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ON SCATTERING THEORY OF A FIRST ORDER ORDINARY DIFFERENTIAL OPERATOR

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In this paper we study the scattering theory of the operator

$$
(0.1)\qquad L_u = iID - i\begin{bmatrix} 0 & u_1 & u_2 \\ u_1^* & 0 & 0 \\ u_2^* & 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},
$$
$$
D = d/dx\,, \quad u = {}^t(u_1, u_2)
$$

where u_1 and u_2 are complex-valued integrable functions, and * denotes complex-conjugate.

The operator L_u has been introduced by Manakov [4] in order to solve the system of non-linear Schrödinger equations

$$
(0.2)\qquad iu_t + 2^{-1}u_{xx} + \|u\|^2 u = 0\,, \quad \|u\|^2 = |u_1|^2 + |u_2|^2
$$

in terms of the scattering data of (0.1).

In [4], Manakov has described the scattering theory of (0.1) formally. The scattering theory of (0.1) resembles in many respects to that of the operator

$$
(0.3)\qquad L_u = i\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} D - i\begin{bmatrix} 0 & u \\ u^* & 0 \end{bmatrix}
$$

which has been introduced by Zakharov and Shabat [7] in order to solve the scalar non-linear Schrödinger equation. The scattering theory of (0.3) has been treated by Tanaka [6] in detail.

The main differences between (0.1) and (0.3) appear in the structure of the scattering matrix (S-matrix) (§2) and in the part of the inverse problem (§5). The S-matrix is the operator which relates the asymptotic behavior of solutions of eigenvalue problem $L_u f = \zeta f$ as $x \to -\infty$ to the asymptotic behavior as $x \to \infty$. As the suitable base of this eigenspace, we take Jost solutions. In the case of (0.3) the S-matrix is 2×2 type and has the form

$$
\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}
$$

with analogous properties as in the case of one-dimensional Schrödinger operator [2]. In our case the S-matrix is a 3×3 unitary matrix $(a_{\alpha\beta})_{1\leqq\alpha,\beta\leqq 3}$. The element $a_{11}(\zeta)$ is holomorhic in the upper half plane $\mathrm{Im}\,\zeta>0$ (ζ is the parameter of the eigenvalue problem $L_u f=\zeta f$), and $a_{\alpha\beta}(\zeta)$ (α, $\beta=2, 3$) are holomorphic in the lower half plane $\mathrm{Im}\,\zeta<0$. Let ζ_j ($j=1, \cdots, N$) be the zeros of $a_{11}(\zeta)$ in the upper half plane. The elements $a_{\alpha\beta}(\zeta)$ (α, $\beta=2, 3$) and their values at $\zeta_j{}^*$ play important roles in deriving the system of integral equations of the inverse problem. At $\zeta=\zeta_j$, $\zeta_j{}^*$, Jost solutions are linearly dependent and coefficients of the linear relation are related to $a_{\alpha\beta}(\zeta_j{}^*)$ (α, $\beta=2, 3$). These relations do not appear explicitly in [4]. Whole S-matrix is determined from its first row and pair of complex numbers associated to each zero ζ_j. Actual construction is much more complicated than the analogous one for the case of (0.3).

In solving the inverse problem, we meet a new problem which does not arise in the cases of one-dimensional Schrödinger operator and (0.3). In our formulation of the problem we must show certain relations among the solutions of the fundamental integral equations of the inverse problem. Though we have no complete proof, we can show that these relations hold for the case of reflectionless potentials which correspond to the S-matrix of the case $a_{12}(\xi)\equiv a_{13}(\xi)\equiv 0$.

In §§1 and 2 we treat the direct scattering theory. In §3 we reconstruct the S-matrix from the scattering data. In §4 we derive the fundamental integral equations of the inverse problem. In §5 we study the inverse problem.

Throughout the paper integrations are taken over $(-\infty, \infty)$ unless explicitly indicated. For a complex matrix M, $M^\dagger$ denotes its adjoint matrix *i.e.* $M^\dagger={}^tM^*$.

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1. Jost solutions

We consider on the whole real line the eigenvalue problem

$$(1.1)\qquad L_u f=\zeta f\,,\quad \zeta=\xi+i\eta\,,\quad \xi,\,\eta\in\boldsymbol{R}$$

where the operator L_u is defined by (0.1).

In case of $u_1=u_2=0$, the functions

$$\theta_1(x,\zeta)={}^t(1,0,0)\exp(-i\zeta x)$$
$$\theta_2(x,\zeta)={}^t(0,1,0)\exp(i\zeta x)$$
$$\theta_3(x,\zeta)={}^t(0,0,1)\exp(i\zeta x)$$

form a complete system of solutions of (1.1).

Put $v(x)=|u_1(x)|+|u_2(x)|$, and throughout the paper we assume that $v(x)\in L^1(-\infty,\infty)$. Then put

$$\sigma_\pm(x) = \pm\int_x^{\pm\infty} v(y)dy\,.$$

For a three-dimensional vector $y={}^t(y_1, y_2, y_3)$ we denote

$$|y| = \max(|y_1|, |y_2|, |y_3|)\,.$$

First we have

Theorem 1.1. *There exist unique solutions* ϕ_α, ψ_β $(\alpha, \beta=1, 2, 3)$ *of* (1.1) *such that*

$$\begin{aligned}
&|\phi_1(x, \zeta)-\theta_1(x, \zeta)| \leqq C\sigma_+(x)\exp(\eta x) && \eta\leqq 0 \\
&|\phi_\alpha(x, \zeta)-\theta_\alpha(x, \zeta)| \leqq C\sigma_+(x)\exp(-\eta x) && \eta\geqq 0 \quad \alpha = 2, 3
\end{aligned}$$

as $x\to\infty$ *and*

$$\begin{aligned}
&|\psi_1(x, \zeta)-\theta_1(x, \zeta)| \leqq C\sigma_-(x)\exp(\eta x) && \eta\geqq 0 \\
&|\psi_\alpha(x, \zeta)-\theta_\alpha(x, \zeta)| \leqq C\sigma_-(x)\exp(-\eta x) && \eta\leqq 0 \quad \alpha = 2, 3
\end{aligned}$$

as $x\to-\infty$. ψ_1, ϕ_2 *and* ϕ_3 *are holomorphic in* ζ *in the upper half plane and* ϕ_1, ψ_2 *and* ψ_3 *are holomorphic in* ζ *in the lower half plane.*

Proof. Substituting the expression

$$f(x) = \alpha(x)\theta_1(x, \zeta)+\beta(x)\theta_2(x, \zeta)+\gamma(x)\theta_3(x, \zeta)$$

into (1.1), we have

$$\begin{aligned}
\alpha'(x) &= (u_1(x)f_2(x)+u_2(x)f_3(x))\exp(i\zeta x) \\
\beta'(x) &= -u_1{}^*(x)f_1(x)\exp(-i\zeta x) \\
\gamma'(x) &= -u_2{}^*(x)f_1(x)\exp(-i\zeta x)\,.
\end{aligned}$$

From these relations we obtain the integral equations for ϕ_α:

(1.2) $$\phi_\alpha(x, \zeta)-\theta_\alpha(x, \zeta) = \begin{pmatrix} -\int_x^\infty (u_1(y)\phi_{\alpha 2}(y, \zeta)+u_2(y)\phi_{\alpha 3}(y, \zeta))\exp(i\zeta(y-x))dy \\ \int_x^\infty u_1{}^*(y)\phi_{\alpha 1}(y, \zeta)\exp(i\zeta(x-y))dy \\ \int_x^\infty u_2{}^*(y)\phi_{\alpha 1}(y, \zeta)\exp(i\zeta(x-y))dy \end{pmatrix}.$$

These integral equations can be solved by the method of successive approximation yielding the required estimates as in [1, p. 22]. Q.E.D.

We call the solutions ϕ_α, ψ_β Jost solutions of (1.1).

Put

$$\begin{aligned}
&\phi_1(x, \zeta) = \exp(-i\zeta x) k_1(x, \zeta) \\
&\phi_\alpha(x, \zeta) = \exp(i\zeta x) k_\alpha(x, \zeta) \qquad \alpha = 2, 3 \\
&\psi_1(x, \zeta) = \exp(-i\zeta x) h_1(x, \zeta) \\
&\psi_\alpha(x, \zeta) = \exp(i\zeta x) h_\alpha(x, \zeta) \qquad \alpha = 2, 3 .
\end{aligned} \tag{1.3}$$

For k_α, h_β, assume that the integral representations

$$\begin{aligned}
&k_1(x, \zeta) = {}^t(1, 0, 0) + \int_0^\infty K_1(x, y) \exp(-2i\zeta y)\, dy \\
&k_2(x, \zeta) = {}^t(0, 1, 0) + \int_0^\infty K_2(x, y) \exp(2i\zeta y)\, dy \\
&k_3(x, \zeta) = {}^t(0, 0, 1) + \int_0^\infty K_3(x, y) \exp(2i\zeta y)\, dy \\
&h_1(x, \zeta) = {}^t(1, 0, 0) + \int_{-\infty}^0 H_1(x, y) \exp(-2i\zeta y)\, dy \\
&h_2(x, \zeta) = {}^t(0, 1, 0) + \int_{-\infty}^0 H_2(x, y) \exp(2i\zeta y)\, dy \\
&h_3(x, \zeta) = {}^t(0, 0, 1) + \int_{-\infty}^0 H_3(x, y) \exp(2i\zeta y)\, dy
\end{aligned} \tag{1.4}$$

hold. Putting them into (1.2), we obtain the integral equations for K_α:

$$\begin{aligned}
&K_{11}(x, y) + \int_x^\infty (u_1(z) K_{12}(z, y) + u_2(z) K_{13}(z, y))\, dz = 0 \\
&K_{12}(x, y) - \int_x^{x+y} u_1^*(z) K_{11}(z, x+y-z)\, dz = u_1^*(x+y) \\
&K_{13}(x, y) - \int_x^{x+y} u_2^*(z) K_{11}(z, x+y-z)\, dz = u_2^*(x+y) \\
&K_{\alpha 1}(x, y) + \int_x^{x+y} (u_1(z) K_{\alpha 2}(z, x+y-z) + u_2(z) K_{\alpha 3}(z, x+y-z))\, dz \\
&= -u_{\alpha-1}(x+y) \\
&K_{\alpha 2}(x, y) - \int_x^\infty u_1^*(z) K_{\alpha 1}(z, y)\, dz = 0 \\
&K_{\alpha 3}(x, y) - \int_x^\infty u_2^*(z) K_{\alpha 1}(z, y)\, dz = 0 \qquad \alpha = 2, 3 .
\end{aligned}$$

Let $v(x)$ be in $L^{1,\infty} = L^1 \cap L^\infty$ (in the following we use this notation) and put

$$m_\pm(x) = \text{ess.}\sup_{\pm y \geq \pm x} v(y) .$$

Then the integral equations for kernels K_α can be solved by the method of successive approximation and the solutions are estimated as

$$|K_\alpha(x, y)| \leqq Cm_+(x+y) \exp(\sigma_+(x)) \qquad \alpha = 1, 2, 3 .$$

We also have the integral equations for kernels H_α and the estimates

$$|H_\alpha(x, y)| \leqq Cm_-(x+y) \exp(\sigma_-(x)) \qquad \alpha = 1, 2, 3$$

hold. Thus we have

Theorem 1.2. *Suppose that v is in $L^{1,\infty}$ and $m_\pm(\pm x)$ are in $L^{1,\infty}(a, \infty)$ for some a. Then Jost solutions have the integral representations* (1.3). *And the functions u_1 and u_2 are reconstructed by*

$$u_1(x) = K_{12}^*(x, 0) = -K_{21}(x, 0) = -H_{12}^*(x, 0) = H_{21}(x, 0)$$
$$u_2(x) = K_{13}^*(x, 0) = -K_{31}(x, 0) = -H_{13}^*(x, 0) = H_{31}(x, 0) .$$

2. Scattering matrix and scattering data

In this section we define the scattering matrix and the scattering data of the operator (0.1), and investigate the properties of them.

Let $y(x)$, $z(x)$ and $w(x)$ be solutions of (1.1), then as is well known the wronskian $|y, z, w| = \det(y, z, w)$ satisfies the differential equation $f' = i\zeta f$. From this fact and the asymptotic behavior of Jost solutions, we have

(2.1) $$|\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi)| = |\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi)| = \exp(i\xi x) .$$

Hence $\psi_\alpha(x, \xi)$ are expressed by means of $\phi_\alpha(x, \xi)$:

(2.2) $$\psi_\alpha(x, \xi) = \textstyle\sum_{\beta=1}^{3} a_{\alpha\beta}(\xi)\phi_\beta(x, \xi) \qquad \alpha = 1, 2, 3$$

i.e.

$$(\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi)) = (\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^t(a_{\alpha\beta}(\xi)) .$$

On the other hand, let $L_u y = \zeta_1 y$, $L_u z = \zeta_2 z$. Then we can show

$$(z'^\dagger y + z^\dagger y') + i(\zeta_1 - \zeta_2^*) z^\dagger I y = 0$$

where I is the matrix in (0.1). Therefore in case of $\zeta_1 = \zeta_2^*$ we have $(d/dx)(z^\dagger y) = 0$. Using this fact, we have

(2.3) $$\phi_\alpha^\dagger(x, \xi)\phi_\beta(x, \xi) = \psi_\alpha^\dagger(x, \xi)\psi_\beta(x, \xi) = \delta_{\alpha\beta}$$

(2.4) $$a_{\alpha\beta}(\xi) = \phi_\beta^\dagger(x, \xi)\psi_\alpha(x, \xi)$$

i.e.

(2.5) $${}^t(a_{\alpha\beta}(\xi)) = (\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^\dagger(\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi))$$

and

$$\sum_{\gamma=1}^{3} a_{\alpha\gamma}^{*}(\xi) a_{\beta\gamma}(\xi) = \delta_{\alpha\beta}\,.$$

These relations and (2.1) imply that the matrix $(a_{\alpha\beta}(\xi))$ is unitary and unimodular (*i.e.* det $(a_{\alpha\beta}(\xi))=1$), and the matrices $(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))$, $(\psi_1(x, \xi), \psi_2(x, \xi), \psi_3(x, \xi))$ are unitary ([4]).

Putting the integral representations for ϕ_α, ψ_β into (2.4), we obtain the integral representations for $a_{\alpha\beta}$:

$$\begin{aligned}
&a_{11}(\xi) = 1+\int_0^\infty A_{11}(y) \exp(2i\xi y)\,dy \\
&a_{\alpha\beta}(\xi) = \delta_{\alpha\beta}+\int_0^\infty A_{\alpha\beta}(y) \exp(-2i\xi y)\,dy \qquad \alpha, \beta = 2, 3 \\
&a_{1\alpha}(\xi) = \int A_{1\alpha}(y) \exp(-2i\xi y)\,dy \qquad \alpha = 2, 3 \\
&a_{\alpha 1}(\xi) = \int A_{\alpha 1}(y) \exp(2i\xi y)\,dy \qquad \alpha = 2, 3\,.
\end{aligned} \tag{2.6}$$

We write explicite formula only for A_{11}:

$$A_{11}(y) = H_{11}(x, -y)+K_{11}^{*}(x, y)+\sum_{\beta=1}^{3}\int_0^y K_{1\beta}^{*}(x, y-z) H_{1\beta}(x, -z)\,dz\,.$$

The functions $A_{\alpha\beta}$ are bounded integrable functions estimated as

$$\begin{aligned}
&|A_{11}(y)|,\ |A_{\alpha\beta}(y)| \leqq C(m_-(-2^{-1}y)+m_+(2^{-1}y)) \qquad y>0,\ \alpha, \beta = 2, 3 \\
&|A_{1\alpha}(y)|,\ |A_{\alpha 1}(y)| \leqq Cm_\pm(\pm y) \qquad \pm y>0 \quad \alpha = 2, 3\,.
\end{aligned}$$

The coefficient $a_{11}(\xi)$ is the boundary value of the holomorphic function in the upper half plane and $a_{\alpha\beta}(\xi)$ $(\alpha, \beta=2, 3)$ are the boundary values of the holomorphic functions in the lower half plane. These holomorphic functions are defined by

$$\begin{aligned}
&a_{11}(\zeta) = \phi_1{}^\dagger(x, \zeta^*)\psi_1(x, \zeta) \qquad \mathrm{Im}\,\zeta \geqq 0 \\
&a_{\alpha\beta}(\zeta) = \phi_\beta{}^\dagger(x, \zeta^*)\,\psi_\alpha(x, \zeta) \qquad \mathrm{Im}\,\zeta \leqq 0\,, \quad \alpha, \beta = 2, 3\,.
\end{aligned} \tag{2.7}$$

Next we derive the relations among Jost solutions. For this purpose we begin with following lemma, which is obtained by direct calculation.

Lemma 2.1. *Let $y(x)$ and $z(x)$ be solutions of* (1.1), *then the function $w(x)=\exp(i\zeta^* x)\,(y^*(x)\times z^*(x))$ satisfies the equation $L_u w=\zeta^* w$, where the expression $y\times z$ denotes the vector product of three-dimensional vectors y, z.*

For Jost solutions we have

Lemma 2.2. *The relations*

(2.8) $\phi_1(x, \zeta^*) = \exp(i\zeta^* x)(\phi_2^*(x, \zeta) \times \phi_3^*(x, \zeta))$
(2.9) $\psi_1(x, \zeta) = \exp(i\zeta x)(\psi_2^*(x, \zeta^*) \times \psi_3^*(x, \zeta^*))$ $\quad \operatorname{Im} \zeta \geqq 0$

hold.

Proof. As we have seen at the beginning of this section, the matrix $(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))$ is unitary *i.e.*

$$(\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^{-1} = (\phi_1(x, \xi), \phi_2(x, \xi), \phi_3(x, \xi))^\dagger .$$

Calculating the left-hand side according to the formula for the inverse matrix, we have (2.8) for real ξ:

$$\phi_1(x, \xi) = \exp(i\xi x)(\phi_2^*(x, \xi) \times \phi_2^*(x, \xi)) .$$

From this relation on the real line we obtain relations among integral kernels:

$$K_{11}(x, y) = \int_0^y (K_{22}^*(x, y-z) K_{33}^*(x, z) - K_{23}^*(x, y-z) K_{32}^*(x, z))\, dz$$
$$+ K_{22}^*(x, y) + K_{33}^*(x, y)$$

and similar relations for K_{12} and K_{13}. Using these relations, we have (2.8).

Q.E,D.

REMARK 1. In case of $u_2 \equiv 0$, we have

$$\phi_3(x, \zeta) = \psi_3(x, \zeta) = \theta_3(x, \zeta)$$

and

$$\exp(i\zeta^* x)(y^*(x) \times \theta_3^*(x, \zeta)) = {}^t(y_2^*(x), -y_1^*(x), 0) .$$

Thus the operation $\exp(i\zeta^* x)(y^*(x) \times \theta_3^*(x, \zeta))$ corresponds to the operation $\#$ in [6], [7].

From (2.7), (2.8) and (2.9) we obtain another expressions of $a_{11}(\zeta)$:

$$a_{11}(\zeta) = \exp(-i\zeta x) |\psi_1(x, \zeta), \phi_2(x, \zeta), \phi_3(x, \zeta)|$$
$$a_{11}^*(\zeta^*) = \exp(-i\zeta^* x) |\phi_1(x, \zeta^*), \psi_2(x, \zeta^*), \psi_3(x, \zeta^*)| \qquad \operatorname{Im} \zeta \geqq 0 .$$

Therefore if ζ_0 is a zero of $a_{11}(\zeta)$ in the upper half plane, ψ_1, ϕ_2, ϕ_3 are linearly dependent at ζ_0 and ϕ_1, ψ_2, ψ_3 are linearly dependent at ζ_0^*:

$$(2.10) \quad \begin{aligned} \psi_1(x, \zeta_0) &= c_1 \phi_2(x, \zeta_0) + c_2 \phi_3(x, \zeta_0) \\ \phi_1(x, \zeta_0^*) &= d_1 \psi_2(x, \zeta_0^*) + d_2 \psi_3(x, \zeta_0^*) . \end{aligned}$$

There exist relations among c_α and d_β, which we need later. The following theorem gives the relations.

Theorem 2.3. *The relations*

$$
(2.11)\qquad \begin{array}{ll} c_1{}^*d_1 = -a_{33}(\zeta_0{}^*) & c_1{}^*d_2 = a_{23}(\zeta_0{}^*) \\ c_2{}^*d_1 = a_{32}(\zeta_0{}^*) & c_2{}^*d_2 = -a_{22}(\zeta_0{}^*) \end{array}
$$

hold.

Proof. By lemma 2.2 and (2.10), we have

$$
(2.12)\qquad \exp(i\zeta_0 x)(\psi_2{}^*(x, \zeta_0{}^*)\times\psi_3{}^*(x, \zeta_0{}^*)) = c_1\phi_2(x, \zeta_0)+c_2\phi_3(x, \zeta_0)
$$

$$
(2.13)\qquad \exp(i\zeta_0{}^*x)(\phi_2{}^*(x, \zeta_0)\times\phi_3{}^*(x, \zeta_0)) = d_1\psi_2(x, \zeta_0{}^*)+d_2\psi_3(x, \zeta_0{}^*)\,.
$$

Taking the vector product of $\phi_2(x, \zeta_0)$ and (2.12), we have

$$
\phi_2{}^*(x, \zeta_0)\times(\psi_2(x, \zeta_0{}^*)\times\psi_3(x, \zeta_0{}^*)) = c_2{}^*\exp(i\zeta_0{}^*x)(\phi_2{}^*(x, \zeta_0)\times\phi_3{}^*(x, \zeta_0))\,.
$$

From this and (2.13), we have

$$
(2.14)\qquad \phi_2{}^*(x, \zeta_0)\times(\psi_2(x, \zeta_0{}^*)\times\psi_3(x, \zeta_0{}^*)) = c_2{}^*d_1\psi_2(x, \zeta_0{}^*)+c_2{}^*d_2\psi_3(x, \zeta_0{}^*).
$$

Using the identity for vector products

$$
y\times(z\times w) = ({}^t wy)z-({}^t yz)w
$$

and the relations (2.7), we conclude that the left-hand side of (2.14) is equal to

$$
a_{32}(\zeta_0{}^*)\psi_2(x, \zeta_0{}^*)-a_{22}(\zeta_0{}^*)\psi_3(x, \zeta_0{}^*)\,.
$$

Q.E.D.

We now assume that

$$
(2.15)\qquad a_{11}(\xi)\neq 0
$$

for any real ξ. As the integral representation (2.6) holds with integral kernel A_{11}, $a_{11}(\zeta)\to 1$ as $|\zeta|\to\infty$. By this property and (2.15) $a_{11}(\zeta)$ has only finite number of zeros in the upper half plane. We assume further that those zeros are simple. We denote them by ζ_j $(j=1, \cdots, N)$ and denote by $c_{j\alpha}$, $d_{j\alpha}$ $(j=1, \cdots, N, \alpha=1, 2)$ the number associated with ζ_j by the relations (2.10).

We define $r_{\pm\alpha}(\xi)$ $(\alpha=1, 2)$ by

$$
\begin{aligned} r_{+\alpha}(\xi) &= a_{11}(\xi)^{-1}a_{1,\alpha+1}(\xi) \\ r_{-\alpha}(\xi) &= a_{11}^*(\xi)^{-1}a_{\alpha+1,1}^*(\xi)\,. \end{aligned}
$$

We call the matrix $(a_{\alpha\beta}(\xi))$ the scattering matrix and the collection $\{r_{+\alpha}(\xi), \zeta_j, c_j(\alpha=1, 2, j=1,\cdots, N)\}$ (resp. $\{r_{-\alpha}(\xi), \zeta_j{}^*, d_{j\alpha}(\alpha=1, 2, j=1, \cdots, N)\}$) the right (resp. left) scattering data of the operator L_u or of the potential $u={}^t(u_1, u_2)$.

We note here that the functions $r_{\pm\alpha}$ have the expressions

$$
(2.16)\qquad r_{\pm\alpha}(\xi) = \int F_{\pm 0\alpha}(x)\exp(-2i\xi x)\,dx\,,\quad F_{\pm 0\alpha}\in L^{1,\infty}\,.
$$

For, by (2.15) and the Wiener's theorem [3, Theorem W], there exists a function $g\in L^1$ which satisfies

$$a_{11}(\xi)^{-1} = 1+\int g(x)\exp(2i\xi x)\,dx\,,$$

and in view of the fact $A_{11}\in L^{1,\infty}$, we can show that $g\in L^{1,\infty}$.

REMARK 2. In case of $u_2\equiv 0$ the scattering matrix turns out to be

$$\begin{bmatrix} a(\xi) & b(\xi) & 0 \\ -b^*(\xi) & a^*(\xi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $a(\xi)$ and $b(\xi)$ are the coefficients which appear in [6], [7].

REMARK 3. Our arguments in this section work for the operator

$$(2.17)\qquad L_u = i\begin{bmatrix} 1 & 0 \\ 0 & -I_n \end{bmatrix} D - i\begin{bmatrix} 0 & {}^tu \\ u^* & 0 \end{bmatrix},\quad u={}^t(u_1,\cdots,u_n)\,,$$

where I_n is the identity matrix of degree n, by suitable modifications.

3. Reconstruction of the scattering matrix from the scattering data

Let the collection $\{r_\alpha(\xi),\ \zeta_j,\ c_{j\alpha}\ (\alpha=1,2,j=1,\cdots,N)\}$ have the following properties: the functions $r_\alpha(\xi)$ have the expressions

$$r_\alpha(\xi) = \int R_\alpha(x)\exp(2i\xi x)\,dx \quad R_\alpha\in L^{1,\infty}\,,$$

ζ_j are complex numbers, which satisfy $\operatorname{Im}\zeta_j>0$, $\zeta_j\neq\zeta_k$ $(j\neq k)$, and $c_{j\alpha}$ are complex numbers, which satisfy $\prod_{j=1}^N(|c_{j1}|+|c_{j2}|)\neq 0$.

Our purpose in this section is to determine the unitary and unimodular matrix $(a_{\alpha\beta}(\xi))$ $(\alpha,\beta=1,2,3)$ and complex numbers $d_{j\alpha}$ from this collection so that they satisfy following requirements:

i) the elements $a_{\alpha\beta}$ have the integral representations (2.6) with bounded integrable kernels $A_{\alpha\beta}$,

ii) the numbers ζ_j are the zeros of $a_{11}(\zeta)$ in the upper half plane and the zeros ζ_j of $a_{11}(\zeta)$ are simple,

iii) the numbers $c_{j\alpha}$ and $d_{j\alpha}$ satisfy the relations

$$\begin{aligned} c_{j1}^*d_{j1} &= -a_{33}(\zeta_j^*) & c_{j1}^*d_{j2} &= a_{23}(\zeta_j^*) \\ c_{j2}^*d_{j1} &= a_{32}(\zeta_j^*) & c_{j2}^*d_{j2} &= -a_{22}(\zeta_j^*)\,, \end{aligned}$$

iv) the functions r_α are reconstructed by

$$r_\alpha(\xi)=a_{11}(\xi)^{-1}a_{1,\alpha+1}(\xi)\,.$$

At first by virtue of Wiener's theorem, there exists a function $g\in L^{1,\infty}$ which satisfies

$$1+\int g(x)\exp(2i\xi x)\,dx=(1+|r_1(\xi)|^2+|r_2(\xi)|^2)^{-1}\,.$$

We denote the left-hand side of the above relation by $f(\xi)$. Since the matrix-valued function

$$F(\xi)=\begin{bmatrix}1-f(\xi)|r_1(\xi)|^2 & -f(\xi)r_1{}^*(\xi)r_2(\xi)\\ -f(\xi)r_1(\xi)r_2{}^*(\xi) & 1-f(\xi)|r_2(\xi)|^2\end{bmatrix}$$

is positive definite, by the factorization theorem of positive definite matrix function [3, Theorem 8.2], there exists a matrix-valued function

$$G_1(\zeta)=\begin{bmatrix}1+\int_0^\infty g_{22}(x)\exp(-2i\zeta x)\,dx & \int_0^\infty g_{23}(x)\exp(-2i\zeta x)\,dx\\ \int_0^\infty g_{32}(x)\exp(-2i\zeta x)\,dx & 1+\int_0^\infty g_{33}(x)\exp(-2i\zeta x)\,dx\end{bmatrix}$$

where $g_{\alpha\beta}(x)$ (α, $\beta=2$, 3) belong to L^1, which satisfies

$$G_1{}^\dagger(\xi)G_1(\xi)=F(\xi)$$

and $\det G_1(\zeta)\neq 0$ for $\operatorname{Im}\zeta\leqq 0$. The matrix function $G_1(\zeta)$ is uniquely determined. By the proof of the theorem we quoted above and the fact g_1, R_1, $R_2\in L^{1,\infty}$, we can show that the function $g_{\alpha\beta}$ are in $L^{1,\infty}$.

Put

$$b_{11}(\zeta)=\Pi_{j=1}^N(\zeta-\zeta_j)(\zeta-\zeta_j{}^*)^{-1}\,,\quad b_j=\dot{b}_{11}(\zeta_j)^{-1}$$

where $\dot{f}=df/d\zeta$. We consider the system of linear equations for the unknowns $d_{j\alpha}$

(3.1) $$-c'^*_{k2}d_{k2}=1-\sum_{j=1}^N b_jc'_{j1}d^*_{j1}(\zeta^*_k-\zeta_j)^{-1}$$

(3.2) $$c'^*_{k1}d_{k2}=-\sum_{j=1}^N b_jc'_{j2}d^*_{j1}(\zeta^*_k-\zeta_j)^{-1}$$

(3.3) $$c'^*_{k2}d_{k1}=-\sum_{j=1}^N b_jc'_{j1}d^*_{j2}(\zeta^*_k-\zeta_j)^{-1}$$

(3.4) $$-c'^*_{k1}d_{k1}=1-\sum_{j=1}^N b_jc'_{j2}d^*_{j2}(\zeta^*_k-\zeta_j)^{-1}$$

where $c'_{j\alpha}$ are defined by

(3.5) $${}^t(c'_{j1},c'_{j2})=(\det G^*_1(\zeta^*_j))^{-1}G^*_1(\zeta^*_j)^t(c_{j1},c_{j2})\,,$$

or in matrix notation

(3.1)′ $\quad -C_2{}^*d_2 = J - ABC_1d_1{}^*$

(3.2)′ $\quad C_1{}^*d_2 = -ABC_2d_1{}^*$

(3.3)′ $\quad C_2{}^*d_1 = -ABC_1d_1{}^*$

(3.4)′ $\quad -C_1{}^*d_1 = J - ABC_2d_2{}^*$

where $A=((\zeta_j{}^*-\zeta_k)^{-1})$, $J={}^t(1, \cdots, 1)$ $d_\alpha={}^t(d_{1\alpha}, \cdots, d_{N\alpha})$ and B, C_α are diagonal matrices with the entries (b_j), $(c'_{j\alpha})$ respectively.

This system consists of $4N$-equations for $2N$-unknowns. However, by virtue of the relations $(AB)^{-1}=(AB)^*$ and $ABJ=-J$ (see [7, Appendix]), we conclude that (3.1)′ is equivalent to (3.4)′ and (3.2)′ is equivalent to (3.3)′. Thus this system reduces to $2N$-equations. Next by elimination, we have

$$(C_1A^*C_1{}^*+C_2A^*C_2{}^*)B^*d_\alpha = C_\alpha J \quad \alpha = 1, 2\,.$$

Since

$$i(C_1A^*C_1{}^*+C_2A^*C_2{}^*) = (i(c'_{j1}c'^*_{k1}+c'_{j2}c'^*_{k2})(\zeta_j-\zeta_k{}^*)^{-1})\,,$$

we have

$$-i((f_j, f_k))B^*d_\alpha = C_\alpha J$$

where $((f_j, f_k))$ denotes the Gram matrix of vector-valued functions f_j;

$$f_j(x) = {}^t(c'_{j1}, c'_{j2}) \exp(i\zeta_j x) \quad x \geqq 0\,.$$

Therefore under our assumptions the numbers $d_{j\alpha}$ are uniquely determined and satisfy the equations (3.1)—(3.4).

Let $d_{j\alpha}$ be determined as above, and put

$$\begin{aligned}
b_{22}(\zeta) &= 1-\textstyle\sum_{j=1}^N b_jc'_{j1}d^*_{j1}(\zeta-\zeta_j)^{-1}\\
b_{23}(\zeta) &= -\textstyle\sum_{j=1}^N b_jc'_{j2}d^*_{j1}(\zeta-\zeta_j)^{-1}\\
b_{32}(\zeta) &= -\textstyle\sum_{j=1}^N b_jc'_{j1}d^*_{j2}(\zeta-\zeta_j)^{-1}\\
b_{33}(\zeta) &= 1-\textstyle\sum_{j=1}^N b_jc'_{j2}d^*_{j2}(\zeta-\zeta_j)^{-1}\,.
\end{aligned}$$

Consider the function $f(\zeta)=b_{22}(\zeta)b_{33}(\zeta)-b_{23}(\zeta)b_{32}(\zeta)$. This function is a rational function of ζ, and by the relations (3.1)—(3.4) $f(\zeta)$ vanishes at $\zeta_j{}^*$ $(j=1, \cdots, N)$ and has poles of at most first order at ζ. Therefore the function $f(\zeta)b^*_{11}(\zeta^*)^{-1}$ is a bounded entire function which tends to 1 as $|\zeta|\to\infty$. Thus we have

(3.6) $\quad b^*_{11}(\zeta^*) = b_{22}(\zeta)b_{33}(\zeta)-b_{23}(\zeta)b_{32}(\zeta)\,.$

Using (3.1)—(3.4) and another expression for $b_{11}(\zeta)$

$$b_{11}(\zeta) = 1+\textstyle\sum_{j=1}^N b_j{}^*(\zeta-\zeta_j{}^*)^{-1}\,,$$

we have, by direct calculation,

$$(3.7)\qquad b_{22}(\zeta) = b_{11}^*(\zeta^*)b_{33}^*(\zeta^*) \quad b_{32}(\zeta) = -b_{11}^*(\zeta^*)b_{23}^*(\zeta^*)\,.$$

From (3.6) and (3.7) we conclude that the matrix

$$G_2(\xi) = (b_{\alpha\beta}(\xi)) \quad \alpha,\ \beta = 2,\ 3$$

is unitary.

Put $G(\zeta) = G_2(\zeta)G_1(\zeta)$, then the matrix-valued function $G(\zeta)$ is holomorphic in the lower half plane and by (3.5)

$$G(\zeta_j{}^*) = \begin{bmatrix} -c_{j2}^* d_{j2} & c_{j1}^* d_{j2} \\ c_{j2}^* d_{j1} & -c_{j1}^* d_{j1} \end{bmatrix}$$

holds.

Define $a_{\alpha\beta}(\zeta)$ (α, $\beta = 2$, 3) by

$$(a_{\alpha\beta}(\zeta)) = G(\zeta)\,.$$

By the argument of [3, p. 230], the functions $a_{\alpha\beta}(\zeta)$ (α, $\beta = 2$, 3) satisfy our requirements i), iii).

Next define $a_{11}(\zeta)$, $a_{12}(\xi)$, $a_{13}(\xi)$, $a_{21}(\xi)$ and $a_{31}(\xi)$ by the following:

$$\begin{aligned} a_{11}(\zeta) &= a_{22}^*(\zeta^*)a_{33}^*(\zeta^*) - a_{23}^*(\zeta^*)a_{32}^*(\xi^*) \\ a_{1\alpha}(\xi) &= a_{11}(\xi)r_{\alpha-1}(\xi) \quad \alpha = 2,\ 3 \\ a_{21}(\xi) &= a_{32}^*(\xi)a_{13}^*(\xi) - a_{12}^*(\xi)a_{33}^*(\xi) \\ a_{31}(\xi) &= a_{12}^*(\xi)a_{23}^*(\xi) - a_{22}^*(\xi)a_{13}^*(\xi)\,. \end{aligned}$$

It is easy to see that the matrix $(a_{\alpha\beta})$ which we have constructed satisfies i), ii), iii), iv) and (2.15).

REMARK. In the general case (2.17) the determination of $G(\zeta)$ is done by a quite analogous way. But on the determination of $d_{j\alpha}$ there appears a system of higher order algebraic equations. We do not know the solvability and the uniqueness of the solution of that system.

4. Fundamental integral equations

In this section we derive the system of integral equations which connects kernels K_α, H_α with the scattering data.

From (2.16) we have

$$F_{\pm 0\alpha}(x) = \pi^{-1}\int r_{\pm\alpha}(\xi)\exp(2i\xi x)\,d\xi \quad \alpha = 1,\ 2\,.$$

Now on the identity

$$\begin{aligned}&a_{11}(\xi)^{-1}h_1(x,\xi)-{}^t(1,0,0)\\&=(k_1(x,\xi)-{}^t(1,0,0))+r_{+1}(\xi)\exp(2i\xi x)k_2(x,\xi)\\&\quad+r_{+2}(\xi)\exp(2i\xi x)k_3(x,\xi)\end{aligned}$$

multiply $\pi^{-1}\exp(2i\xi y)$ and integrate over $(-\infty,\infty)$. The right-hand side gives

$$\begin{aligned}K_1(x,y)&+{}^t(0,F_{+01}(x+y),F_{+02}(x+y))\\&+\int_0^\infty F_{+01}(x+y+z)K_2(x,z)\,dz+\int_0^\infty F_{+02}(x+y+z)K_3(x,z)\,dz\,.\end{aligned}\tag{4.1}$$

For $y>0$ the left-hand side gives

$$\begin{aligned}&2i\textstyle\sum_{j=1}^N \dot{a}_{11}(\zeta_j)^{-1}h_1(x,\zeta_j)\exp(2i\zeta_j y)\\&=2i\textstyle\sum_{j=1}^N \dot{a}_{11}(\zeta_j)^{-1}(c_{j1}k_2(x,\zeta_j)+c_{j2}k_3(x,\zeta_j))\exp(2i\zeta_j(x+y))\,.\end{aligned}\tag{4.2}$$

Put

$$\begin{aligned}&f_{+j\alpha}(x)=-2i\dot{a}_{11}(\zeta_j)^{-1}c_{j\alpha}\exp(2i\zeta_j x)\quad j=1,\cdots,N,\quad \alpha=1,2\,,\\&F_{+\alpha}(x)=F_{+0\alpha}(x)+\textstyle\sum_{j=1}^N f_{+j\alpha}(x)\,.\end{aligned}$$

Putting these into the equality (4.1)=(4.2), we have

$$\begin{aligned}K_1(x,y)&+\int_0^\infty F_{+1}(x+y+z)K_2(x,z)\,dz+\int_0^\infty F_{+2}(x+y+z)K_3(x,z)\,dz\\&+{}^t(0,F_{+1}(x+y),F_{+2}(x+y))=0\,.\end{aligned}\tag{4.3}$$

Analogously

$$\begin{aligned}H_1(x,y)&+\int_{-\infty}^0 F_{-1}(x+y+z)H_2(x,z)\,dz+\int_{-\infty}^0 F_{-2}(x+y+z)H_3(x,z)\,dz\\&+{}^t(0,F_{-1}(x+y),F_{-2}(x+y))=0\end{aligned}$$

holds where

$$\begin{aligned}&F_{-\alpha}(x)=F_{-0\alpha}(x)+\textstyle\sum_{j=1}^N f_{-j\alpha}(x)\\&f_{-j\alpha}(x)=-2i\dot{a}_{11}^*(\zeta_j)^{-1}d_{j\alpha}\exp(2i\zeta_j^*x)\,.\end{aligned}$$

The derivation of (4.3) is similar to that in the case (0.3). The derivation of the following equation (4.5) does not appear in the case of (0.3).

We denote $(a_{\alpha\beta}(\xi))^{-1}=(b_{\alpha\beta}(\xi))$ $(\alpha,\beta=2,3)$. Since the scattering matrix $(a_{\alpha\beta}(\xi))$ $(\alpha,\beta=1,2,3)$ is unitary and unimodular, it follows that $\det(a_{\alpha\beta}(\xi);\ \alpha,\beta=2,3)=a_{11}^*(\xi)$. Accordingly the coefficients $b_{\alpha\beta}$ are the boundary values of the functions meromorphic in the lower half plane. These functions are defined by the relations

$$(4.4)\quad \begin{array}{ll} b_{22}(\zeta)=a_{11}^*(\zeta^*)^{-1}a_{33}(\zeta) & b_{23}(\zeta)=-a_{11}^*(\zeta^*)^{-1}a_{23}(\zeta) \\ b_{32}(\zeta)=-a_{11}^*(\zeta^*)^{-1}a_{32}(\zeta) & b_{33}(\zeta)=a_{11}^*(\zeta^*)^{-1}a_{22}(\zeta)\,. \end{array}$$

From the identities

$$\begin{aligned} h_\alpha(x,\xi)&=a_{\alpha1}(\xi)k_1(x,\xi)\exp(-2i\xi x)+a_{\alpha2}(\xi)k_2(x,\xi)\\ &\quad+a_{\alpha3}(\xi)k_3(x,\xi)\quad \alpha=2,3\,, \end{aligned}$$

by using the matrix $(b_{\alpha\beta}(\xi))$, we obtain

$$\begin{aligned} &b_{22}(\xi)h_2(x,\xi)+b_{23}(\xi)h_3(x,\xi)-{}^t(0,1,0)\\ &=k_2(x,\xi)-{}^t(0,1,0)-r_{+1}^*(\xi)k_1(x,\xi)\exp(-2i\xi x)\,. \end{aligned}$$

Multiply $\pi^{-1}\exp(-2i\xi y)$ on this identity and integrate over $(-\infty,\infty)$. Then the right-hand side gives

$$K_2(x,y)-\int_0^\infty F_{+01}^*(x+y+z)K_1(x,z)\,dz-{}^t(F_{+01}^*(x+y),0,0)\,.$$

By (4.4), for $y>0$ the left-hand side gives

$$2i\textstyle\sum_{j=1}^N \dot{a}_{11}^*(\zeta_j)^{-1}(a_{33}(\zeta_j^*)h_2(x,\zeta_j^*)-a_{23}(\zeta_j^*)h_3(x,\zeta_j^*))\exp(-2i\zeta_j^*y)$$

using the relations (2.11)

$$=2i\textstyle\sum_{j=1}^N \dot{a}_{11}^*(\zeta_j)^{-1}(-c_{j1}^*d_{j1}h_2(x,\zeta_j^*)-c_{j1}^*d_{j2}h_3(x,\zeta_j^*))\exp(-2i\zeta_j^*y)$$

by (2.10)

$$=-2i\textstyle\sum_{j=1}^N \dot{a}_{11}^*(\zeta_j)^{-1}c_{j1}^*k_1(x,\zeta_j^*)\exp(-2i\zeta_j^*(x+y))\,.$$

Thus we have

$$(4.5)\qquad K_2(x,y)-\int_0^\infty F_{+1}^*(x+y+z)K_1(x,z)\,dz-{}^t(F_{+1}^*(x+y),0,0)=0\,.$$

Analogously we have

$$K_3(x,y)-\int_0^\infty F_{+2}^*(x+y+z)K_1(x,z)\,dz-{}^t(F_{+2}^*(x+y),0,0)=0$$

$$H_2(x,y)-\int_{-\infty}^0 F_{-1}^*(x+y+z)H_1(x,z)\,dz-{}^t(F_{-1}^*(x+y),0,0)=0$$

$$H_3(x,y)-\int_{-\infty}^0 F_{-2}^*(x+y+z)H_1(x,z)\,dz-{}^t(F_{-2}^*(x+y),0,0)=0\,.$$

By the standard arguments as in [1], [5], we can show

$$|F_{+\alpha}(x)|\leqq C_+(x)m_+(x)\,,\quad |F_{-\alpha}(x)|\leqq C_-(x)m_-(x)\quad \alpha=1,2$$

where $C_\pm(x)$ are some non-increasing functions as $x\to\pm\infty$ respectively.

5. Inverse problem

In this section we study the inverse problem for the operator L_u.

Let $\{r_{\pm\alpha}(\xi), \zeta_j, c_{j\alpha}, (\alpha=1, 2, j=1, \cdots, N)\}$ be the right scattering data. We construct the scattering matrix according to the construction in §3. Then the functions

$$F_{\pm 0\alpha}(x) = \pi^{-1}\int r_{\pm\alpha}(\xi) \exp(2i\xi x) d\xi$$

belong to $L^{1,\infty}$. Here we assume that the functions $m_{\pm\alpha}(\pm x)=\sup_{\pm y\geq\pm x}|F_{\pm 0\alpha}(\pm y)|$ belong to $L^1(a, \infty)$ for some a.

The solvability of the system of integral equations in §4 for kernels $F_{\pm\alpha}$, the uniqueness of the solutions and the estimates for the solutions can be shown by an analogous way which has been stated in [6]. We denote its solutions by K_α, H_α and define k_α, h_α, ϕ_α, ψ_α by (1.4), (1.3). If for these solutions the equalities

$$\begin{aligned} &K_{12}^*(x, 0) = -K_{21}(x, 0) \quad K_{13}^*(x, 0) = -K_{31}(x, 0) \\ &-H_{12}^*(x, 0) = H_{21}(x, 0) \quad -H_{13}^*(x, 0) = H_{31}(x, 0) \end{aligned} \tag{5.1}$$

hold, we put

$$\begin{aligned} u_{+1}(x) &= K_{12}^*(x, 0) \qquad u_{+2}(x) = K_{13}^*(x, 0) \\ u_{-1}(x) &= -H_{12}^*(x, 0) \quad u_{-2}(x) = -H_{13}^*(x, 0)\,. \end{aligned}$$

In that case by analogous arguments as in [1], [5], we can show that ϕ_α, ψ_α satisfy (1.1) with $u=u_\pm={}^t(u_{\pm1}, u_{\pm2})$ respectively, and that under certain additional analytical conditions on $F_{\pm 0\alpha}$ $u_\pm(x)$ coincide.

At present we have no proof for (5.1). Here we show that for reflectionless case $(r_{+1}(\xi)=r_{+2}(\xi)=0)$ (5.1) can be verified. In that case as the kernels $F_{\pm\alpha}$ degenerate, the system of integral equations reduces to system of linear equations. We seek the solutions K_α in the form

$$\begin{aligned} K_1(x, y) &= \textstyle\sum_{l=1}^N f_{1l}(x) \exp(2i\zeta_l y) \\ K_j(x, y) &= \textstyle\sum_{l=1}^N f_{jl}(x) \exp(-2i\zeta_l^* y) \quad j = 2, 3 \end{aligned}$$

where

$$f_{jl}(x) = {}^t(f_{jl}^{(1)}(x)\,,\ f_{jl}^{(2)}(x)\,,\ f_{jl}^{(3)}(x)) \quad j = 1, 2, 3\,,\quad l = 1, \cdots, N\,.$$

Put

$$g_{jk}(x) = {}^t(f_{j1}^{(k)}(x), \cdots, f_{jN}^{(k)}(x)) \quad j, k = 1, 2, 3\,.$$

Then the system of integral equations takes the following form, which is the system of linear equations for unknowns $f_{jl}^{(k)}$:

$$g_{11}+BC_1EA^*g_{21}+BC_2EA^*g_{31}=0$$
$$g_{12}+BC_1EA^*g_{22}+BC_2EA^*g_{32}-2iBC_1EJ=0$$
$$g_{13}+BC_1EA^*g_{23}+BC_2EA^*g_{33}-2iBC_2EJ=0$$
$$g_{j1}-B^*C^*_{j-1}E^*Ag_{11}-2iB^*C^*_{j-1}E^*J=0$$
$$g_{j2}-B^*C^*_{j-1}E^*Ag_{12}=0$$
$$g_{j3}-B^*C^*_{j-1}E^*Ag_{13}=0 \quad j=2,3$$

where A, B are defined in §3 and C_j, E are the diagonal matrices with the entries (c_{lj}), $(\exp(2i\zeta_l x))$ respectively.

From above equations we obtain equations for g_{12}, g_{13}, g_{21}, g_{31}

$$D_1^*g_{1j}=2iBC_{j-1}EJ \quad j=2,3$$
$$D_2\begin{bmatrix} g_{21} \\ g_{31}\end{bmatrix}=2i\begin{bmatrix} B^*C_1^*E^*J \\ B^*C_2^*E^*J\end{bmatrix}$$

where

$$D_1=I_N+E^*C_1^*B^*AEC_1BA^*+E^*C_2^*B^*AEC_2BA^*$$
$$D_2=\begin{bmatrix} I_N+E^*C_1^*B^*AEC_1BA^* & E^*C_1^*B^*AEC_2BA^* \\ E^*C_2^*B^*AEC_1BA^* & I_N+E^*C_2^*B^*AEC_2BA^*\end{bmatrix}$$

(I_N is the identity matrix of degree N).

We calculate the determinants and the cofactors of matrices D_1 and D_2, by using the formula

$$\det(\delta_{jk}+x_{jk})=1+\sum_{l=1}^{m}X_l \quad j,k=1,\cdots,m$$

where X_l is the sum of the principal minors of matrix (x_{jk}) of degree l and the modification of this formula. We denote the (α,β)-cofactors of the matrices D_j by $d^{(j)}_{\alpha\beta}$ respectively. Then the expressions

$$K^*_{1j}(x,0)=-2i(\det D_1)^{-1}\sum_{\alpha,\beta=1}^{N}d^{(1)}_{\alpha\beta}c^*_{\alpha,j-1}b_\alpha^*\exp(-2i\zeta_\alpha^*x)$$
$$K_{21}(x,0)=2i(\det D_2)^{-1}\sum_{\alpha,\beta=1}^{N}(d^{(2)}_{\alpha\beta}c^*_{\alpha1}+d^{(2)}_{\alpha+N,\beta}c^*_{\alpha2})b_\alpha^*\exp(-2i\zeta_\alpha^*x)$$
$$K_{31}(x,0)=2i(\det D_2)^{-1}\sum_{\alpha,\beta=1}^{N}(d^{(2)}_{\alpha,\beta+N}c^*_{\alpha1}+d^{(2)}_{\alpha+N,\alpha+N}c^*_{\beta2})b_\alpha^*\exp(-2i\zeta_\alpha^*x)$$

hold.

After direct calculation, we have

$$\begin{aligned}\det D_1&=\det D_2\\&=\sum_{0\leq j\leq N,\ 1\leq k_1<\cdots<k_j\leq N,\ 1\leq l_1<\cdots<l_j\leq N}b^*_{k_1}\cdots b^*_{k_j}b_{l_1}\cdots b_{l_j}\times\\&\quad\times\exp(2i(\zeta_{l_1}+\cdots+\zeta_{l_j}-\zeta^*_{k_1}-\cdots-\zeta^*_{k_j})x)\times\\&\quad\times\det((\zeta_r-\zeta_s^*)^{-1})\det((c^*_{s,1}c_{r,1}+c^*_{s,2}c_{r,2})(\zeta_s^*-\zeta_r)^{-1})\end{aligned}$$

where $r=l_1, \cdots, l_j$, $s=k_1, \cdots, k_j$.

And for cofactors we obtain the relations

$$
\begin{aligned}
&d^{(1)}_{\alpha\beta}c^*_{\alpha 1}b_\alpha{}^* \exp(-2i\zeta_\alpha{}^*x) = (d^{(2)}_{\beta\alpha}c^*_{\beta 1}+d^{(2)}_{\beta+N,\alpha}c^*_{\beta 2})b_\beta{}^* \exp(-2i\zeta_\beta{}^*x) \\
&d^{(1)}_{\alpha\beta}c^*_{\beta 2}b_\alpha{}^* \exp(-2i\zeta_\alpha{}^*x) = (d^{(2)}_{\beta,\alpha+N}c^*_{\beta 1}+d^{(2)}_{\beta+N,\alpha+N}c^*_{\beta 2})b_\beta{}^* \exp(-2i\zeta_\beta x) \\
&\alpha, \beta = 1, \cdots, N.
\end{aligned}
$$

Thus we have (5.1).

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