/y^3$, $x'=x/y^2$, $y'=1/y$, we have

$$t'^2 = x'^3+\bar{\phi}(y') \qquad \text{with } \deg_{y'}\bar{\phi}(y')\leqq 4\,.$$

Now assuming that $d\leqq 4$ and $\phi(y)$ has no monomial terms whose degrees are congruent to zero modulo 3, we are going to compute D_2-D_3 and K_H explicitly. Let ν be the number of distinct roots of $\phi'(y)=0$. If $\nu=1$, we may assume that $\phi(y)=y^d$ after a suitable change of variables. Let P: $(x, y)=(0, 0)$. P is a singular point of C with multiplicity $(2, 1, \cdots)$ if $d=2$; $(3, 1, \cdots)$ if $d=4$. Then $D=2E$ with $E=\sigma^{-1}(P)$ if $d=2$; $D=3E$ if $d=4$. Then $D_1=0$, $D_2=D_3=E$ if $d=2$; $D_1=D_2=D_3=E$ if $d=4$. In each case $D_2-D_3=0$. If $\nu=2$, let α_1 and α_2 be distinct roots. We have two possible casse: (i) Both α_1 and α_2 are simple roots; (ii) One of α_1 and α_2 is a double root and the other one is a simple root. However neither case can occur. Indeed, $d=3$ in the first case, and the second case is impossible. If $\nu=3$, let α_1, α_2 and α_3 be distinct roots. Then $d=4$, and

(*) A point P of F is a branch point, *i.e.*, $P\in B$ if the normalization of $\mathcal{O}_{P,F}$ in K is a local ring.

(**) A point P of B has negligible singularity if and only if it is of one of the following types: (i) a simple point of B, (ii) a double point of B, (iii) a triple point of B with at most a double point (not necessarily ordinary) infinitely near (cf. Artin [1]). For the arithmetic genus formula, see also [B. Iversen: Numerical invariants and multiple planes, Amer. J. Math., 92 (1970), 968–996].

α_1, α_2 and α_3 are all simple roots. Let $P_i (i=1, 2, 3)$ be the singular point of C with y-coordinate α_i. The multiplicity of P_i is $(2, 1, \cdots)$. Hence $D=2(\sigma^{-1}(P_1)+\sigma^{-1}(P_2)+\sigma^{-1}(P_3))$, $D_1=0$ and $D_2=D_3=\sigma^{-1}(P_1)+\sigma^{-1}(P_2)+\sigma^{-1}(P_3)$. Thus $D_2-D_3=0$. Therefore $p_a(H)=0$.

On the other hand, since $K_H \sim \pi^{-1}(K_F+Z)$, we see from the above observations on K_F+Z that $K_F+Z<0$ if $d\leqq 4$. Hence $K_H<0$ and $P_2(H)=0$. Therefore H is rational by virtue of Castelnuovo's criterion of rationality. Q.E.D.

2. Let us consider the following conditions on $\phi(y)$:

(1) For *every* root α of $\phi'(y)=0$, $v_\alpha(\phi(y)-\phi(\alpha))\leqq 5$, where v_α is the $(y-\alpha)$-adic valuation of $k[y]$ with $v_\alpha(y-\alpha)=1$.

(2) If, moreover, $\phi(y)-\phi(\alpha)=a(y-\alpha)^3+$(terms of higher degree in $y-\alpha$) for *some* root α of $\phi'(y)=0$ and $a\in k-(0)$ then $v_\alpha(\phi(y)-\phi(\alpha)-a(y-\alpha)^3)\leqq 5$.

Assume that $v_\alpha(\phi(y)-\phi(\alpha))\geqq 6$ for some root α of $\phi'(y)=0$. Since $d>0$, this assumption implies $d\geqq 6$. Then by a birational transformation $(t, x, y)\mapsto (t_1=t/(y-\alpha)^3, x_1=(x+\phi(\alpha)^{1/3})/(y-\alpha)^2, y_1=y-\alpha)$, we have

$$t_1^2 = x_1^3+\phi_1(y_1) \quad \text{with} \quad \deg_{y_1}\phi_1 = \deg_y\phi - 6 .$$

Assume next that $\phi(y)-\phi(\alpha)=a(y-\alpha)^3+$(terms of higher degree in $y-\alpha$) for some root α of $\phi'(y)=0$ and that $v_\alpha(\phi(y)-\phi(\alpha)-a(y-\alpha)^3)\geqq 6$. Then by a birational transformation $(t, x, y)\mapsto (t_1=t, x_1=x+a^{1/3}(y-\alpha), y_1=y)$ we have

$$t_1^2 = x_1^3+\phi_1(y_1) \quad \text{with} \quad \deg_{y_1}\phi_1 = d \quad \text{and} \quad v_\alpha(\phi_1(y_1)-\phi_1(\alpha))\geqq 6 .$$

Therefore the argument in the former case applies, and we can drop the degree of ϕ_1 by 6. Therefore we may assume that $d\geqq 7$ and that the conditions (1) and (2) hold. Hereafter we assume these conditions for $\phi(y)$. Then we have:

Lemma 3. *With the notations as above, $D_2=D_3$.*

Proof. Let α be a root of $\phi'(y)=0$, and let P: $(x, y)=(-\phi(\alpha)^{1/3}, \alpha)$ be the corresponding singular point of C. Let $e=v_\alpha(\phi(y)-\phi(\alpha))$. Since the conditions (1) and (2) hold, we may assume that $e=2$, 4 or 5. In fact, the case where $e=3$ can be reduced to the case where $e=4$ or 5 by a birational transformation $(t, x, y)\mapsto (t, x+a^{1/3}(y-\alpha), y)$, which is biregular at P. P is then a cuspidal singular point with multiplicity $(2, 1, \cdots)$ if $e=2$; $(3, 1, \cdots)$ if $e=4$; $(3, 2, 1, \cdots)$ if $e=5$. Hence $\sigma^{-1}(P)=E_1$ (irreducible) if $e=2$ or 4; $\sigma^{-1}(P)=E_1+E_2$ (E_1 and E_2 are irreducible) if $e=5$. Then $D_2=D_3=E_1$ if $e=2$ or 4; $D_2=D_3=E_1+2E_2$ if $e=5$. In both cases, $D_2=D_3$. Q.E.D.

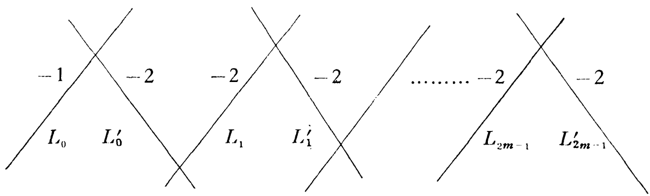
Corollary. *Let m be the quotient of d divided by 6. If one assumes the conditions* (1) *and* (2) *on $\phi(y)$, $p_a(H)=m$.*

The canonical model H of K might contain the exceptional curves of the first kind. When $p_a(H)=m>0$ (*i.e.*, $d\geqq 7$), let H_0 be the minimal non-singular model of K, which is, needless to say, obtained from H by contracting all exceptional curves of the first kind. We shall describe the canonical divisor K_{H_0} of H_0.

Lemma 4. *Assume that $d=6m+1$ with $m>0$. Thren we have:*

(i) $\pi^{-1}(\bar{l}_\infty\cap\bar{E}_1)=L'_0$, $\pi^{-1}(\bar{E}_1\cap\bar{E}_2)=L'_1$, $\cdots$, $\pi^{-1}(\bar{E}_{n-1}\cap\bar{E}_n)=L'_{n-1}$ *where* L'_i $(0\leqq i\leqq n-1)$ *is an irreducible non-singular rational curve with* $(L'^2_i)=-2$ *and* $n=2m$.

(ii) $\pi^{-1}(\bar{l}_\infty)=2L_0+L'_0$, $\pi^{-1}(\bar{E}_i)=L'_{i-1}+2L_i+L'_i$ $(1\leqq i\leqq n-1)$, *where* L_i $(0\leqq i\leqq n-1)$ *is an irreducible non-singular rational curve such that* $(L_0^2)=-1$, $(L_i^2)=-2$ $(1\leqq i\leqq n-1)$.



(Fig. 3)

(iii) $K_H\sim\pi^{-1}(K_F+Z)\sim(m-1)\pi^{-1}\sigma^{-1}(l)+4mL_0+(4m-1)L'_0+(4m-2)L_1$
$+(4m-3)L'_1+\cdots+3L'_{2m-2}+2L_{2m-1}+L'_{2m-1}$.

(iv) $W:=L_0+L'_0+L_1+\cdots+L'_{2m-1}$ *is contractible. Let* $\tau\colon H\to H_0$ *be the contraction of* W. *Then* H_0 *is a minimal model of* K. *Hence* $K_{H_0}\sim(m-1)\tau\pi^{-1}\sigma^{-1}(l)$.

(v) *For every positive integer* r *the* r*-genus* $P_r(H_0)$ *of* H_0 *is* $r(m-1)+1$. *In particular,* $p_g(H_0)=p_a(H_0)=m$ *and* $q=0$.

(vi) *If* $m=1$, i.e., $d=7$, H_0 *is a* $K3$*-surface. If* $m>1$, $\kappa(H_0)=1$.

Proof. First of all note that $B=\bar{C}+\bar{S}_\infty+(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_n)+D_1$ and $K_F+Z\sim(m-1)\sigma^{-1}(l)+(2m\bar{l}_\infty+(2m-1)\bar{E}_1+\cdots+\bar{E}_{2m-1})$. Let $\sigma_1\colon F_1\to F$ be the blowings-up with centers at $\bar{l}_\infty\cap\bar{E}_1$, $\bar{E}_1\cap\bar{E}_2$, $\cdots$, $\bar{E}_{n-1}\cap\bar{E}_n$ (cf. Fig. 1). Then $\pi\colon H\to F$ factors as $\pi\colon H\xrightarrow{\pi_1}F_1\xrightarrow{\sigma_1}F$, *i.e.*, $\pi=\sigma_1\pi_1$. Since the branch locus B_1 on F_1 is of the form $B_1=\sigma'_1(\bar{l}_\infty)+\sigma'_1(\bar{E}_1)+\cdots+\sigma'_1(\bar{E}_{n-1})+B'_1$ with B'_1 having no intersections with $\sigma'_1(\bar{l}_\infty+\bar{E}_1+\cdots+\bar{E}_{n-1})$, π_1 conicides with $\bar{\pi}\colon H\to\bar{F}$, which is the canonical normalization morphism, on a small open neighbourhood of $\sigma_1^{-1}(\bar{l}_\infty\cup\bar{E}_1\cup\cdots\cup\bar{E}_{n-1})$. Now writing locally the equations of $\pi^{-1}(\bar{l}_\infty\cap\bar{E}_1)=\pi_1^{-1}(\sigma_1^{-1}(\bar{l}_\infty\cap\bar{E}_1))$, $\pi^{-1}(\bar{E}_1\cap\bar{E}_2)$, $\cdots$, $\pi^{-1}(\bar{E}_{n-1}\cap\bar{E}_n)$, it is not hard to show that L'_0, $\cdots$, L'_{n-1} are irreducible non-singular rational curves. For $0\leqq i\leqq n-1$, $(L'^2_i)=2(\sigma_1^{-1}(\bar{E}_i\cap\bar{E}_{i+1})^2)=-2$. This proves the assertion (i).

To show the assertion (ii), note that $\bar{l}_\infty$, $\bar{E}_1$, $\cdots$, $\bar{E}_{n-1}$ are components of the branch locus B. Therefore $\pi^{-1}(\bar{l}_\infty)=2L_0+L'_0$ and $\pi^{-1}(\bar{E}_i)=L'_{i-1}+2L_i+L'_i$ $(1\leqq i\leqq n-1)$ with non-singular irreducible rational curves L_i $(0\leqq i\leqq n-1)$. Since $(\sigma'_1(\bar{l}_\infty)^2)=-2$ and $\pi_1^{-1}(\sigma'_1(\bar{l}_\infty))=2L_0$, we have $4(L_0^2)=-4$. Hence $(L_0^2)=-1$.

Similarly, $(\sigma_1'(\bar{E}_i)^2)=-4$ and $\pi_1^{-1}(\sigma_1'(\bar{E}_i))=2L_i$ for $1\leqq i\leqq n-1$. Hence $(L_i^2)=-2$ for $1\leqq i\leqq n-1$.

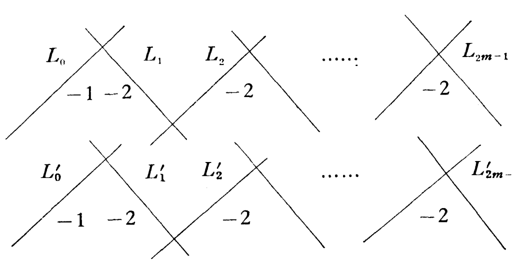
By virtue of the assertions (i) and (ii), $K_H\sim\pi^{-1}(K_F+Z)\sim(m-1)\pi^{-1}\sigma^{-1}(l)+\pi^{-1}(2m\bar{l}_\infty+(2m-1)\bar{E}_1+\cdots+\bar{E}_{2m-1})=(m-1)\pi^{-1}\sigma^{-1}(l)+4mL_0+(4m-1)L_0'+(4m-2)L_1+\cdots+3L_{2m-2}'+2L_{2m-1}+L_{2m-1}'$. Since L_i's and L_i''s $(0\leqq i\leqq 2m-1)$ have the configuration as indicated in the Fig. 3, it is easy to show that W is contractible, and $(m-1)\pi^{-1}\sigma^{-1}(l)$ is the moving part of $|K_H|$. Let $\tau: H\to H_0$ be the contraction of W. Then $K_{H_0}=\tau((m-1)\pi^{-1}\sigma^{-1}(l))$. Hence $\dim|K_{H_0}|\geqq 0$ and $|K_{H_0}|$ has no fixed components if $m\geqq 1$. This implies that H_0 is a minimal model of K. Thus the assertions (iii) and (iv) are proven.

Let us show that $P_r(H_0)=(m-1)r+1$ for every positive integer r. There exists a non-singular irreducible rational curve $\tilde{S}_\infty$ on H such that $\pi(\tilde{S}_\infty)=\bar{S}_\infty$, $\pi^{-1}(\bar{S}_\infty)>2\tilde{S}_\infty$ and $\tilde{S}_\infty\cap\mathrm{Supp}(W)=\phi$. Let $\hat{S}_\infty=\tau(\tilde{S}_\infty)$. Then $\hat{S}_\infty$ is a non-singular irreducible rational curve. Since $\dim|rK_{H_0}|=\dim Tr_{\hat{S}_\infty}|rK_{H_0}|+\dim|rK_{H_0}-\hat{S}_\infty|+1$, we compute $\dim Tr_{\hat{S}_\infty}|rK_{H_0}|$ and $\dim|rK_{H_0}-\hat{S}_\infty|$. Suppose that $|rK_{H_0}-\hat{S}_\infty|\neq\phi$, and let $M\in|rK_{H_0}|$ be such that $M>\hat{S}_\infty$. Then $\tau^{-1}M>\tau^{-1}\hat{S}_\infty=\tilde{S}_\infty$, and $\tau^{-1}M\sim r(m-1)\pi^{-1}\sigma^{-1}(l)$. Then $\sigma\pi(\tau^{-1}M)>\sigma\pi\tilde{S}_\infty=S_\infty$, and $\sigma\pi(\tau^{-1}M)\sim 2r(m-1)l$. This is a contradiction since no members of $|2r(m-1)l|$ on $F_0=\boldsymbol{P}^1\times\boldsymbol{P}^1$ contain S_∞. Thus $\dim|rK_{H_0}-\hat{S}_\infty|=-1$. On the other hand, since $\hat{S}_\infty\cong\boldsymbol{P}^1$ and $\deg Tr_{\hat{S}_\infty}|rK_{H_0}|=r(m-1)^{(*)}$ and $Tr_{\hat{S}_\infty}|rK_{H_0}|$ is apparently complete we have $\dim Tr_{\hat{S}_\infty}|rK_{H_0}|=r(m-1)$. Therefore $P_r(H_0)=r(m-1)+1$. In particular, $p_g(H_0)=P_1(H_0)=m=p_a(H_0)$. Hence $q=\dim H^1(H_0,\mathcal{O}_{H_0})=p_g(H_0)-p_a(H_0)=0$. Thus H_0 is a regular surface. If $m=1$, H_0 is a $K3$-surface. If $m>1$, $\kappa(H_0)=1$ since $P_r(H_0)$ is a linear polynomial in r. This completes the proof of the assertions (v) and (vi). Q.E.D.

In a similar fashion we can show:

Lemma 5. *Assume that $d=6m+2$ with $m>0$. Then we have:*

(i) $\pi^{-1}(\bar{l}_\infty)=L_0+L_0'$, $\pi^{-1}(\bar{E}_i)=L_i+L_i'$ $(1\leqq i\leqq 2m-1)$ *where L_i's and L_i''s are irreducible non-singular rational curves such that $(L_0^2)=(L_0'^2)=-1$ and $(L_i^2)=(L_i'^2)=-2$ $(1\leqq i\leqq 2m-1)$. They have the following configuration:*



(Fig. 4)

(*) Cf. $2(\hat{S}_\infty\cdot\tau\pi^{-1}\sigma^{-1}(l))=2(\tilde{S}_\infty\cdot\pi^{-1}\sigma^{-1}(l))=(2\tilde{S}_\infty\cdot\pi^{-1}\sigma^{-1}(l))=2(\bar{S}_\infty\cdot\sigma^{-1}(l))=2.$

(ii) $K_H \sim \pi^{-1}(K_F+Z) \sim (m-1)\pi^{-1}\sigma^{-1}(l)+2mL_0+(2m-1)L_1+\cdots+L_{2m-1}$
$+2mL_0'+(2m-1)L_1'+\cdots+L_{2m-1}'$.

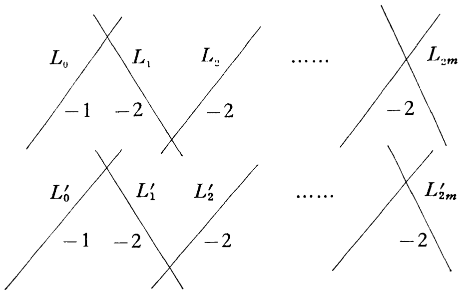
(iii) *Let* $W:=L_0+L_1+\cdots+L_{2m-1}+L_0'+L_1'+\cdots+L_{2m-1}'$. *Then* W *is contractible, and if* τ: $H \to H_0$ *is the contraction of* W, H_0 *is a minimal model of* K. *Hence* $K_{H_0} \sim (m-1)\tau\pi^{-1}\sigma^{-1}(l)$.

(iv) *For every positive integer* r, $P_r(H_0)=r(m-1)+1$. *In particular*, $p_g(H_0)$ $=p_a(H_0)=m$ *and* $q=0$.

(v) *If* $m=1$, i.e., $d=8$, H_0 *is a* $K3$-*surface. If* $m>1$, $\kappa(H_0)=1$.

Lemma 6. *Assume that* $d=6m+4$ *with* $m>0$. *Then we have*:

(i) $\pi^{-1}(\bar{l}_\infty)=L_0+L_0'$, $\pi^{-1}(\bar{E}_i)=L_i+L_i'$ $(1 \leqq i \leqq 2m)$, *where* L_i*'s and* L_i'*'s are irreducible non-singular rational curves such that* $(L_0^2)=(L_0'^2)=-1$, $(L_i^2)=(L_i'^2)=-2$ $(1 \leqq i \leqq 2m)$. *They have the following configuration*:



(Fig. 5)

(ii) $K_H \sim \pi^{-1}(K_F+Z) \sim (m-1)\pi^{-1}\sigma^{-1}(l)+(2m+1)L_0+2mL_1+\cdots+L_{2m}$
$+(2m+1)L_0'+2mL_1'+\cdots+L_{2m}'$.

(iii) *Let* $W:=L_0+L_1+\cdots+L_{2m}+L_0'+L_1'+\cdots+L_{2m}'$. *Then* W *is contractible, and if* τ: $H \to H_0$ *is the contraction of* W, H_0 *is a minimal model of* K. *Hence* $K_{H_0} \sim (m-1)\tau\pi^{-1}\sigma^{-1}(l)$.

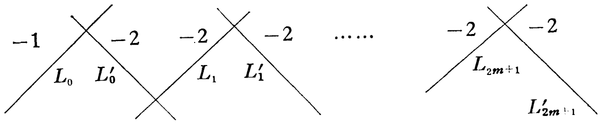
(iv) *For every positive integer* r, $P_r(H_0)=r(m-1)+1$. *In particular*, $p_g(H_0)$ $=p_a(H_0)=m$ *and* $q=0$.

(v) *If* $m=1$, i.e., $d=10$, H_0 *is a* $K3$-*surface. If* $m>1$, $\kappa(H_0)=1$.

Lemma 7. *Assume that* $d=6m+5$ *with* $m>0$. *Then we have*:

(i) $\pi^{-1}(\bar{l}_\infty \cap \bar{E}_1)=L_0'$, $\pi^{-1}(\bar{E}_1 \cap \bar{E}_2)=L_1'$, $\cdots$, $\pi^{-1}(\bar{E}_n \cap \bar{E}_{n+1})=L_n'$, *where* $n=2m+1$ *and* L_i' $(0 \leqq i \leqq n)$ *is an irreducible non-singular rational curve with* $(L_i'^2)=-2$.

(ii) $\pi^{-1}(\bar{l}_\infty)=2L_0+L_0'$ *and* $\pi^{-1}(\bar{E}_i)=L_{i-1}'+2L_i+L_i'$ $(1 \leqq i \leqq n)$, *where* L_i $(0 \leqq i \leqq n)$ *is an irreducible non-singular rational curve such that* $(L_0^2)=-1$ *and* $(L_i^2)=-2$ $(0<i \leqq n)$. L_i*'s and* L_i'*'s have the following configuration*:



(Fig. 6)

(iii) $K_H \sim \pi^{-1}(K_F+Z) \sim (m-1)\pi^{-1}\sigma^{-1}(l)+(4m+4)L_0+(4m+3)L_0'$
$+\cdots+2L_{2m+1}+L_{2m+1}'$.

(iv) *Let* $W:=L_0+L_0'+\cdots+L_{2m+1}+L_{2m+1}'$. *Then* W *is contractible. If* τ: $H \to H_0$ *is the contraction of* W, H_0 *is the minimal model of* K. *Hence* $K_{H_0} \sim (m-1)\tau\pi^{-1}\sigma^{-1}(l)$.

(v) *For every positive integer* r, $P_r(H_0)=r(m-1)+1$. *In particular,* $p_g(H_0)=p_a(H_0)=m$ *and* $q=0$.

(vi) *If* $m=1$, i.e., $d=11$, H_0 *is a* $K3$*-surface. If* $m>1$, $\kappa(H_0)=1$.

Combining the above results, we have our main theorem.

REMARK. If $m>1$, H_0 is not birational to an elliptic surface. Assume the contrary, and let ρ: $H' \to H_0$ be a birational morphism with a non-singular projective surface H' endowed with an elliptic pencil $\mathcal{L}=\{C_\alpha;\ \alpha\in \boldsymbol{P}^1\}$. Then $K_{H'} \sim (m-1)\rho^{-1}\tau\pi^{-1}\sigma^{-1}(l)+E$, where $E\geqq 0$ with Supp(E) the union of exceptional curves arising from ρ. For a general member C of $\mathcal{L}$ we have $(C^2)=0$, and $C\cdot K_{H'}\geqq 0$ because C is a non-singular irreducible curve distinct from components of E. Since $1=p_a(C)=(C^2+C\cdot K_{H'})/2+1$ we have $C\cdot K_{H'}=0$. Hence C coincides with a component of a member of $|(m-1)\rho^{-1}\tau\pi^{-1}\sigma^{-1}(l)|$, *i.e.,* $C=\rho^{-1}\tau\pi^{-1}\sigma^{-1}(l)\cong\tau\pi^{-1}\sigma^{-1}(l)$ for some l. This is absurd because $\tau\pi^{-1}\sigma^{-1}(l)$ is rational.

OSAKA UNIVERSITY

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