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Osaka J. Math.
10 (1973), 115-130

ON SOME PARABOLIC EQUATIONS OF EVOLUTION IN HILBERT SPACE

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(Received February 14, 1972)

1. Introduction

Let X and V be two Hilbert spaces such that V is a dense subspace of X with continuous imbedding $V \rightarrow X$. Identifying X with its antidual (= the set of continuous antilinear forms on X) we may consider $V \subset X \subset V^*$ algebraically and topologically where V^* is the antidual of V. As is easily seen V is a dense subspace of V^*. The inner product and norm in X are denoted by (f, g) and $|f|$, and those in V are by $((u, v))$ and $||u||$. For $f \in X$ and $u \in V$, (f, u) is equal to the value at u of f considered as an element of V^*, so we denote the V^*-V duality by (f, u) without causing any confusion. Sometimes we write also (u, f) instead of $\overline{(f, u)}$. The norm in V^* is denoted by $||f||_*$.

Let $a(t; u, v)$, $0 \leqq t \leqq T$, be a family of sesquilinear forms defined on $V \times V$ satisfying the following assumptions:

there exist positive constants M, δ, K and $0 < \rho \leqq 1$ such that

$$|a(t; u, v)| \leqq M||u|| \, ||v|| \,, \tag{1.1}$$

$$\mathrm{Re}\; a(t; u, u) \geqq \delta ||u||^2, \tag{1.2}$$

$$|a(t; u, v) - a(s; u, v)| \leqq K|t-s|^{\rho} ||u|| \, ||v|| \tag{1.3}$$

for any $u, v \in V$ and $t, s \in [0, T]$.

We define the operator $A(t)$ in the following manner;

the element $u \in V$ belongs to $D(A(t))$, the domain of $A(t)$, and $A(t)u = f \in X$ if and only if $a(t; u, v) = (f, v)$ for any $v \in V$.

It is well-known that $-A(t)$ generates an anlytic semigroup of bounded operators in X. We consider the initial value problem of the evolution equation in X

$$du(t)/dt + A(t)u(t) = f(t)\,, \tag{1.4}$$

*) Part of the contents of this paper was talked by the second author at the Conference on Evolution Equations and Functional Analysis held at the University of Kansas, Lawrence, Kansas in June-July, 1970.

$$u(0) = \phi. \tag{1.5}$$

To solve this equation we extend the operator $A(t)$ to an operator on V into V^*, and first solve (1.4)—(1.5) in the larger space V^*. This extension of $A(t)$ which is again denoted by $A(t)$ is defined by $a(t; u, v) = (A(t)u, v)$ for $u, v \in V$. The operator $-A(t)$ thus extended generates an analytic semigroup in V^* and furthermore it has the constant domain V, and hence we may apply the result of [7] to construct the evolution operator $U(t, s)$ of (1.4) considered as an equation in V^*. The restriction of $U(t, s)$ to X is uniformly bounded in the space of bounded operators from X to itself. If $\rho > 1/2$, one can show that it is the desired evolution operator of the problem (1.4)—(1.5) in X. This result is closely related to the results of T. Kato [2] and P.E. Sobolevskii [5], [6] since under our hypotheses $A(t)^\theta$ has a constant domain for any $\theta < 1/2$ (T. Kato [1]) and the assumption $\rho > 1/2$ is considered reasonable compared with the results of [2], [5], [6].

Next we show the existence and uniqueness of a mild solution of the semilinear equation

$$du(t)/dt + A(t)u(t) = f(t, u(t)), \tag{1.6}$$

$$u(0) = \phi, \tag{1.7}$$

namely the solution of the integral equation

$$u(t) = U(t, 0)\phi + \int_0^t U(t, s) f(s, u(s)) ds.$$

Here $f(t, u)$ is a mapping from $[0, T] \times X$ into V^* satisfying the monotonicity condition with respect to u as well as some continuity condition in (t, u), This kind of theorem was first established by T. Kato [3] where $-A(t)$ was assumed to be the infinitesimal generator of a contraction semigroup but not of an anlaytic semigroup. However in [3] $A(t)^{-1}$ must be continuously differentiable in t and f is a mapping from $[0, T] \times X$ to X. In the proof of our result we approximate the equation (1.6) by a sequence of equations to which the result of T. Kato [3] can be applied and then go to the limit. In this result we do not assume $\rho > 1/2$.

Finally we describe another proof of Théorème 1.1 of Chap. IV of J.L. Lions [4] which asserts the existence of the solution of (1.4)-(1.5) in the sense that

$$\int_0^T a(t; u(t), \psi(t)) dt - \int_0^T (u(t), \psi'(t)) dt = \int_0^T (f(t), \psi(t)) dt + (\phi, \psi(0))$$

for any ψ such that $\psi \in L^2(0, T; V)$, $\psi' \in L^2(0, T; X)$ and $\psi(T) = 0$. As in the theorem of Lions we assume in this result that $a(t; u, v)$ is a measurable function of t for each fixed $u, v \in V$ instead of (1.3).

2. Some lemmas

In this section we consider sesquilinear forms and operators associated with them and prove some lemmas which will be used in the subsequent sections.

Let $a(u, v)$ be a sesquilinear form defined on $V\times V$. We assume

$$|a(u, v)| \leqq M\|u\| \, \|v\| , \tag{2.1}$$

$$\mathrm{Re}\ a(u, u) \geqq \delta\|u\|^2, \qquad \delta>0 , \tag{2.2}$$

for any $u, v\in V$. We define an operator A on V to V^* by

$$a(u, v) = (Au, v) , \qquad u, v\in V .$$

Lemma 2.1. *If* Re $\lambda\leqq 0$, *then the operator* $A-\lambda$ *has an inverse defined in the whole of* V^* *which saitsfies the following estimates:*

$$|(A-\lambda)^{-1}f| \leqq M_1|\lambda|^{-1}|f| , \tag{2.3}$$

$$|(A-\lambda)^{-1}f| \leqq M_2|\lambda|^{-1/2}\|f\|_* , \tag{2.4}$$

$$\|(A-\lambda)^{-1}f\| \leqq M_2|\lambda|^{-1/2}|f| , \tag{2.5}$$

$$\|(A-\lambda)^{-1}f\| \leqq \delta^{-1}\|f\|_* , \tag{2.6}$$

$$\|(A-\lambda)^{-1}f\|_* \leqq M_1|\lambda|^{-1}\|f\|_* \tag{2.7}$$

for any $f\in X$ *or* V^*, *where* $M_1=1+M/\delta$, $M_2=\{(1+M/\delta)/\delta\}^{1/2}$.

Proof. The first part of the assertion follows from the Lax-Milgram theorem. If $f=(A-\lambda)u$, Re $\lambda\leqq 0$, then

$$(f, v) = a(u, v)-\lambda(u, v) \tag{2.8}$$

for any $v\in V$. Hence taking $v=u$ and using (2.2) we get

$$\mathrm{Re}\,(f, u) \geqq \delta\|u\|^2-\mathrm{Re}\ \lambda\,|u|^2 \geqq \delta\|u\|^2 , \tag{2.9}$$

from which (2.6) follows. (2.4) is an immediate consequence of

$$|\lambda|\,|u|^2 \leqq \|f\|_*\|u\|+M\|u\|^2 \leqq \frac{1}{\delta}\left(1+\frac{M}{\delta}\right)\|f\|_*^2 .$$

Noting $\delta\|u\|^2 \leqq \mathrm{Re}\,(f, u) \leqq |f|\,|u|$ we find

$$|\lambda|\,|u|^2 \leqq |f|\,|u|+M\|u\|^2 \leqq |f|\,|u|+(M/\delta)|f|\,|u| = M_1|f|\,|u|$$

which implies (2.3). Moreover in view of (2.3) and (2.9)

$$\delta\|u\|^2 \leqq M_1|\lambda|^{-1}|f|^2$$

which implies (2.5). From (2.8) and (2.6) it follows that

$$|\lambda|\,|(u, v)| \leqq \|f\|_*\|v\| + M\|u\|\,\|v\| \leqq M_1\|f\|_*\|v\|$$

and hence (2.7) is proved.

In view of Lemma 2.1 $-A$ generates an analytic semigroup both in X and V^* which is defined by

$$\exp(-tA) = -\frac{1}{2\pi i}\int_\Gamma e^{-\lambda t}(\lambda - A)^{-1}d\lambda$$

where Γ is a smooth contour running in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for some $\theta \in (0, \pi/2)$. It is known that

$$|A \exp(-tA)f| \leqq Ct^{-1}|f|, \tag{2.10}$$

$$\|A \exp(-tA)f\|_* \leqq Ct^{-1}\|f\|_*. \tag{2.11}$$

Here and in what follows we use the notation C to denote constants which depend only on the assumptions we are making at each occasion.

Lemma 2.2. *For any* $u \in V$

$$\delta\|u\| \leqq \|Au\|_* \leqq M\|u\|.$$

Proof is easy and omitted.

Lemma 2.3. *For* $t > 0$

$$|\exp(-tA)f| \leqq Ct^{-1/2}\|f\|_*, \tag{2.12}$$

$$|A \exp(-tA)f| \leqq Ct^{-3/2}\|f\|_*, \tag{2.13}$$

$$\|\exp(-tA)f\| \leqq Ct^{-1/2}|f|, \tag{2.14}$$

$$\|\exp(-tA)f\| \leqq Ct^{-1}\|f\|_*, \tag{2.15}$$

$$\|A \exp(-tA)f\| \leqq Ct^{-3/2}|f|, \tag{2.16}$$

$$\|A \exp(-At)f\|_* \leqq Ct^{-1/2}|f|. \tag{2.17}$$

Proof. (2.12) and (2.13) are easily shown with the aid of (2.4). (2.15) is a direct consequence of (2.11) and Lemma 2.2. The remaining ones can be easily established using (2.5) and Lemma 2.2.

3. Construction of evolution operator

In virtue of the results of the preceding section $-A(t)$ generates an analytic semigroup also in V^* and moreover it has the constant domain V. It immediately follows from (1.3) that

$$\|A(t)u-A(s)u\|_* \leqq K|t-s|^\rho \|u\| \tag{3.1}$$

which together with Lemma 2.2 implies that the bounded operator valued function $A(t)A(0)^{-1}$ in V^* is Hölder continuous. Thus we can apply the result of [7] to construct the evolution operator $U(t, s)$ of the equation $u'+A(t)u=0$ in V^* in the following manner:

$$U(t, s) = \exp(-(t-s)A(s))+W(t, s)\,, \tag{3.2}$$

$$W(t, s) = \int_s^t \exp(-(t-\tau)A(\tau))R(\tau, s)d\tau\,, \tag{3.3}$$

$$R(t, s) = \textstyle\sum_{m=1}^\infty R_m(t, s)\,, \tag{3.4}$$

$$R_1(t, s) = -(A(t)-A(s))\exp(-(t-s)A(s))\,, \tag{3.5}$$

$$R_m(t, s) = \int_s^t R_1(t, \tau)R_{m-1}(\tau, s)d\tau\,. \tag{3.6}$$

Lemma 3.1. $\|R(t, s)f\|_* \leqq C(t-s)^{\rho-1}\|f\|_*\,,$ (3.7)

$$\|R_1(t, s)f\|_* \leqq C(t-s)^{\rho-1/2}|f|\,, \tag{3.8}$$

$$\|R(t, s)f\|_* \leqq C(t-s)^{\rho-1/2}|f|\,. \tag{3.9}$$

Proof. (3.7) is known from [7]. (3.8) follows from (3.1) and (2.14). For proof of (3.9) it suffices to show

$$\|R_m(t, s)f\|_* \leqq C_0 C_1^{m-1}(t-s)^{m\rho-1/2}|f|\,\Gamma(\rho)^{m-1}\Gamma(\rho+1/2)/\Gamma(m\rho+1/2)$$

which can be shown by induction, where C_0 and C_1 are constants independent of the m and f.

Lemma 3.2. $\|R_1(t, s)f-R_1(\tau, s)f\|_*$

$$\leqq C\{(t-\tau)^\rho(t-s)^{-1}+(t-\tau)(t-s)^{-1}(\tau-s)^{\rho-1}\}\|f\|_*\,.$$

This is proved in [7].

Lemma 3.3. $\|R(t, s)f-R(\tau, s)f\|_*$

$$\begin{aligned}&\leqq C(t-\tau)^\rho(t-s)^{-1/2}|f|+C(t-\tau)(t-s)^{-1}(\tau-s)^{\rho-1/2}|f|\\&\quad+C\int_\tau^t(t-\sigma)^{\rho-1}(\sigma-s)^{\rho-1/2}|f|\,d\sigma\\&\quad+C\int_s^\tau\{(t-\tau)^\rho(t-\sigma)^{-1}+(t-\tau)(t-\sigma)^{-1}(\tau-\sigma)^{\rho-1}\}(\sigma-s)^{\rho-1/2}|f|\,d\sigma\,.\end{aligned} \tag{3.10}$$

Proof. In view of (2.16) we have

$$\begin{aligned}&\|\{\exp(-(t-s)A(s))-\exp(-(\tau-s)A(s))\}f\|\\&=\|-\int_\tau^t A(s)\exp(-(r-s)A(s))f\,dr\|\\&\leqq C\int_\tau^t (r-s)^{-3/2}|f|\,dr\leqq C(t-\tau)(t-s)^{-1}(\tau-s)^{-1/2}|f|\,.\end{aligned}$$

Hence noting (3.1) we get

$$\begin{aligned}&\|(A(\tau)-A(s))\{\exp(-(t-s)A(s))-\exp(-(\tau-s)A(s))\}f\|_*\\&\leqq C(t-\tau)(t-s)^{-1}(\tau-s)^{\rho-1/2}|f|\,.\end{aligned}\tag{3.11}$$

On the other hand by (3.1) and (2.14) we obtain

$$\|(A(t)-A(\tau))\exp(-(t-s)A(s))f\|_*\leqq C(t-\tau)^\rho(t-s)^{-1/2}|f|\,.\tag{3.12}$$

It follows from (3.11) and (3.12) that $\|R_1(t, s)f-R_1(\tau, s)f\|_*$ is bounded by the sum of the first two terms of the right hand side of (3.10). Therefore we can show (3.10) without any difficulty taking into consideration the identity

$$\begin{aligned}R(t, s)-R(\tau, s)&=R_1(t, s)-R_1(\tau, s)+\int_\tau^t R_1(t, \sigma)R(\sigma, s)d\sigma\\&+\int_s^\tau (R_1(t, \sigma)-R_1(\tau, \sigma))R(\sigma, s)d\sigma\end{aligned}$$

as well as Lemmas 3.1 and 3.2

Lemma 3.4. $|W(t, s)f|\leqq C(t-s)^\rho|f|\,,$ (3.13)

$$\|W(t, s)f\|\leqq C(t-s)^{\rho-1/2}|f|\,.\tag{3.14}$$

Proof. (3.13) follows from Lemmas 2.3 and 3.1. Set

$$M(t, \tau)=\exp(-(t-\tau)A(\tau))-\exp(-(t-\tau)A(t))\,.$$

Then noting

$$M(t, \tau)=\frac{1}{2\pi i}\int_\Gamma e^{-\lambda(t-\tau)}(\lambda-A(\tau))^{-1}(A(\tau)-A(t))(\lambda-A(t))^{-1}d\lambda\,,$$

we can easily show

$$\|M(t, \tau)f\|\leqq C(t-\tau)^{\rho-1}\|f\|_*$$

with the aid of Lemma 2.1 and (3.1). This inequality together with Lemmas 3.1, 3.3, 2.2 and (2.15) enables us to rewrite

$$\begin{aligned}W(t, s)&=\int_s^t M(t, \tau)R(\tau, s)d\tau+\int_s^t \exp(-(t-\tau)A(t))(R(\tau, s)-R(t, s))d\tau\\&+A(t)^{-1}\{I-\exp(-(t-s)A(t))\}R(t, s)\end{aligned}$$

and to derive (3.14).

Lemma 3.5. $$|U(t, s)f| \leqq C(t-s)^{-1/2}\|f\|_*\,, \tag{3.15}$$

$$\|U(t, s)f\| \leqq C(t-s)^{-1/2}|f|\,. \tag{3.16}$$

Proof. (3.15) immediately follows from Lemmas 2.3 and 3.1, while (3.16) from Lemmas 2.3 and 3.4.

If we set

$$S(t, s) = A(t)\exp(-(t-s)A(t)) - A(s)\exp(-(t-s)A(s))\,,$$

then we get the following

Lemma 3.6. $$|S(t, s)f| \leqq C(t-s)^{\rho-1}|f|\,, \tag{3.17}$$

$$|S(t, s)f| \leqq C(t-s)^{\rho-3/2}\|f\|_*\,. \tag{3.18}$$

Proof. Since

$$S(t, s) = \frac{1}{2\pi i}\int_\Gamma \lambda e^{-\lambda(t-s)}\{(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\}d\lambda\,,$$

we may establish (3.17) with the aid of

$$|\{(\lambda - A(t))^{-1} - (\lambda - A(s))^{-1}\}f| \leqq C(t-s)^\rho|\lambda|^{-1}|f|$$

which follows from Lemma 2.1 and (3.1). (3.18) is proved analogously.

Henceforth in this section we suppose

$$\rho > 1/2\,. \tag{3.19}$$

Lemma 3.7. $|(\partial/\partial t)W(t, s)f| \leqq C(t-s)^{\rho-1}|f|\,.$

Proof. Since we can write

$$\begin{aligned}(\partial/\partial t)W(t, s) &= \int_s^t S(t, \tau)R(\tau, s)d\tau \\ &\quad - \int_s^t A(t)\exp(-(t-\tau)A(t))(R(\tau, s) - R(t, s))d\tau \\ &\quad + \exp(-(t-s)A(t))R(t, s)\,,\end{aligned}$$

we get the desired estimate with the aid of Lemmas 2.3, 3.1, 3.3 and 3.6.

Theorem 3.1 *Suppose the assumptions* (1.1), (1.2), (1.3), (3.19) *are satisfied. Then the operator valued function* $U(t, s)$ *constructed above is the evolution operator for the initial value problem* (1.4)–(1.5), *i.e., it is continuously differentiable in* t *and* s, $0 \leqq s < t \leqq T$, *in the uniform operator topology of the space of bounded operators on* X *to itself, its range is contained in* $D(A(t))$ *if* $t > s$, *and it satisfies*

$$(\partial/\partial t)U(t, s)+A(t)U(t, s) = 0\,, \qquad 0 \leqq s < t \leqq T\,, \tag{3.20}$$

$$U(s, s) = I, \qquad 0 \leqq s \leqq T\,, \tag{3.21}$$

$$U(t, s)U(s, r) = U(t, r)\,, \qquad 0 \leqq r \leqq s \leqq t \leqq T\,, \tag{3.22}$$

$$(\partial/\partial s)U(t, s) - \overline{U(t, s)A(s)} = 0\,, \qquad 0 \leqq s < t \leqq T\,. \tag{3.23}$$

Furthermore the following estimates hold:

$$|(\partial/\partial t)U(t, s)| = |A(t)U(t, s)| \leqq C(t-s)^{-1}\,, \tag{3.24}$$

$$|(\partial/\partial s)U(t, s)| = |\overline{U(t, s)A(s)}| \leqq C(t-s)^{-1}\,. \tag{3.25}$$

In the above $\overline{U(t, s)A(s)}$ *denotes the unique bounded extension of the operator* $U(t, s)A(s)$ *to the whole space* X, *and the notation* $|\quad|$ *used in* (3.24) *and* (3.25) *stands for the norms of bounded operators on* X *to itself.*

Proof. Now that we have proved the preceding Lemmas, (3.20), (3.21), (3.22) and (3.24) can be verified without difficulty. It is also easily seen that the operator $A(t)^*$ associated with the adjoint form $a^*(t; u, v) = \overline{a(t; v, u)}$ is the adjoint operator of $A(t)$ in either space of X and V^*. We notice that the evolution operator $V(t, s)$ of the adjoint equation

$$-dv(s)/ds + A(s)^* v(s) = 0\,,$$

that is, the operator valued function satisfying

$$-(\partial/\partial s)V(t, s) + A(s)^* V(t, s) = 0\,, \qquad 0 \leqq s < t \leqq T\,, \tag{3.26}$$

$$V(t, t) = I\,, \qquad 0 \leqq t \leqq T\,, \tag{3.27}$$

can be constructed similarly to $U(t, s)$. We have

$$U(t, s) = V(t, s)^*\,. \tag{3.28}$$

For we find for $s < r < t$

$$\begin{aligned}&(\partial/\partial r)(U(r, s)f,\ V(t, r)g)\\ &= -(A(r)U(r, s)f,\ V(t, r)g) + (U(r, s)f,\ A(r)^* V(t, r)g)\\ &= 0\,,\end{aligned}$$

which implies

$$(f,\ V(t, s)g) = (U(t, s)f,\ g)\,.$$

In view of (3.26), (3.28) and the inequality

$$|(\partial/\partial s)V(t, s)| \leqq C(t-s)^{-1}$$

which can be shown in the same way as (3.24) was established, we can verify

(3.23) and (3.25).

Theorem 3.2. *Let the assumptions of Theorem* 3.1 *be satisfied. For* $f \in C([0, T]; X)$, *the problem* (1.4)–(1.5) *has at most one solution given by*

$$u(t) = U(t, 0)\phi + \int_0^t U(t, s) f(s) ds. \tag{3.29}$$

If f is Hölder continuous in the strong topology of X, then the function defined by (3.29) *is indeed the solution of* (1.4)–(1.5).

Proof. The uniqueness follows from (3.21) and (3.23). We know that the second term of the right hand side of (3.29) is differentiable in X noting Lemma 3.6 and that

$$\begin{aligned} &\frac{\partial}{\partial t}\int_0^t \exp(-(t-s)A(s)) f(s) ds \\ &= \int_0^t S(t, s) f(s) ds - \int_0^t A(t) \exp(-(t-s)A(t))(f(s)-f(t)) ds \\ &\quad + \exp(-(t-s)A(t)) f(t). \end{aligned}$$

Thus the proof is complete.

4. Semilinear equations

In this section it is assumed that X is separable.

Let $I=[0, T]\,(0<T<\infty)$ and f be a mapping of $I \times X$ into V^* which maps bounded sets into bounded sets such that its restriction to $I \times V$ is demicontinous from $I \times V$ to V^* and satisfies

$$\operatorname{Re}(f(t, u)-f(t, v), u-v) \leqq 0, \qquad u, v \in V. \tag{4.1}$$

Under these assumptions together with those of the preceding sections we consider a mild solution of the semilinear equation

$$u'(t) + A(t) u(t) = f(t, u(t)), \tag{4.2}$$

$$u(0) = \phi, \tag{4.3}$$

i.e., the solution of

$$u(t) = U(t, 0)\phi + \int_0^t U(t, s) f(s, u(s)) ds. \tag{4.4}$$

Theorem 4.1. *Under our hypotheses the mild solution of* (4.2)–(4.3) *exists in* $C(I; X) \cap L^2(I; V)$. *The solution is unique and the mapping* $\phi \mapsto u$ *is continuous from* X *to* $C(I; X) \cap L^2(I; V)$.

Lemma 4.1. *If we set*

$$u(t) = U(t, 0)\phi + \int_0^t U(t, s) f(s) ds \tag{4.5}$$

for $\phi\in X$ *and* $f\in L^2(I;\, V^*)$, *then* $u\in C(I;\, X)\cap L^2(I;\, V)$,

$$\frac{1}{2}\,|u(t)|^2+\int_0^t \mathrm{Re}\, a(s;\, u(s),\, u(s))ds=\frac{1}{2}\,|\phi|^2+\int_0^t \mathrm{Re}\,(f(s),\, u(s))ds\,, \tag{4.6}$$

$$|u(t)|^2+\delta\int_0^t ||u(s)||^2 ds\leqq|\phi|^2+\frac{1}{\delta}\int_0^t ||f(s)||_*^2 ds\,. \tag{4.7}$$

Proof. First assume that $f\in C^1(I;\, V^*)$. Then $u\in C((0,\, T];\, V)\cap C^1((0,\, \mathrm{T}];\, V^*)$ and $u'(t)+A(t)u(t)=f(t)$, $t>0$. Hence $|u(t)|^2$ is differentiable and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\,|u(t)|^2 &= \mathrm{Re}\,(u'(t),\, u(t))\\ &= \mathrm{Re}\,(-A(t)u(t)+f(t),\, u(t))\\ &= -\mathrm{Re}\, a(t;\, u(t),\, u(t))+\mathrm{Re}\,(f(t),\, u(t))\,.\end{aligned}$$

For $0<\varepsilon<t\leqq T$ we get by integration

$$\frac{1}{2}\,|u(t)|^2+\int_\varepsilon^t \mathrm{Re}\, a(s;\, u(s),\, u(s))ds=\frac{1}{2}\,|u(\varepsilon)|^2+\int_\varepsilon^t \mathrm{Re}\,(f(s),\, u(s))ds\,, \tag{4.8}$$

so that

$$|u(t)|^2+\delta\int_\varepsilon^t ||u(s)||^2 ds\leqq|u(\varepsilon)|^2+\frac{1}{\delta}\int_\varepsilon^t ||f(s)||_*^2 ds. \tag{4.9}$$

It follows from Lemmas 3.4 and 3.5 that $u(\varepsilon)\to\phi$ in X as $\varepsilon\to 0$. Hence letting $\varepsilon\to 0$ in (4.8) and (4.9) we get $u\in L^2(I;\, V)$ and (4.6), (4.7). For a general $f\in L^2(I;\, V^*)$ we take a sequence $\{f_n\}\subset C^1(I;\, V^*)$ tending to f in $L^2(I;\, V^*)$ and let u_n be a function defined by (4.5) with f replaced by f_n. Applying (4.7) to u_n-u_m we get

$$|u_n(t)-u_m(t)|^2+\delta\int_0^t ||u_n(s)-u_m(s)||^2 ds\leqq\frac{1}{\delta}\int_0^t ||f_n(s)-f_m(s)||_*^2 ds$$

which implies that $\{u_n\}$ is a Cauchy sequence both in $C(I;\, X)$ and $L^2(I;\, V)$. It is easily seen that $u_n(t)\to u(t)$ in V^* for each t, and hence $u_n\to u$ both in $C(I;\, X)$ and $L^2(I;\, V)$. (4.6) and (4.7) follow from the corresponding relations for u_n.

Let Λ be the operator defined by $(\Lambda u,\, v)=((u,\, v))$. It is rather well-known that Λ is a positive definite self-adjoint operator both in X and V^*. It is not difficult to show that the domain of $\Lambda^{1/2}$ coincides with V(resp. X) when it is considered as an operator in X(resp. V^*). Hence $I_n=(1+n^{-1}\Lambda^{1/2})^{-1}$ is a contraction and converges strongly to the indetity as $n\to\infty$ in either space. Furthermore I_n maps X onto V and V^* onto X. We can show without difficulty

$$(I_n l, f)=(l,\, I_n f) \text{ for } l\in V^* \text{ and } f\in X\,. \tag{4.10}$$

Put $f_n(t,\, u)=I_n f(t,\, I_n u)$. Then f_n is demicotinuous form $I\times X$ to X and maps bounded sets of $I\times X$ into bounded sets of X. Furthermore in view of (4.1) and (4.10) we have

$$\operatorname{Re}(f_n(t, u)-f_n(t, v), u-v)\leqq 0, \ u, v\in X. \tag{4.11}$$

Take a regularizing function $j\in C_0^\infty(-\infty, \infty)$ with $j(t)\geqq 0$, $j(t)=0$ for $|t|\geqq 1$ and $\int_{-\infty}^{\infty} j(t)dt=1$, and let $j_n(t)=nj(nt)$. The sesquilinear form defined by

$$a_n(t; u, v) = \int_{-\infty}^{\infty} j_n(t-s)\, a(s; u, v)ds \tag{4.12}$$

satisfies the conditions (1.1)–(1.3) with the same constants M, δ, K and ρ. Let $A_n(t)$ be the operator associated with (4.12). Then $A_n(t)^{-1}$ is a continuously differentiable function of t with values in the space of bounded operators in X. Thus we may apply Theorem 4 of T. Kato [3] to the problem (4.2)–(4.3) with $A_n(t)$ and $f_n(t, u)$ instead of $A(t)$ and $f(t, u)$, and the existence of the unique mild solution follows:

$$u_n(t) = U_n(t, 0)\phi+\int_0^t U_n(t, s)f_n(s, u_n(s))ds\,, \tag{4.13}$$

where $U_n(t, s)$ is the evolution operator of $u'+A_n(t)u=0$.

Lemma 4.2. *$\|(A_n(t)-A(t))u\|_*\leqq Kn^{-\rho}\|u\|$ for $u\in V$.*

Proof. We have only to notice (3.1) and

$$a_n(t; u, v)-a(t; u, v) = \int_{-\infty}^{\infty} j_n(t-s)(a(s; u, v)-a(t; u, v))ds\,.$$

Lemma 4.3. *$|U_n(t, s)\phi- U(t, s)\phi|\leqq Cn^{-\rho}|\phi|$ for $\phi\in X$.*

Proof. It is obvious that the statement of Lemma 3.5 remain to hold for $U_n(t, s)$ with the same constants. So we obtain the above estimate from Lemmas 3.5, 4.2 and

$$U_n(t, s)\phi- U(t, s)\phi = \int_s^t U_n(t, r)(A(r)-A_n(r))U(r, s)\phi dr.$$

In view of Lemma 4.1 and (4.11) we have

$$\begin{aligned}
&\frac{1}{2}|u_n(t)|^2-\frac{1}{2}|\phi|^2+\int_0^t \operatorname{Re} a_n(s; u_n(s), u_n(s))ds \\
&= \operatorname{Re}\int_0^t (f_n(s, u_n(s)), u_n(s))ds \\
&= \operatorname{Re}\int_0^t (f_n(s, u_n(s))-f_n(s, 0), u_n(s))ds+\operatorname{Re}\int_0^t (f_n(s, 0), u_n(s))ds \\
&\leqq \operatorname{Re}\int_0^t (f_n(s, 0), u_n(s))ds\,,
\end{aligned}$$

so that

$$|u_n(t)|^2+\delta\int_0^t\|u_n(s)\|^2 ds$$
$$\leqq |\phi|^2+\frac{1}{\delta}\int_0^t\|f_n(s, 0)\|_*^2 ds\leqq |\phi|^2+\frac{1}{\delta}\int_0^t\|f(s, 0)\|_*^2 ds\,.$$

This shows that $\{u_n\}$ is bounded both in $C(I; X)$ and $L^2(I; V)$, and so is $g_n(t)=f_n(t, u_n(t))$ in $L^\infty(I; V^*)$. Consequently we can select their subsequences which we again denote by $\{u_n\}$, $\{g_n\}$ for simplicity such that $u_n \rightharpoonup u$ in $L^2(I; V)$ and $g_n \rightharpoonup g$ in $L^2(I; V^*)$ where $\rightharpoonup$ denotes the weak convergence in the corresponding spaces.

We know that $U(t, s)^*$ is the evolution operator of the adjoint equation

$$-v'(s)+A(s)^*v(s)=0 \tag{4.14}$$

by Theorem 3.1 and has the same properties as those of $U(t, s)$. In particular $U(t, s)^*$ is a bounded operator on V^* to V if $t>s$.

If we define

$$(Uf)(t)=\int_0^t U(t, s)f(s)ds\,, \tag{4.15}$$

$$(U^*f)(s)=\int_s^T U(t, s)^*f(t)dt \tag{4.16}$$

for $f\in L^2(I; V^*)$, then U and U^* are both bounded operators on $L^2(I; V^*)$ into $C(I; X)\cap L^2(I; V)$ by Lemma 4.1 and the remark just mentioned. Similarly we define U_n and U_n^* by (4.15) and (4.16) with $U(t, s)$ and $U(t, s)^*$ replaced by $U_n(t, s)$ and $U_n(t, s)^*$, respectively.

Lemma 4.4. *For each $f\in L^2(I; V^*)$, $U_nf\to Uf$ and $U_n^*f\to U^*f$ in $C(I: X)\cap L^2(I; V)$ as $n\to\infty$.*

Proof. Let $v_n=U_nf$ and $v=Uf$. Arguing as in the proof of Lemma 4.1 we get

$$|v_n(t)-v(t)|^2+\delta\int_0^t\|v_n(s)-v(s)\|^2 ds\leqq\frac{1}{\delta}\int_0^t\|(A_n(s)-A(s))v(s)\|_*^2 ds\,.$$

The right hand side tends to 0 as $n\to\infty$ by Lemma 4.2 and $v\in L^2(I; V)$.

Lemma 4.5 $u(t)=U(t, 0)\phi+\int_0^t U(t, s)g(s)ds.$

Proof. We write

$$(U_ng_n)(t)-(Ug)(t)=\int_0^t(U_n(t, s)-U(t, s))g_n(s)ds+\int_0^t U(t, s)(g_n(s)-g(s))ds\,.$$

The first term on the right tends weakly to 0 in $L^2(I; V)$ as $n\to\infty$ by Lemma 4.2.

Since U is a bounded operator on $L^2(I; V^*)$ to $L^2(I; V)$ the second term also goes to 0. Moreover by Lemma 4.3 $U_n(t, 0)\phi \to U(t, 0)\phi$ in X. Thus we obtain the desired identity letting $n\to\infty$ in (4.13).

$$\text{Let } f\in L^2(I; V^*) \text{ and } w(t) = U(t, 0)\phi+\int_0^t U(t, s)f(s)ds\,.$$

Arguing as in the proof of Lemma 4.1 we find

$$\begin{aligned}&\frac{1}{2}\,|u_n(T)-w(T)|^2+\mathrm{Re}\int_0^T a_n(t; u_n(t)-w(t), u_n(t)-w(t)dt\\ &= -\mathrm{Re}\int_0^T((A_n(t)-A(t))w(t), u_n(t)-w(t))dt \qquad (4.17)\\ &\quad+\mathrm{Re}\int_0^T(f_n(t, u_n(t))-f(t), u_n(t)-w(t))dt\,.\end{aligned}$$

Noting here that

$$\begin{aligned}&(f_n(t, u_n(t))-f(t), u_n(t)-w(t))\\ &\quad= (f(t, I_nu_n(t))-f(t, w(t)), I_nu_n(t)-w(t))+(I_nf(t)-f(t), u_n(t)-w(t))\\ &\qquad+(f(t, I_nu_n(t))-f(t), w(t)-I_nw(t))+(f(t, w(t))-f(t), I_nu_n(t)-w(t))\,,\end{aligned}$$

we get from (4.1) and (4.17)

$$\begin{aligned}0\leqq&-\mathrm{Re}\int_0^T((A_n(t)-A(t))w(t), u_n(t)-w(t))dt\\ &+\mathrm{Re}\int_0^T(I_nf(t)-f(t), u_n(t)-w(t))dt\\ &+\mathrm{Re}\int_0^T(f(t, I_nu_n(t))-f(t), w(t)-I_nw(t))dt \qquad (4.18)\\ &+\mathrm{Re}\int_0^T(f(t, w(t))-f(t), I_nu_n(t)-w(t))dt\,.\end{aligned}$$

Since $\{f(\cdot, I_nu_n(\cdot))\}$ is bounded in $L^\infty(I; V^*)$ and $f(\cdot, w(\cdot))\in L^2(I; V^*)$, we get

$$0\leqq\mathrm{Re}\int_0^T(f(t, w(t))-f(t), u(t)-w(t))dt \qquad (4.19)$$

by letting $n\to\infty$ in (4.18).

Let h be an arbitrary element of $L^2(I; V^*)$ and apply (4.19) to $g-n^{-1}h$ instead of f. Then in view of Lemma 4.5 we get

$$\mathrm{Re}\int_0^T(f(t, u(t)-n^{-1}z(t))-g(t)+n^{-1}h(t), z(t))dt\geqq 0 \qquad (4.20)$$

where $z=Uh$. Since $z\in C(I; X)\cap L^2(I; V)$, $f(t, u(t)-n^{-1}z(t))$ is uniformly bounded in V^* and converges to $f(t, u(t))$ in the weak topology of V^* for almost every $t\in I$. Hence letting $n\to\infty$ in (4.20) we obtain

$$\mathrm{Re}\int_0^T (h(t), (U^*\tilde{f})(t))dt = \mathrm{Re}\int_0^T (\tilde{f}(t), z(t))dt \geqq 0$$

where $\tilde{f}(t)=f(t, u(t))-g(t)$. In virtue of the arbitrariness of h we get

$$U^*\tilde{f} = 0. \tag{4.21}$$

That u is the desired solution of (4.4) follows from (4.21), Lemma 4.5 and the following lemma since the lemma clearly remains valid if U is replaced by U^*.

Lemma 4.6. *$Uf=0$, $f\in L^2(I; V^*)$, implies $f=0$.*

Proof. Suppose $Uf=0$, i.e.,

$$\int_0^t U(t, s)f(s)ds \equiv 0. \tag{4.22}$$

Operating $U(t', t)$ with $t\leqq t'$, t': rational, to both sides of (4.22), we get

$$\int_0^t U(t', s)f(s)ds \equiv 0, \qquad 0\leqq t\leqq t',$$

from which it follows that $U(t', t)f(t)=0$ at almost all $t\in[0, t']$. Hence there exists a null set N of I such that for every $t\in I-N$ and rational number $t'>t$, $t'\in I$, $f(t)$ is an element of V^* and $U(t', t)f(t)=0$. Letting $t'\to t$ we get $f(t)=0$ almost everywhere in I.

REMARK. In case when V is separable, the above lemma follows from Théorème 1.1. of Chap. IV of J.L. Lions [4] since as we shall show in the next section Uf is the solution of (1.4)–(1.5) with $f=0$, $\phi=0$ in a certain weak sense.

End of the proof of Theorem 4.1. Let u_1 and u_2 be the solutions of (4.4) with ϕ_1 and ϕ_2 in place of ϕ. Then arguing as in the proof of Lemma 4.1 we get

$$|u_1(t)-u_2(t)|^2+2\delta\int_0^t ||u_1(s)-u_2(s)||^2 ds \leqq |\phi_1-\phi_2|^2.$$

The assertion of the theorem is an immediate consequence of this inequality.

5. Another proof of a theorem of J. L. Lions

In this section we assume (1.1), (1.2) and that $a(t; u, v)$ is a measurable function of t for each fixed $u, v\in V$. Moreover we suppose that V is separable.

We give another proof of the following theorem established by J.L. Lions ([4]: p.46).

Theorem 5.1. *Under the the assumptions indicated above, for any $f\in L^2(I; V^*)$ and $\phi\in X$, there exists a function $u\in L^2(I; V)$ satisfying*

$$\int_0^T a(t; u(t), \psi(t))dt-\int_0^T (u(t), \psi'(t))dt = \int_0^T (f(t), \psi(t))dt+(\phi, \psi(0)) \tag{5.1}$$

for any $\psi\in\Phi$, *where*

$$\Phi=\{\psi;\ \psi\in C(I;V),\ \psi'\in C(I;X),\ \psi(T)=0\}.$$

Let $a_n(t;u,v)$ and $A_n(t)$ be the sesquilinear form and its associated operator defined in the previous section. Similarly let $U_n(t,s)$ be the evolution operator of $u'(t)+A_n(t)u(t)=0$ and

$$(U_nf)(t)=\int_0^t U_n(t,s)f(s)ds$$

for $f\in L^2(I;V^*)$.

Lemma 5.1. *If* $f\in L^2(I;V^*)$, *then* $u_n=U_nf$ *satisfies*

$$\int_0^T a_n(t;u_n(t),\psi(t))dt-\int_0^T(u_n(t),\psi'(t))dt=\int_0^T(f(t),\psi(t))dt \tag{5.2}$$

for any $\psi\in\Phi$.

Proof. It is easy to show that the assertion of the Lemma is true for $f\in C^1(I;V^*)$. For a general $f\in L^2(I;V^*)$ we can establish the same conclusion approximating f by a sequence of functions in $C^1(I;V^*)$.

Lemma 5.2. *If f is a Bochner integrable function in $a\leqq t\leqq b$ with values in some Banach space, then at almost every t in $[a,b]$*

$$\lim_{h\to0}\frac{1}{h}\int_0^h\|f(t+s)-f(t)\|ds=0.$$

Proof. If f is a numerical valued function, this is a known theorem of the theory of Lebesgue integral. From the proof of the theorem it is obvious that the same result remains valid for functions of the kind stated in the Lemma.

Proof of Theorem 5.1. For each $u\in V$

$$\|A_n(t)^*u-A(t)^*u\|_*\leqq Cn\int_{-1/n}^{1/n}\|A(t-s)^*u-A(t)^*u\|_*ds, \tag{5.3}$$

and hence in virtue of Lemma 5.2 we have $A_n(t)^*u\to A(t)^*u$ in V^* as $n\to\infty$ except for some null set $N\subset I$. By the separability of V and $\|A_n(t)^*u\|_*\leqq M\|u\|$ it is easy to show that we can take N independently of an individual $u\in V$. Therefore by Lebesgue's theorem we get

$$\int_0^T\|A_n(t)^*\psi(t)-A(t)^*\psi(t)\|_*^2dt\to0\ (n\to\infty) \tag{5.4}$$

for any $\psi\in\Phi$. Now setting $u_n(t)=U_n(t,0)\phi+(U_nf)(t)$, we get by Lemma 4.1 that $u_n\in C(I;X)\cap L^2(I;V)$ and $\{u_n\}$ is bounded in $L^2(I;V)$, and we can find a subsequence which we write again as $\{u_n\}$ converging weakly to some u in $L^2(I;V)$. Hence noting

$$a_n(t; u_n(t), \psi(t)) = (u_n(t), A_n(t)^*\psi(t)),$$
$$a(t; u(t), \psi(t)) = (u(t), A(t)^*\psi(t))$$

and using (5.4) we get

$$\int_0^T a_n(t; u_n(t), \psi(t))dt \to \int_0^T a(t; u(t), \psi(t))dt \quad (n\to\infty), \tag{5.5}$$

$$\int_0^T (u_n(t), \psi'(t))dt \to \int_0^T (u(t), \psi'(t))dt \quad (n\to\infty). \tag{5.6}$$

Since $U_n(t, 0)\phi$ is differentiable in X in $0<t\leqq T$ and $(d/dt)U_n(t, 0)\phi = -A_n(t)U_n(t, 0)\phi$, we get integrating by parts

$$-\int_0^T (U_n(t, 0)\phi, \psi'(t))dt + \int_0^T a_n(t; U_n(t, 0)\phi, \psi(t))dt = (\phi, \psi(0))$$

for any $\psi \in \Phi$. In virtue of this equality and Lemma 5.1 we obtain

$$\int_0^T a_n(t; u_n(t), \psi(t))dt - \int_0^T (u_n(t), \psi'(t))dt$$
$$= \int_0^T (f(t), \psi(t))dt + (\phi, \psi(0)). \tag{5.7}$$

(5.1) follows immediately from (5.5), (5.6) and (5.7).

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