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THE COMPLEX BORDISM OF CYCLIC GROUPS

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Introduction. In their book, *Differentiable Periodic Maps* [2], P.E. Conner and E.E. Floyd initiated the study of cobordism groups of periodic maps and succeeded in determining the additive structure of the cobordism groups of free orientation-preserving $\boldsymbol{Z}_p$-actions on manifolds for odd primes p and of free $\boldsymbol{Z}_p$-actions preserving a stably almost-complex structure for arbitrary primes by calculating $MSO_*(B\boldsymbol{Z}_p)$ and $MU_*(B\boldsymbol{Z}_p)$ respectively. Kamata [5] obtained the same results for $MU_*(B\boldsymbol{Z}_p)$ using slightly different methods. We extend these results to a determination of $MU_*(BG)$ where G is an arbitrary cyclic group. The main result is Proposition 16:

$$MU_{2n+1}(B\boldsymbol{Z}_{p^s}) \cong \sum_{a=1}^{s} \sum_{b=p^{a-1}-1}^{n} \frac{\Gamma_{2(n-b)}(p^a)}{p^{\left[\frac{b-p^{a-1}+1}{p^{a-1}(p-1)}\right]+s-a+1}\,\Gamma_{2(n-b)}(p^a)}$$

where $\Gamma_*(p^a) \simeq MU_*/\langle CP(p-1)^{p^{a-1}}\rangle$ and the square brackets indicate the greatest integer function. We show this by constructing an explicit set of generators coming from the K-theory of the generalized lens spaces $L^n(p^s)$ and computing the order of the group they generate.

I would like to thank the referee for catching several embarrassing errors and suggesting ways of correcting them.

Results. We will have need of several homology and cohomology theories. Following J.F. Adams, let H be the Eilenberg-MacLane spectrum for the integers, K the BU spectrum, and MU the Thom spectrum for the unitary group. The resulting homology theories are denoted by $H_*(\ )$, $K_*(\ )$, and $MU_*(\ )$, and similarly in the case of cohomology theories. When we have need of unreduced theories, we write X^+ for the disjoint union of X and a basepoint, so that $H_*(X^+)$, for example, is ordinary, unreduced, integral homology. In dealing with K-theory, we will be exclusively concerned with $K^0(X)$ which we agree to write as $K(X)$, remembering that this is the reduced group, i.e., what is usually written

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as $\tilde{K}(X)$.

The following description of $MU_*(X)$ will be very convenient. Consider the set of all continuous maps $f\colon M^m \to X$ where M is a stably almost-complex manifold. Two such maps f_1 and f_2 are said to be equivalent if there is a stably almost-complex $(m+1)$-manifold W^{m+1} and a map $f\colon W^{m+1} \to X$ such that the boundary of W is the disjoint union of M_1 and M_2 and f restricted to the boundary of W is the disjoint union of $-f_1$ and f_2. Impose an addition on the set of resulting equivalence classes by the disjoint union of maps. It is a standard result that the resulting graded group is isomorphic to $MU_*(X^+)$.

Recall that the ring of coefficients MU_* is a polynomial ring over the integers on countably many generators, one in each positive, even dimension. There are many ways of choosing such generators, but it is convenient to have a standard set to work with. Following M. Hazewinkel [4] we proceed as follows:

Suppose S is a natural number. An ordered factorization of S is an ordered set $(q_1, \cdots, q_t, d)$ of natural numbers where each q_i is a positive power of a prime and d is not a power of a prime and $S = q_1 \cdots q_t d$. For example, the ordered factorizations of 12 are: (12), (2, 6), (4, 3, 1), (3, 4, 1), (2, 2, 3, 1), (2, 3, 2, 1), and (3, 2, 2, 1).

Associate to each ordered factorization $(q_1, \cdots, q_t, d)$ a positive integer $b(q_1, \cdots, q_t, d)$ as follows:

1) $b(q_1, \cdots, q_t, d) = b(q_1, \cdots, q_t)$
2) If $q_i = p_i^{k_i}$, then $b(q_1, \cdots, q_t) = b(p_1, \cdots, p_t)$
3) $b(p) = 1$ and $b(d) = 1$
4) If $S = (p_1, \cdots, p_t)$, then

$$b(p_1, \cdots, p_t) = \{\prod_{p \in S} c(p, p_t)\} b(p_1, \cdots, p_{t-1})$$

$$\text{where } c(p,q) = \begin{cases} 1 & \text{if } p = q \\ q^{p-1} & \text{if } p \neq q\,. \end{cases}$$

This suffices to give an inductive definition of $b(q_1, \cdots, q_t, d)$. For example: $b(2, 3, 2, 1) = b(2, 3, 2) = 12$.

Proposition 1 (Hazewinkel): *There exist elements $v_i \in MU_{2i}$ such that*

1) $MU_* = \boldsymbol{Z}[v_1, v_2, \cdots]$
2) *If we set $V_i = v_{i-1}$, then, in $MU_* \otimes Q$,*

$$\frac{[CP(s-1)]}{s} = \sum_{\{(q_1, \cdots, q_t, d)\}} \frac{b(q_1, \cdots, q_t)}{p_1 \cdots p_t} V_{q_1} V_{q_2}^{r_1} \cdots V_{q_t}^{r_{t-1}} V_d^{r_t}$$

where q_i is a power of p_i, $r_i = q_1 \cdots q_i$, and the sum is taken over all ordered factorizations of s.

Notation. For a fixed prime p, let

$$d(s)=p^{s-1}+\cdots+1\,.$$

DEFINITION. By $\Gamma_*(p^s)$ we mean $MU_*/\langle v_{p-1}^{d(s)-d(s-1)}\rangle$. This definition bears a few words of explanation. $\Gamma_*(p^s)$ as defined is a graded ring. We are interested, however, only in its structure as a graded abelian group. With this in mind, we will often write $\Gamma_*(p)\subseteq\Gamma_*(p^2)\subseteq\cdots\subseteq MU_*$ even though the inclusion is not true for the rings in question, only the groups. Each $\Gamma_*(p^s)$ is, of course, a graded, free abelian group with a rather complicated number of generators in each dimension.

Proposition 2. *$MU_{2n}(B\boldsymbol{Z}_{p^s})=0$ and $MU_{2n+1}(B\boldsymbol{Z}_{p^s})$ is a finite abelian group of order $p^{s(n)}$, where $s(n)=s\sum_{j=0}^{n}\pi(j)$ and $\pi(n)$ is the number of partitions of n.*

Proof. Consider the Atiyah-Hirzebruch spectral sequence, henceforth denoted AHSS.

$$E^2_{r,q}=H_r(BZ_{p^s};MU_q)=\begin{cases}0 & \text{if } q \text{ is odd or } r \text{ is even}\\ (\boldsymbol{Z}_{p^s})^{\pi(q/2)} & \text{otherwise.}\end{cases}$$

For purely dimensional reasons there can be no non-zero differentials, so the spectral sequence collapses and $E^2=E^\infty$. There is a filtration $MU_t(B\boldsymbol{Z}_{p^s})=F_t\supseteq\cdots\supseteq F_0\supseteq F_{-1}=0$ such that $F_q/F_{q-1}=E^\infty_{q,t-q}$. If t is even, then $E^\infty_{q,t-q}=0$ $\forall q$. Therefore $MU_t(B\boldsymbol{Z}_{p^s})=0$. If t is odd, $E^\infty_{t-q,q}$ is zero for q odd and has order $p^{s\pi(q/2)}$ for q even. Q.E.D.

In order to get the precise structure of the odd dimensional groups, we need some information from K-theory.

There is a natural inclusion $\boldsymbol{Z}_{p^s}\to S^1$ given by $1\mapsto\exp(2\pi i/p^s)$ so that the standard free action of S^1 on S^{2n+1} induces a free action of $\boldsymbol{Z}_{p^s}$ on S^{2n+1}. Denote the resulting $(2n+1)$-dimensional quotient manifold by $L^n(p^s)$, the $(2n+1)$-dimensional lens space. We then have the tower of fibrations.

$$\begin{tikzcd} \boldsymbol{Z}_{p^s} \arrow[r] & S^{2n+1} \arrow[d] \\ S^1=S^1/\boldsymbol{Z}_{p^s} \arrow[r] & L^n(p^s) \arrow[d, "\pi"] \\ & CP(n) \end{tikzcd}$$

Let ξ_n be the canonical line bundle over $CP(n)$, $\eta_n=\pi^*(\xi_n)$ and $(\eta_n-[1])=x\in K(L^n(p^s))$.

Proposition 3: $K(L^n(p^s))=\dfrac{\boldsymbol{Z}[x]}{((1+x)p^s-1,\ x^{n+1})}$.

For the proof, see Atiyah [1], p. 105.

DEFINITION. For a given prime p, let $m(j, s)=p^{\left[\frac{j}{p^{s-1}(p-1)}\right]+1}$, where the square brackets indicate the greatest integer function.

Proposition 4. *Consider $K(L^n(p^s))$. For every $j\geq p^{s-1}$, there is a sequence of integers $\{b_i\}$ such that*

$$m(n-j, s)x^j = pm(n-j, s)\{\sum_{i<j} b_i x^i\}$$

Proof. The proof is by induction on n and j for a fixed s.

The theorem is trivial for $n<p^{s-1}$, since in this case $x^j=0$. Assume the theorem is true for $n-1\geq p^{s-1}-1$ and write

$$K(L^{n-1}(p^s)) = \frac{\boldsymbol{Z}[y]}{((1+y)^{p^s}-1, y^n)} .$$

Mapping $y^i \mapsto x^{i+1}$ induces a group homomorphism $g\colon K(L^{n-1}(p^s))\to K(L^n(p^s))$. By induction, $m(n-1-j, s)y^j=pm(n-1-j, s)\sum_{i<j} b_i y_i, j\geq p^{s-1}$. Applying g to this equality, we obtain $m(n-(j+1), s)x^{j+1}=pm(n-(j+1), s)\sum b_i x^{i+1}$. Thus the theorem is true for n as long as $j<p^{s-1}$.

Suppose then that $j=p^{s-1}$. We know that $\sum_{i=1}^{p^s}\binom{p^s}{i}x^i=0$. For $1\leq j\leq p^s-1$, $\binom{p^s}{j}$ is divisible by p and for $p<j<p^{s-1}$, $\binom{p^s}{j}$ is divisible by p^2. Therefore $m(n-p^s, s)\binom{p^s}{j}=km(n-p^{s-1}, s)=\bar{k}\, m(n-i, s)$, for $i\geq p^{s-1}$. Multiply the above sum by $m(n-p^s, s)$. Then

$$0 = pm(n-p^{s-1}, s)\sum_{i=1}^{p^{s-1}-1} k_i x^i+km(n-p^{s-1}, s)x^{p^{s-1}}+\sum_{i=p^{s-1}+1}^{p^s} k_i m(n-i, s)x^i$$

where $k\not\equiv 0 \pmod p$.

Now $\sum_{i=p^{s-1}+1}^{p^s} k_i m(n-i, s)x^i =$

$$= p\sum_{i<p^{s-1}} \bar{k}_i m(n-p^{s-1}, s)x^i+p\bar{k}m(n-p^{s-1}, s)x^{p^{s-1}}+p\sum_{i=p^{s-1}+1}^{p^s-1} \bar{k}_i m(n-i, s)x^i$$

Therefore

$$0 = pm(n-p^{s-1}, s)\sum_{i<p^{s-1}} h_i x^i+(k+p\bar{k})m(n-p^{s-1}, s)x^{p^{s-1}}+\cdots$$
$$\cdots+\sum_{i=p^{s-1}+1}^{p^s-1} h_i m(n-i, s)x^i$$

Repeating this process as long as there are terms x^i for which $i>p^{s-1}$, we obtain

$$m(n-p^{s-1}, s)(k+pb)x^{p^{s-1}} = pm(n-p^{s-1}, s)\sum_{i<p^{s-1}} \bar{k}_i x^i .$$

Since $(k+pb)$ is a unit mod p, this implies that

$$m(n-p^{s-1}, s)x^{p^{s-1}} = pm(n-p^{s-1}, s) \sum_{i<p^{s-1}} b_i x^i \text{ as claimed.} \qquad \text{Q.E.D.}$$

Corollary. *In $K(L^n(p^s))$ the order of the element $x^{p^{s-1}}-p \sum_{i<p^{s-1}} b_i x^i$ is less than or equal to $m(n-p^{s-1}, s)$.*

Suppose $\mathfrak{H}$ is any complex, n-plane bundle over a space X. The map cf_1: $K(X)\to MU^2(X)$ which associates to $[\mathfrak{H}]$-n the first cobordism chern class of $\mathfrak{H}$ is clearly an homomorphism and was shown by Conner and Floyd [3] to be the injection of a direct summand.

If the space X is an n-dimensional manifold which is MU-orientable and, in particular, if X is a U-manifold such as $L^n(p^s)$, then there is a Poincaré duality isomorphism

$$D\colon MU^k(X) \to MU_{n-k}(X) .$$

DEFINITION. By $X(k, s)\in MU_{2k+1}(B\boldsymbol{Z}_{p^s})$ we mean the bordism element represented by the inclusion i: $L^k(p^s)\to B\boldsymbol{Z}_{p^s}$ of the $(2k+1)$-skeleton. When the context is clear, we will write $X(k)$ for $X(k, s)$.

Proposition 5. $i_*(D(cf_1(x)^k))=X(n-k)$.

Proof. This is Proposition 1.3 of [5].

In order to use the above information, it is necessary to understand some elementary results from the theory of formal groups which we now review.

DEFINITION. Suppose R is a commutative ring with unit. By a formal group over R, we mean a formal power series $F(X_1, X_2)= \sum_{i,j\geqq 0} a_{ij}X_1^iX_2^j$, $a_{ij}\in R$ which satisfies

1) $F(X_1, 0) = X_1$ and $F(0, X_2) = X_2$
2) $F(X_1, F(X_2, X_3)) = F(F(X_1, X_2), X_3)$

We are interested in the following formal group over MU^*. Recall that $MU^*(BS^1)\cong MU^*[[X]]$ and $MU^*(BS^1\times BS^1)\cong MU^*[[X_1, X_2]]$, the rings of formal power series in one and two variables respectively. The multiplication m: $S^1\times S^1\to S^1$ in the group S^1 induces a map Bm: $BS^1\times BS^1\to BS^1$ which classifies the tensor product of line bundles. That is, if π_1, π_2: $BS^1\times BS^1\to BS^1$ are the projections and ξ_1 and ξ_2, the respective pullbacks of the universal line bundle ξ over BS^1, then $Bm^*(\xi)=\xi_1\otimes\xi_2$. The standard result is that $Bm^*(X)$ is a formal group over MU^*, being, in fact, a universal object for formal groups over an arbitrary (commutative) ring. We define elements $a_{ik}\in MU^*$ by setting $Bm^*(X)=X_1+X_2+\sum a_{ij}X_1^iX_2^j=F(X_1, X_2)$.

If $\mathfrak{H}$ is a line bundle, write $\mathfrak{H}^2$ for the tensor product of $\mathfrak{H}$ with itself. then

$\mathfrak{H}^2$ is classified by the map $Bm\cdot\Delta\colon BS^1\to BS^1\times BS^1\to BS^1$, where Δ is the diagonal map. Since $X=cf_1(\xi)$, $cf_1(\xi^2)=F(X, X)$ by naturality.

DEFINITION. Let $[k]X\in MU^*(BS^1)$ be defined inductively as follows:

1) $[1]X=X$
2) $[k]X=F(X, [k-1]X)$.

This definition is rigged, of course, to give us the result we really want, namely, $cf_1(\xi^k)=[k]X$.

Notation. We will write

$$[k]X = a(0, k)X+a(1, k)X^2+\cdots+a(m, k)X^{m+1}+\cdots$$

with $a(m, k)\in MU^*=MU_{-*}$.

In general, it is somewhat difficult to give an explicit description of the $a(m,k)$ as bordism classes of familiar manifolds. There is, however, the following result.

Proposition 6. *Given a prime p, the ideal in MU_* generated by $\{a(m,p)\}$ is the ideal of all manifolds whose chern numbers are all divisible by p. This ideal is in fact generated by $\{a(p^i-1, p)\}$ $i=0, 1, \cdots$*

Proof. See [2], Proposition 41.1.

Proposition 7. *$a(p^s-1, p^s)=cv_{p-1}^{a(s)}+py$ where $c\not\equiv 0 \pmod p$ and $y\in MU_*$. If $j<p^s-1$, $a(j, p^s)$ is divisible by p.*

Proof. The proof is by induction on s. The case $s=1$ is the above mentioned result of Conner and Floyd.

Assume by induction that $a(p^{s-1}-1, p^{s-1})=c_1v_{p-1}^{a(s-1)}+py_1$ and that for $j<p^{s-1}-1$, $a(j, p^{s-1})$ is divisible by p. Now,

$$[p^s]X=[p]([p^{s-1}]X)=\sum_{k\geq 0}a(k, p)\{[p^{s-1}]X\}^{k+1}.$$

Therefore

$$a(p^s-1, p^s)=\sum_{k\geq 0}\sum_{(i_0,\cdots,i_k)}a(k, p)a(i_0, p^{s-1})\cdots a(i_k, p^{s-1}).$$
$$\text{where}\qquad i_0+\cdots+i_j=p^s-1-k.$$

Suppose $k<p-1$. Then $a(k, p)$ is divisible by p. Similarly, if $i_j<p^{s-1}-1$, then $a(i_j, p^{s-1})$ is divisible by p.

If $k\geq p-1$ and $i_j\geq p^{s-1}-1$ for all j, then, since $k+i_0+\cdots+i_k=p^s-1$, $k=p-1$ and $i_j=p^{s-1}-1$ for all j. But $a(p-1, p)a(p^{s-1}-1, p^{s-1})^p=$ $=(c_0v_{p-1}+py_0)(c_1v_{p-1}^{a(s-1)}+py_1)^p=c_0c_1v_{p-1}^{a(s)}+py$ and $c_0c_1\not\equiv 0 \pmod p$. Q.E.D.

Proposition 8. *For each integer* $n \geq p^{s-1}-1$, *there is an element* $Y(n, s) \in MU_{2n+1}(BZ_{p^s})$ *which satisfies*

1) $Y(n, s) = v_{p-1}^{a(s-1)} X(n-p^{s-1}+1) + \sum_k w_{k_s, s} X(n-k)$
   where $w_{k,s} \in MU_*/\langle v_{p-1}^{a(s-1)} \rangle = N_*$
2) $m(n-p^{s-1}+1, s) Y(n, s) = 0$.

Proof. Induction on n. We showed that in $K(L^n(p^s))$,

$$m(n-p^{s-1}, s) x^{p^{s-1}} = pm(n-p^{s-1}, s) \sum_{j<p^{s-1}} b_j x^j .$$

Recall that $x = \eta_n - 1$ and $x^k = \sum_{i=0}^{k} (-1)^i \binom{k}{i} \eta_n^i$.
Apply the map cf_1, yielding

$$m(n-p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} [i](cf_1(x))$$
$$= pm(n-p^{s-1}, s) \sum_{j<p^{s-1}} b_j \left\{ \sum_{i'=1}^{j} (-1)^{i'} \binom{j}{i'} [i'](cf_1(x)) \right\}.$$

Equivalently, applying $i_* \circ D$,

$$m(n-p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} \left\{ \sum_{k=0}^{n-1} a(k, i) X(n-k-1) \right\}$$
$$= pm(n-p^{s-1}, s) \sum_{j<p^{s-1}} b_j \left\{ \sum_{i'=1}^{j} (-1)^{i'} \binom{j}{i'} \left\{ \sum_{k'=0}^{n-1} a(k', i') X(n-k'-1) \right\} \right\}.$$

Note that for $* < 2(p^{s-1}-1)$ $MU_* = N_*$ and we see that

$$m(n-p^{s-1}, s) \sum_{i=1}^{p^{s-1}} (-1)^i \binom{p^{s-1}}{i} \left\{ \sum_{k \geq p^{s-1}-1}^{n-1} a(k, i) X(n-k-1) \right\}$$
$$-pm(n-p^{s-1}, s) \sum_{j<p^{s-1}} b_j \left\{ \sum_{i'=1}^{j} (-1)^{i'} \binom{j}{i'} \right\} \left\{ \sum_{k' \geq p^{s-1}-1}^{n-1} a(k', i') X(n-k'-1) \right\}$$

has the form $m(n-p^{s-1}, s) \sum w_{k,s} X(n-k)$, $w_{k,s} \in N_*$.

Now suppose that $k \geq p^{t-1}$. If we expand the $a(k, i)$ in terms of our chosen basis, we will get sums of monomials in the v_i. If a monomial contains no factor $v_{p-1}^{a(s-1)}$, then the product of that monomial and $X(n-k-1)$ has the required form. Suppose that the monomial has the form $\beta\, v_{p-1}^{a(s-1)} X(n-k-1)$. Since $k \geq p^{q-1}$, the degree of $X(n-k-1)$ is strictly less than that of $X(n-p^{s-1})$. Therefore, by induction,

$$m(n-p^{s-1}, s) \beta v_{p-1}^{a(s-1)} X(n-k-1) = m(n-p^{s-1}, s) \beta \sum_k w_{k,s} X(n-k)$$

with $w_{k,s} \in N_*$.
Repeating the induction if necessary, we have that

$$m(n-p^{s-1}, s)\sum_{i=1}^{p^{s-1}}(-1)^i\binom{p^{s-1}}{i}\left\{a(p^{s-1}-1, i)X(n-p^{s-1})\right\}$$
$$-pm(n-p^{s-1}, s)\sum_{j<p^{s-1}}b_j\left\{\sum_{i'=1}^{j}(-1)^{i'}\binom{j}{i'}a(p^{s-1}-1, i')X(n-p^{s-1})\right\}$$

has the form $m(n-p^{s-1}, s)\sum w_{k,s}X(n-k)$, $w_{k,s}\in N_*$. Utilizing Proposition 7, since for $i<p^{s-1}$, $\binom{p^{s-1}}{i}$ is divisible by p, we have that

$$m(n-p^{s-1}, s)\{(c+pd)v_{p-1}^{a(s-1)}+w\}X(n-p^{s-1})$$
$$-pm(n-p^{s-1}, s)\sum_{j<p^{s-1}}\bar{b}_j a(p^{s-1}-1, j)X(n-p^{s-1}),\ c\not\equiv 0 \bmod p\,,$$

has the same form. Expanding the $\bar{b}_j a(p^{s-1}-1, j)$ in terms of our chosen basis as $\bar{b}_j a(p^{s-1}-1, j)=-c_j v_{p-1}^{a(s-1)}+\cdots$, we see that

$$m(n-p^{s-1}, s)(c+pd+p\sum_{j<p^{s-1}}c_i)v_{p-1}^{a(s-1)}X(n-p^{s-1})$$

has the same form. But $c+p(d+\sum_{i<p^{s-1}}c_i)$ is a unit mod p, so

$$m(n-p^{s-1}, s)v_{p-1}^{a(s-1)}X(n-p^{s-1})=m(n-p^{s-1}, s)\sum_k w_{k,s}X(n-k)\,,$$
$$w_{k,s}\in N_*\,.$$

Set $Y(n-1, s)=v_{p-1}^{a(s-1)}X(n-p^{s-1})-\sum_k w_{k,s}X(n-k)$. This clearly satisfies 1) and 2). Q.E.D.

Proposition 9. *For each integer $a\leq s$ and each integer $n\geq p^{a-1}-1$, there is an element $Y(n, a)\in MU_{2n+1}(B\boldsymbol{Z}_{p^s})$ which satisfies*:

1) $Y(n, a)=v_{p-1}^{a(a-1)}X(n-p^{a-1}+1)+\sum_k w_k X(n-k)$ *with* $w_k\in MU_*/\langle v_{p-1}^{a(a-1)}\rangle$.
2) $p^{s-a}m(n-p^{a-1}+1, a)Y(n, a)=0$.

Proof. Induction on s. The case $s=1$ follows immediately from Proposition 8. Suppose we have defined such elements for $s-1$. For each $a<s$, let

$$Y(n, a)=v_{p-1}^{a(a-1)}X(n-p^{a-1}+1, s)+\sum_k w_{k,a}X(n-k, s)\,.$$

According to [2], page 101, if i: $B\boldsymbol{Z}_{p^{s-1}}\rightarrow BZ_{p^s}$, then

$pi_*(X(n, s-1))=p^2X(n, s)$. Therefore, since $p^{s-a}\,m(n-p^{a-1}+1,a)$ is divisible by p^2, $p^{s-a}m(n-p^{a-1}+1, a)\{v_{p-1}^{a(a-1)}X(n-p^{a-1}+1, s)+$ $+\sum_k w_{k,s}X(n-k, s)\}=i_*(p^{s-1-a}m(n-p^{a-1}+1, a)\{v_{p-1}^{a(a-1)}X(n-p^{a-1}+1, s-1)+$ $+\sum_k w_{k,s}X(n-k, s-1)\}=0$.

Clearly the elements $Y(n, a)$ have the form prescribed by 1).

The case $a=s$ is precisely the substance of Proposition 8. Q.E.D.

Proposition 10. *In $MU_{2p^{s+1}-1}(BZ_{p^s})$, the element $v_{p-1}^{a(s)}X(0)$ is divisible by p.*

Proof. Notice that $\eta_n p^s=1$. Therefore $cf_1(\eta_n p^s)=0$ or, equivalently, $\sum_j a(j-1, p^s)X(n-j)=0$.
By Proposition 7, $a(j-1, p^s)$ is divisible by p for $j<p^s$. Therefore $a(p^s-1, p^s)$ $X(0)$ is divisible by p. But again by Proposition 7, $a(p^s-1, p^s)=cv_{p-1}^{a(s)}+pW$, where c is a unit mod p. Therefore $v_{p-1}^{a(s)}X(0)$ is divisible by p. Q.E.D.

We are now in a position to set up the result we wish to prove. Fix an integer n.

DEFINITION. By $T(a, b)$ we mean

$$\frac{\Gamma_{2(n-b)}(p^a)}{p^{s-a}m(b-p^{a-1}+1, a)\Gamma_{2(n-b)}(p^a)}$$

By T we mean $\sum_{a=1}^{s} \sum_{b=p^{a-1}-1}^{n} T(a, b)$.

Construct a map $f(a, b)\colon T(a, b)\to MU_{2n+1}(BZ_{p^s})$, $f(a, b)\colon w_{n-b}\mapsto w_{n-b}Y(b, a)$ for every $w_{n-b}\in\Gamma_{2(n-b)}(p^a)$. By Proposition 9, this map is a well-difined homomorphism. Let $f=\sum_{a,b} f(a, b)\colon T\to MU_{2n+1}(BZ_{p^s})$. Our aim is to show that f is an isomorphism. To accomplish this, we will first show that f is an epimorphism and then that the orders of T and $MU_{2n+1}(BZ_{p^s})$ are equal.

In order to show that f is an epimorphism, we will consider the groups $MUZ_{p*}(BZ_{p^s})$, that is, complex bordism with Z_p coefficients. For this purpose, let R be a Z_p Moore spectrum and define $MUR_*(X)=S_*(MU\wedge R\wedge X^+)=MU_*(R\wedge X^+)$. The result is a generalized homology theory.

Proposition 11. $MUR_{2n+1}(BZ_{p^s})\cong MU_{2n+1}(BZ_{p^s})\otimes Z_p$.

Proof. There is a Künneth short exact sequence in complex bordism.

$$\begin{aligned}0\to MU_m(BZ_{p^s})\otimes Z_p \to MUR_m(BZ_{p^s}) &\to \\ \to \operatorname{Tor}_Z^1(MU_{m-1}(BZ_{p^s}), Z_p) &\to 0\,.\end{aligned}$$

Since $MU_{2n}(BZ_{p^s})=0$, the result follows.

For further calculations, we need the existence of cap products in the AHSS. The following proposition may be garnered from a paper of R. Kultze [6].

Proposition 12. *Suppose $h^*\otimes k_*\to k_*$ is a pairing of coefficient groups of theories $h^*(\ )$ and $k_*(\ )$. The cap product $H^*(X; h^*)\otimes H_*(X; k_*)\to H_*(X; k_*)$ induces a cap product $\cap_2$ on the E^2 terms of the corresponding Atiyah-Hirzebruch spectral sequences which satisfies:*

1) $\cap_2$ induces cap products $\cap_r$: $E^r \otimes E_r \to E_r$
2) Each differential d^r is a (graded) derivation with respect to $\cap_r$; *i.e.* $d^r(a \cap_r b) = d_r(a) \cap_r b \pm a \cap_r d^r(b)$.

We will generally write $\cap$ for $\cap_r$.

We will apply this proposition to the module pairing arising from $MU \wedge MUR \to MUR$.

REMARK. The more natural thing to do would be to use a ring spectrum pairing $MUR \wedge MUR \to MUR$ here. Unfortunately, the general perversity of the universe demands that $MU\boldsymbol{Z}_2$ not be a ring spectrum. Such is life.

The map of spectra $MU \to MUR$ induces a map t: $MU_*(B\boldsymbol{Z}_{p^s}) \to MUR_*(B\boldsymbol{Z}_{p^s})$. Let $t(X(n)) = Z(2n+1)$.

Proposition 13. *$MUR_{2n+1}(B\boldsymbol{Z}_{p^s})$ is additively generated by elements of the form $w_j Z(2(n-j)+1)$ where $w_j \in MU_*/\langle v_{p-1}^{a(s)} \rangle$.*

Proof. Consider the AHSS for $MUR_*(B\boldsymbol{Z}_{p^s})$ in which

$$E^2_{i,q} = H_i(B\boldsymbol{Z}_{p^s};\, MUR_q) = \begin{cases} 0 & q \quad \text{odd} \\ (\boldsymbol{Z}_p)^{\pi(q/2)} & \text{otherwise.} \end{cases}$$

Let $r \geq 2$ be the smallest integer such that $E^r \neq E^{r+1}$. Since $E^2_{p,q} = 0$ for q odd and d^r has bidegree $(r, r-1)$, r must be odd.

Let $\bar{E}$ be the AHSS for $MU_*(B\boldsymbol{Z}_{p^s})$. The map t is induced on the E^2 level by the reduction $\bar{t}$: $H_*(B\boldsymbol{Z}_{p^s};\, MU_*) \to H_*(B\boldsymbol{Z}_{p^s};\, MUR_*)$. Since $H_{2n}(B\boldsymbol{Z}_{p^s};\, \boldsymbol{Z}) = 0$, the universal coefficient theorem says that $\bar{t}$ is an epimorphism in odd dimensions. Therefore $MUR_{2n+1}(B\boldsymbol{Z}_{p^s})$ is at least generated by the elements $b_j Z(2(n-j)+1)$, as b_j ranges over MU_*.

1) Claim $d^r(Z(2j+1) \otimes b_k) = 0$ $\forall j \geq 0$ and $b_k \in MU_*$. In fact we have already noticed that the spectral sequence $\bar{E}$ is trivial for dimensional reasons. Therefore
$$d^r(Z(2j+1) \otimes b_k) = d^r(\bar{t}(X(j) \otimes b_k)) = \bar{t}(\bar{d}^r(X(j) \otimes b_k)) = 0\,.$$
2) Claim d^r: $E^r_{r+1,0} \to E^r_{1,r-1}$ is non-zero. For there is an integer j and a $b_k \in MU_*$ such that
$0 \neq d^r(Z(2j) \otimes b_k) = d^r(Z(2j)) \otimes b_k$. Then $d^r(Z(2j)) \neq 0$. There is a class $u \in H^2(B\boldsymbol{Z}_{p^s};\, \boldsymbol{Z})$ which gives the periodicity of $H_*(B\boldsymbol{Z}_{p^s};\, \boldsymbol{Z}_p)$ via cap products, i.e. $H_m(B\boldsymbol{Z}_{p^s};\, \boldsymbol{Z}_p) = \boldsymbol{Z}_p$ on a generator w_m and $w_m = u \cap w_{m+2}$. A similar periodicity holds for $H_*(B\boldsymbol{Z}_{p^s};\, \boldsymbol{Z})$ with respect to the same u. Denote also by u the corresponding generator in $\bar{E}_2^{2,0}$ of the AHSS for $MU^*(B\boldsymbol{Z}_{p^s})$. Then
$$d^r(Z(2j-2)) = d^r(u \cap Z(2j)) = d_r(u) \cap Z(2j) \pm u \cap d^r(Z(2j)) = \\ = \pm u \cap d^r(Z(2j))\,.$$

But, for $2j \geq r+3$, $u \cap_r = u \cap_2$ is an isomorphism. Therefore, in this range $d^r(Z(2j-2)) \neq 0$. By induction $d^r(Z(r+1)) \neq 0$ as claimed.

3) Claim $d^r(Z(r+1)) = Z(1) \otimes v_{p-1}^{(r-1)/2(p-1)}$. For, since $d^r(Z(r+1)) \neq 0$, there is a $b_k \in MU_*$, $b_k \neq 0$, such that $d^r(Z(r+1)) = Z(1) \otimes b_k$. Then, for any $b_j \in MU_*$, $b_j \neq 0$, $d^r(Z(r+1) \otimes b_j) = Z(1) \otimes b_j b_k \neq 0$. In $Z(1) \otimes b_k = d^r(Z(r+1)) = d^r(u \cap (Z(r+3))) = u \cap d^r(Z(r+3))$. But $u \cap (Z(3) \otimes b_k) = Z(1) \otimes b_k$ and $u \cap$ is an isomorphism. Therefore $d^r(Z(r+3)) = Z(3) \otimes b_k$. Arguing inductively $d^r(Z(r+2j+1) \otimes b_j) = Z(2j+1) \otimes b_j b_k$.

This has two consequences. First, $d^r_{2j,*}$ is a monomorphism for $2j \geq r+1$, so that $E^{r+1}_{2j,*} = 0$ and $d^i_{2j,*} = 0$ for all $i \geq r+1$. Therefore $E^{r+1}_{2j+1,*} = E^\infty_{2j-1,*}$. Secondly, for $j \geq 0$, $Z(2j+1) \otimes b = 0$ in E^∞ if and only if b is in the ideal $\langle b_k \rangle$ generated by b_k. For suppose $b = b_k a \in MU_*$. Then $Z(2j+1) \otimes b = d^r(Z(2j+r+1) \otimes a)$, so that $Z(2j+1) \otimes b = 0$ in $E^{r+1}_{2j+1,*} = E^\infty_{2j+1,*}$. On the other hand, if $b \neq b_k a$, then $Z(2j+1) \otimes b$ cannot be the image of any d^r and we have shown that $d^i_{2i,*} = 0$ for all $i \geq r+1$. Therefore, in this case $Z(2j+1) \otimes b \neq 0$.

Now $v_{p-1}^{a(s)} X(0)$ is divisible by p by Proposition 10. Therefore $t(v_{p-1}^{a(s)} X(0)) = v_{p-1}^{a(s)} Z(1) = 0$ in $MUR_{2p^s-1}(BZ_{p^s}) = MU_{2p^s-1}(BZ_{p^s}) \otimes Z_p$, so that $Z(1) \otimes v_{p-1}^{a(s)} = 0$ in E^∞. Thus $v_{p-1}^{a(s)} \in \langle b_k \rangle$, i.e. b_k is a power of v_{p-1}. For dimensional reasons

$$b_k = v_{p-1}^{\frac{r-1}{2(p-1)}}.$$

REMARK. It turns out that $r = 2p^s - 1$ and $b_k = v_{p-1}^{a(s)}$, but this is not necessary for the proof.

Now, the only non-zero groups appearing in the associated graded of $MUR_{2n+1}(BZ_{p^s})$ are of the form $E^\infty_{2j+1,2(n-j)}$. But we have just shown these groups to be generated by the elements $Z(2j+1) \otimes b$ with $b \in MU_* - \langle b_k \rangle \subseteq MU_* - \langle v_{p-1}^{a(s)} \rangle$. Q.E.D.

Proposition 14. *The map f: $T \to MU_{2n+1}(BZ_{p^s})$ is an epimorphism.*

Proof. Consider the commutative diagram:

$$\begin{array}{ccc} T & \longrightarrow & T \otimes Z_p \\ f \downarrow & & \downarrow f \otimes Z_p \\ MU_{2n+1}(BZ_{p^s}) & \xrightarrow[t]{} & MUR_{2n+1}(BZ_{p^s}), \end{array}$$

and suppose that $t \circ f$ were an epimorphism. Then $f \otimes Z_p$ would be an epimorphism. Since both T and $MU_{2n+1}(BZ_{p^s})$ are finite abelian p-groups, f would also be an epimorphism.

We must show, therefore, that $t \circ f$ is an epimorphism. This is equivalent to showing that image $f \supseteq \{b_k X(n-k) \colon b_k \in MU_* - \langle v_{p-1}^{a(s)} \rangle\}$. Consider the increasing sequence of groups $MU_*-\langle v_{p-1}^{d(1)}\rangle \subseteq MU_*-\langle v_{p-1}^{d(2)}\rangle \subseteq \cdots \subseteq MU_*-\langle v_{p-1}^{d(s)}\rangle$ and suppose that $a\in MU_*-\langle v_{p-1}^{d(i+1)}\rangle$, $a\notin MU_*-\langle v_{p-1}^{d(i)}\rangle$. Then $a=v_{p-1}^{d(i)}\cdot v_{p-1}^{c}\cdot b_j$ where $c<d(i+1)-d(i)$ and $b_j\in\Gamma_{2j}(p)$. In other words $v_{p-1}^{c}\cdot b_j\in\Gamma_*(p^{i+1})$. Recall that

$$Y(n-j-c(p-1),\, i) = v_{p-1}^{t(i)}X(n-j-c(p-1)-p^i+1)+\sum a_k X(n-j-c(p-1)-k)$$

where $a_k\in MU_*/\langle v_{p-1}^{d(i)}\rangle$.

Since we may assume by induction on the power of v_{p-1} appearing in a given monomial that $v_{p-1}^{c}\cdot b_j\cdot a_k X(n-j-c(p-1)-k)$ is in the image of f, it follows that $aX(n-|a|/2)=v_{p-1}^{c}\cdot b_j\cdot Y(n-j-c(p-1))$ modulo the image of f. Therefore $aX(n-|a|/2)$ is in the image of f. Q.E.D.

DEFINITION. By $\pi(n;\, m,\, r)$ we mean the number of partitions of n which contain no more than m terms equal to r.

EXAMPLE. Let $m=1$, $r=2$. Then (3,2) is an allowable partition of 5, but (2,2,1) is not. $\pi(5;\, 1,\, 2)=6$ and $\pi(5;\, 2,\, 1)=5$.

Proposition 15. $\sum_{k=0}^{n}\pi(k) = \sum_{j=0}^{n}\left(\left[\frac{n-j}{(m+1)r}\right]+1\right)\pi(j;\, m,\, r)$

Proof. First notice that the number of partitions of n containing exactly m terms equal to r is equal to the number of unrestricted partitions of $n-mr$. Furthermore, $\pi(k)$ is equal to the sum of the number of partitions of k containing no terms equal to r, those with exactly one r, and so forth. Therefore

$$\pi(k) = \pi(k;\, 0,\, r)+\pi(k-r;\, 0,\, r)+\pi(k-2r;\, 0,\, r)+\cdots .$$

Similarly,

$$\pi(k;\, m,\, r) = \pi(k;\, 0,\, r)+\pi(k-r;\, 0,\, r)+\cdots+\pi(k-mr;\, 0,\, r)\,.$$

Therefore

$$\begin{aligned}&\pi(k;\, m,\, r) = \pi(k)-\pi(k-(m+1)r) \quad\text{and}\\ &\pi(k) = \pi(k;\, m,\, r)+\pi(k-(m+1)r;\, m,\, r)+\pi(k-2(m+1)r;\, m,\, r)+\cdots .\end{aligned}$$

Summing over k,

$$\begin{aligned}\sum_{k=0}^{n}\pi(k) &= \sum_{k=0}^{n}\sum_{a}\pi(k-a(m+1)r;\, m,\, r)\\ &= \sum_{j}(max\{a\colon j = k-a(m+1)r\}+1)\pi(j;\, m,\, r)\\ &= \sum_{j}\left(\left[\frac{n-j}{(m+1)r}\right]+1\right)\pi(j;\, m,\, r) \qquad \text{Q.E.D.}\end{aligned}$$

Proposition 16.

$$MU_{2n+1}(BZ_{p^s})\cong \sum_{a=1}^{s}\sum_{b=p^{a-1}-1}^{n}\frac{\Gamma_{2(n-b)}(p^a)}{p^{\left[\frac{b-p^{a-1}+1}{p^{a-1}(p-1)}\right]+s-a-1}\Gamma_{2(n-b)}(p^a)}$$

Proof. The proposition states that the map $f\colon T\to MU_{2n+1}(BZ_{p^s})$ is an isomorphism. Since we have already shown it to be an epimorphism, it suffices to verify that the two groups involved have the same order.

According to Proposition 2, the order of $MU_{2n+1}(BZ_{p^s})$ is $p^{A(s)}$ where $A(s)=\sum_{k=0}^{n} s\pi(k)$. The order of T on the other hand is clearly $p^{B(s)}$ where

$$B(s)=\sum_{a=1}^{s}\sum_{b=p^{a-1}-1}^{n}\left\{\left[\frac{b-p^{a-1}+1}{p^{a-1}(p-1)}\right]+s-a+1\right\}\pi(n-b;\,p^{a-1}-1,\,p-1)\,.$$

We must show $A(as)=B(s)$.

Proceed by induction on s. The case $s=1$ is an example of Proposition 15. Write $\pi(n;\,m)$ for $\pi(n;\,m,\,p-1)$. Now

$$B(s)=B(s-1)+\sum_{a=1}^{s-1}\sum_{b=p^{a-1}-1}^{n}\pi(n-b;\,p^{a-1}-1)+\cdots$$

$$\cdots+\sum_{b=p^{s-1}-1}^{n}\left\{\left[\frac{b-p^{s-1}+1}{p^{s-1}(p-1)}\right]+1\right\}\pi(n-b;\,p^{s-1}-1)\,.$$

By Proposition 15,

$$\sum_{b=p^{s-1}-1}^{n}\left\{\left[\frac{b-p^{s-1}+1}{p^{s-1}(p-1)}\right]+1\right\}\pi(n-b;\,p^{s-1}-1)=$$

$$=\sum_{b=0}^{n-p^{s-1}+1}\left\{\left[\frac{b}{p^{s-1}(p-1)}\right]+1\right\}\pi(n-b-p^{s-1}+1;\,p^{s-1}-1)$$

$$=\sum_{b=0}^{n-p^{s-1}+1}\pi(b)\,.$$

Remember from the proof of Proposition 15 that

$$\pi(k)=\sum_{a\geq 0}\pi(k-ap^{a-1}(p-1);\,p^{a-1}-1)\,.$$

Therefore

$$\sum_{b=p^{a-1}-1}^{n}\pi(n-b;\,p^{a-1}-1)=\sum_{b=n-p^{a}+2}^{n-p^{a-1}+1}\pi(b)$$

and so

$$B(s)-B(s-1)=\sum_{b=0}^{n-p^{s-1}+1}\pi(b)+\sum_{a=1}^{s-1}\sum_{b=n-p^{a}+2}^{n+p^{a-1}+1}\pi(b)=\sum_{b=0}^{n}\pi(b)=A(s)-A(s-1)\,.$$

Since $A(1)=B(1)$, induction shows that $A(s)=B(s)$ for all s. Q.E.D.

Proposition 17. *Suppose r and s are relatively prime.*

Then $MU_*(BZ^+_{rs})\simeq MU_*(BZ^+_r)\otimes_{MU_*}MU_*(BZ^+_s)$.

Proof. This proposition follows almost immediately from a theorem of Landweber [8] to the effect that if X and Y are CW-complexes such that the AHSS for $MU_*(X)$ is trivial, then there is a natural short exact sequence

$$0\to MU_*(X^+)\otimes MU_*(Y^+)\to MU_*(X^+\wedge Y^+)\to$$
$$\to \mathrm{Tor}_1^{MU_*}(MU_*(X^+),\ MU_*(Y^+))\to 0\,.$$

Since the AHSS for $MU_*(BZ_r)$ collapses for dimensional reasons and the torsion of $MU_*(BZ_r)$ and $MU_*(BZ_s)$ are of relatively prime order, $\mathrm{Tor}_1^{MU_*}$ $(MU_*(BZ^+_r),\ MU_*(BZ^+_s))=0$. Q.E.D.

Corollary. *If r and s are relatively prime, then*

$$MU_{2n+1}(BZ_{rs})=MU_{2n+1}(BZ_r)\oplus MU_{2n+1}(BZ_s)\,.$$

Taken in conjunction, these last two propositions clearly suffice to give the complex bordism of any (finite) cyclic group.

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