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EXTENDIBLE VECTOR BUNDLES OVER LENS SPACES MOD 3

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1. Introduction. In [5] Schwarzenberger investigated the problem of determing whether a real vector bundle over the real projective space RP^n can be extended to a real vector bundle over RP^m $(n<m)$. In [3], he also investigated the case of the complex tangent bundle of the complex projective space.

The purpose of this note is to prove the non-extendibility of a bundle over lens spases mod 3 by making use of Schwarzenberger's technique ([5]).

Let S^{2n+1} be the unit $(2n+1)$-sphere. That is

$$S^{2n+1}=\{(z_0,\cdots,z_n);\sum_{i=0}^{n}|z_i|^2=1,\ z_i\in C\quad\text{for all } i\}$$

Let γ be the rotation of S^{2n+1} defined by

$$\gamma(z_0,\cdots,z_n)=(e^{2\pi i/p}z_0\cdots,e^{2\pi i/p}z_n)\,.$$

Then γ generates the differentiable transformation group Γ of S^{2n+1} of order p, and lens space mod p is defined to be the orbit space $L^n(p)=S^{2n+1}/\Gamma$ It is a compact differentiable $(2n+1)$-manifold without boundary and $L^n(2)=RP^{2n+1}$. The Grothendieck rings $\widetilde{KO}(L^n(p))$, $\tilde{K}(L^n(p))$ were determined by T. Kambe [4]. We recall them in **2**. Let $\{z_0,\cdots,z_n\}\in L^n(p)$ denote the equivalence class of $(z_0,\cdots,z_n)\in S^{2n+1}$. $L^n(p)$ is naturally embedded in $L^{n+1}(p)$ by identifying $\{z_0,\cdots,z_n\}$ with $\{z_0,\cdots,z_n,0\}$. Hence $L^n(p)$ is embedded in $L^m(p)$ for $n<m$. Throughout this note we suppose $p=3$. Now we state our theorems which shall be proved in **3** and **4**.

Let ζ be any t-dimensional real bundle over $L^n(3)$. Let $p(\zeta)$ be the mod 3 Pontryagin class of ζ

$$p(\zeta)=\sum_j p_j(\zeta)\ \text{where}\ p_j(\zeta)=(-1)^j C_{2j}(\zeta\otimes C)\quad\text{mod } 3\,.$$

From the property of the cohomology algebra $H^*(L^n(3)\colon Z_3)$, we have

$$p_j(\zeta)=d_jx^{2j}\,,$$

where $d_j \in Z_3$ and x is a generator of $H^2(L^n(3): Z)$. Then there exists an integer s such that

$$(1) \qquad p(\zeta) = 1 + d_1 x^2 + \cdots + d_s x^{2s} \quad \text{for} \quad 0 \leqq 2s \leqq t\,.$$

Then we have the following

Theorem 1. *Let ζ be a t-dimensional real vector bundle over $L^n(3)$. If $2t < n+1$, then we have*

$$p(\zeta) = (1+x^2)^s \quad \textit{mod } 3 \textit{ for some integer } s \quad 0 \leqq 2s \leqq t\,.$$

Corollary 2. *Under the assumptions of Theorem 1,*

$$p(\zeta) = p(\eta_{L^n} \oplus \cdots (s) \cdots \oplus \eta_{L^n}) \quad \textit{for some } 0 \leqq 2s \leqq t\,,$$

where we denote by $\oplus$ a Whitney sum of η_{L^n}. (See 2 for the definition of η_{L^n}.)

For a pair (X, Y) of compact spaces, a bundle ζ_Y over Y is said to be *extendible to X* provided there exists a bundle ζ_X over X such that

$$\zeta_X|_Y \cong \zeta_Y\,,$$

where we denote by $|_Y$ the restriction to Y.

Let a be a real number. We denote by $[a]$ the integral part of a. Let b be an integer. We denote by $\nu_3(b)$ an integer q such that

$$b = r \cdot 3^q, \quad \text{where } (r, 3) = 1\,.$$

For integers t and m, define

$$\beta_3(t, m) = \mathrm{Min}\left[\left(i - \left[\frac{i}{2}\right] - 1\right) - \nu_3\left(i - \left[\frac{i}{2}\right]\right) + \nu_3\left\{\binom{i-[i/2]}{[i/2]}\right\}\right]$$

where $t < i < m$, $i \equiv 0 \bmod 2$ and $i \equiv 1 \bmod 6$.

Theorem 3. *Assume that n, m and t are the positive integers such that*

$$(2) \qquad 2t < m+1$$

$$(3) \qquad n \not\equiv 0 \textit{ mod } 4$$

$$(4) \qquad m \not\equiv 0 \textit{ mod } 4$$

$$(5) \qquad \left[\frac{m}{2}\right] \geqq \left[\frac{n}{2}\right] + \beta_3(t, m)\,.$$

Let ζ be a t-dimentional real vector undle over $L^n(3)$ which is extendible to $L^m(3)$ $(n < m)$. Then ζ is stably equivalent to

$$\eta_{L^n}\oplus\cdots(s)\cdots\oplus\eta_{L^n} \quad \textit{for some integer } s\ (0\leqq 2s\leqq t)\,.$$

As an application of Th. 3, we obtain the following

Theorem 4. *Let ζ be a t-dimensional real vector bundle over $L^n(3)$ ($n\not\equiv 0$ mod 4). Assume that ζ is stably equivalent to*

$$\eta_{L^n}\oplus\cdots(s)\cdots\oplus\eta_{L^n} \quad \textit{for some } s>\left[\frac{t}{2}\right].$$

Then ζ is not extendible to $L^{\phi(t,n)}(3)$, where

$$\phi(t,n)=\mathrm{Min}\left\{m\geqq 2t;\ m\not\equiv 0 \ \mathit{mod}\ 4,\ \left[\frac{m}{2}\right]-\beta_3(t,m)\geqq\left[\frac{n}{2}\right]\right\}.$$

Next we show

Theorem 5. *The tangent bundle $\tau(L^n(3))$ of $L^n(3)$ is not extendible to $L^{\phi(2n+1,n)}(3)$ for $n\not\equiv 0$ mod 4. And $\tau(L^n(3))$ is not extendible to $L^{4n+2}(3)$ ($n\not\equiv 0$ mod 4).*

2. The structure of $\widetilde{KO}(L^n(p))$. The structure of $\widetilde{KO}(L^n(p))$ is stated as follows [4]. Let CP^n be the complex projective space of complex n-dimension. Let η be the canonical complex line bundle over CP^n, $r(\eta)$ the real restriction of η. Consider the natural projection

$$\pi\colon L^n(p)\to CP^n\,.$$

Define $\eta_{L^n}=\pi^*(r(\eta))\in KO(L^n(p))$ where $\pi^*\colon KO(CP^n)\to KO(L^n(p))$ is the induced homomorphism of π. Let $\bar\sigma_n$ denote the stable class of η_{L^n}, i.e., $\bar\sigma_n=\eta_{L^n}-2\in\widetilde{KO}(L^n(p))$. We recall $\tau_{L^n}\oplus 1=(n+1)\eta_{L^n}$ where τ_{L^n} is the tangent bundle of $L^n(p)$. The theorem of T. Kambe (Th. 2, [4]) is as follows:

Theorem (*Kambe*). *Let p be an odd prime, $q=(p-1)/2$ and $n=s(p-1)+r$ ($0\leqq r<p-1$). Then*

$$\widetilde{KO}(L^n(p))\cong\begin{cases}(Z_{p^{s+1}})^{[r/2]}+(Z_{p^s})^{q-[r/2]}\cdots(\textit{if } n\not\equiv 0 \ \mathit{mod}\ 4)\\ Z_2+(Z_{p^{s+1}})^{[r/2]}+(Z_{p^s})^{q-[r/2]}\cdots(\textit{if } n\equiv 0 \ \mathit{mod}\ 4)\end{cases}$$

and the direct summand $(Z_{p^{s+1}})^{[r/2]}$ and $(Z_{p^s})^{q-[r/2]}$ are additively generated by $\bar\sigma_n,\cdots,\bar\sigma_n^{[r/2]}$ and $\bar\sigma_n^{[r/2]+1},\cdots,\bar\sigma_n^{q}$ respectively. Moreover its ring structure is given by

$$\bar\sigma_n^{q+1}=\sum_{i=1}^{q}\frac{-(2q+1)}{(2i-1)}\binom{q+i-1}{2i-2}\bar\sigma_n^{i},\ \bar\sigma_n^{[n/2]+1}=0\,.$$

In the theorem, $(Z_a)^b$ indicates the direct sum of b-copies of cyclic group of order a. Let $p=3$ in the above theorem. If $n\not\equiv 0$ mod 4 then

$$\widetilde{KO}(L^n(3)) \cong Z_{3^s}, \quad s=\left[\frac{n}{2}\right].$$

and Z_{3^s} is generated by $\bar{\sigma}_n$. Its ring structure is given by

$$\bar{\sigma}_n{}^2=(-3)\bar{\sigma}_n, \quad \bar{\sigma}^{s+1}=0.$$

3. The proofs of Theorem 1 and Corollary 2. From Th. 11.3 in [2], we obtain the following equality. (For the proof, see Proposition 5, in the last part of this section.) Let $\mathcal{P}_3^k$: $H^q(L^n(3): Z_3)\to H^{q+4k}(L^n(3): Z_3)$ the k-th reduced power operation mod 3. Then we have

$$\mathcal{P}_3^k(p_s(\zeta))=(\sum_{n+m=k} p_n(\zeta)p_m(\zeta))p_s(\zeta)+\sum_{l>s} p_l(\zeta)\,(\cdots) \tag{6}$$

for $0\leqq k\leqq s$.

Let s be an integer such as (1) in **1**. Since $d_j\equiv 0$ for all $j>s$ and $d_s\not\equiv 0$, then (6) gives

$$\mathcal{P}_3^k(p_s(\zeta))=(\sum_{n+m=k} p_n(\zeta)p_m(\zeta))p_s(\zeta). \tag{7}$$

For an element x^{2s} of $H^s(L^n(3): Z_3)$, we have

$$\mathcal{P}_3^k(x^{2s})=\binom{2s}{k}x^{2s+2k} \quad\text{and}\quad d_s\binom{2s}{k}x^{2s+2k}=(\sum_{n+m=k} d_n d_m)d_s x^{2s+2k}.$$

From $\quad 2s+2k\leqq 4s\leqq 2t<n+1$, $x^{2s+2k}\neq 0$. Hence $\binom{2s}{k}=\sum_{n+m=k} d_n d_m$.

By induction, we obtain $d_j\equiv\binom{s}{j}$ mod 3. Therefore

$$\begin{aligned} p(\zeta) &= 1+\binom{s}{1}x^2+\cdots+\binom{s}{s}x^{2s} \mod 3 \\ &= (1+x^2)^s \mod 3. \end{aligned}$$

The proof of Theorem 1 is completed if we prove Proposition 5. Now, it is well known that the bundle η_{L^n} over $L^n(3)$ has the total Pontryagin class mod 3 $p(\eta_{L^n})=1+x^2$. Thus the proof of Corollary 2 is completed.

Now, in order to prove the formula (6) in the proof of Theorem 1, we conisder a following symmetric polynomial. Let $\sum x_1^p x_2^p\cdots x_k^p x_{k+1}\cdots x_s$ be a homogeneous symmetric polynomial in variables $x_1, x_2, \cdots, x_t$ of degree $N=(p-1)k+s$ where p, k and N are positive integers.

To prove (6), we show the following propositions.

Proposition 1.

$$\sum x_1^2 x_2^2 \cdots x_k^2 x_{k+1} \cdots x_s = \sum_{i=0}^{k} A(i)\sigma_{k-i}\sigma_{s+i}$$

where $A(i)=(-1)\sum_{j=1}^{i}\binom{s-k+2i}{j} A(i-j)$, $1 \leqq i \leqq k$ *and* $A(0)=1$.

Proof. Ptu $f(k, s)= \sum x_1^2 \cdots x_k^2 x_{k+1} \cdots x_s$. By an easy calculation,

$$f(k, s) = (\sum x_1 \cdots x_k)(\sum x_1 \cdots x_s) - \sum_{j_1=1}^{k}\binom{s-k+2j_1}{j_1} f(k-j_1, s+j_1)\,. \tag{8}$$

By making use of (8) repeatedly, we have

$$f(k, s) = \sigma_k \sigma_s + \sum_{l=1}^{k} F_l$$

where $F_l=(-1)^l \sum_{j_1=1}^{k} \sum_{j_2=1}^{k-j_1} \cdots \sum_{j_l=1}^{k-\sum_{i=1}^{l-1} j_i} \binom{s-k+2j_1}{j_1} \cdots \binom{s-k+2\sum_{i=1}^{l} j_i}{j_l} \sigma_{k-\sum_{i=1}^{l} j_i} \sigma_{s+\sum_{i=1}^{l} j_i}$ and $1 \leqq j_l \leqq k - \sum_{i=1}^{l-1} j_i \leqq k-(l-i)$. If $l=k$, then $k= \sum_{i=1}^{k} j_i$.

Let $A_l(i)$ be the coefficient of $\sigma_{k-i}\sigma_{s+i}$ in F_l, then

$$A_l(i) = (-1)^l \sum_{n=1}^{i-1} \sum_{j_1+\cdots+j_{l-1}=i-n} \binom{s-k+2j_1}{j_1} \cdots \binom{s-k+2\sum_{i=1}^{l-1} j_i}{j_{l-1}} \binom{s-k+2i}{n}\,.$$

Put $A(i)=A_1(i)+\cdots+A_i(i)$. Then

$$A(i) = (-1)\sum_{n=1}^{i-1}\binom{s-k+2i}{n}\sum_{l=2}^{i-n+1} A_{l-1}(i-n) - \binom{s-k+2i}{i}\,.$$

Since $\sum_{l=2}^{j+1} A_{l-1}(j)$ is a coefficient $A(j)$ of $\sigma_{k-j}\sigma_{s+j}$ in $f(k, s)$, we have

$$A(i) = (-1)\sum_{j=1}^{i}\binom{s-k+2i}{j} A(i-j)\,.$$

This completes the proof of Proposition 1.

Proposition 2.

$$\sum x_1^2 \cdots x_k^2 = \sum_{i=0}^{k} A(i)\sigma_{k-i}\sigma_{k+i}$$

where (a) $\sum_{j=0}^{i} A(i-j)\binom{2i}{j} = 0$ *and* $A(0)=1$

(b) $A(i) \equiv (-1)^{i+1}$ *mod* 3 *for* $i \neq 0$.

Proof. The part of (a) is completed by Proposition 1. The proof of (b) is obtained by induction. For $i=1, 2$, $A(1)=(-1)\binom{2}{1}\equiv 1$ and $A(2)=2$. Assuming the equation (b) for integers $i\leqq 2q$, we have

$$A(i+1)=2\sum_{j=1}^{i}\binom{2i+2}{j}(-1)^{i-j+2}+2\binom{2i+2}{i+1}$$

By making use of $\sum_{j=1}^{i}\binom{2i+2}{j}(-1)^{j+1}\equiv\binom{2j+2}{i+1}+2\binom{2i+2}{2i+2}+2\binom{2i+2}{0}+2\binom{2i+2}{i+1}$, we have

$$A(i+1)\equiv 1 \bmod 3 .$$

Assuming the equation (b) for integers $i\leqq 2q+1$, we can obtain $A(i+1)\equiv 2 \bmod 3$. Thus Proposition 2 is obtained by induction.

Proposition 3.

$$\sum x_1^3x_1^3\cdots x_k^3x_{k+1}\cdots x_s=(\sum x_1^2\cdots x_k^2)\sigma_s-\sum_{l>s}\sigma_l(\cdots) .$$

Proof. Put $f(a, b)=\sum x_1^3\cdots x_a^3x_{a+1}^2\cdots x_{a+b}^2x_{a+b+1}\cdots x_c$ with $c=N-2a-b$. By calculation, we have the following equality;

$$(9)\qquad f(k, 0)=(\sum x_1\cdots x_k)\sigma_s-\sum_{\alpha_1=1}^{k}f(k-\alpha_1, \alpha_1) .$$

Define $a_0=k$, $b_0=0$ and $c_0=N-2a_0-b_0$. Then (9) is reformed as follows:

$$(10)\quad f(a_0, b_0)=f(0, a_0)\sigma_{c_0}-\sum_{\alpha_1=1}^{a_0}f(a_1, b_1) \text{ where } a_1=a_0-\alpha_1,\ b_1=\alpha_1+\beta_1(\beta_1=0)$$

Now for each term $f(a_1, b_1)$ in (10), we obtain

$$(11)\qquad f(a_1, b_1)=f(0, a_1)\sigma_{N-2a_1-b_1}-\sum_{\alpha_2=0}^{a_1}\sum_{\beta_2=0}^{b_1}A(\alpha_2, \beta_2)f(a_1-\alpha_2, \alpha_2+\beta_2)$$

for some integers $A(\alpha_2, \beta_2)$ and $A(0, b_1)=0$. We can inductively define two sequences $\{a_i\}$, $\{b_i\}$ satisfying the followings

$$(12)\qquad f(a_{i-1}, b_{i-1})=f(0, a_{i-1})\sigma_{N-2a_{i-1}-b_{i-1}}-\sum_{\alpha_i=0}^{a_{i-1}}\sum_{\beta_i=0}^{b_{i-1}}A(\alpha_i, \beta_i)f(a_i, b_i)$$

$$(13)\qquad a_i=a_{i-1}-\alpha_i,\ b_i=\alpha_i+\beta_i$$

with some integers $A(\alpha_i, \beta_i)$ and $A(0, b_{i-1})=0$.
Put $c_{i-1}=N-2a_{i-1}-b_{i-1}$. Then we have $s<c_1<c_2<\cdots<c_i<\cdots$. From (13), $a_{i+1}\leqq a_i$ for all i. Hence conisder the following cases:

(14) there exists an integer n such as $a_{i+1}<a_i$ for all $i\geqq n$,

(15) there exists an integer m such as $a_m=\cdots=a_i=\cdots$ for all $i\geqq m$.

If (14) is satisfied, then $a_q=0$ for some integer q. From (12) andProposition 1, we have

$$(16)\qquad f(a_{q-1}, b_{q-1}) = f(0, a_{q-1})\sigma_{c_{q-1}}\sum_{\beta_q=1}^{b_{q-1}} A(0, \beta_q)f(0, b_q)$$
$$= f(0, a_{q-1})\sigma_{c_{q-1}} - \sum_{\beta_q=0}^{b_{q-1}}\sum_{i=0}^{b_q} A(0, \beta_q)A(i)\sigma_{b_q-i}\sigma_{c_q+i}$$

If (15) is satisfied, then $b_i>b_{i+1}$ for all $i\geqq m$.
Therefore $b_r=0$ for some integer r. From (12) we have

$$f(a_{r-1}, b_{r-1}) = f(0, a_{r-1})\sigma_{a_{r-2}} - \sum_{\alpha_r=0}^{a_{r-1}} A(\alpha_r, 0)f(a_r, 0)\,.$$

Since $a_r<a_0$, the above discussion is also applied to $f(a_r, 0)$ in this case. Hence, by making use of (9) repeatedly, we have finally

$$f(k, 0) = f(0, k)\sigma_s - \sum_{l>s}(\cdots)\sigma_l\,.$$

From Proposition 2 and Proposition 3 we have the following

Proposition 4.

$$\sum x_1^3\cdots x_k^3 x_{k+1}\cdots x_s = (\sigma_k^2 + \sum_{j=1}^{k}(-1)^{j+1}\sigma_{k-j}\sigma_{k+j})\sigma_s + \sum_{l>s}(\cdots)\sigma_l\,.$$

Now, we can prove the formula (6) in the proof of Theorem 1.

Proposition 5. $\mathcal{P}_3^k(p_s(\zeta)) = (\sum_{n+m=k} p_n(\zeta)p_m(\zeta))p_s(\zeta) + \sum_{l>s} p_l(\zeta)(\cdots)\,.$

Proof. Let $C_i\in H^{2i}(B_{U(t)} : Z_3)$ be the i-th Chern class mod 3. By Th. 11.3 ([2]) and Proposition. 4, we have

$$(17)\qquad \mathcal{P}_3^k(C_i) = (C_k^2 + \sum_{j=1}^{k}(-1)^{j+1}C_{k-j}C_{k+j})C_i + \sum_{l>i}(\cdots)C_l\,.$$

Let $p_s(\zeta)$ be the s-th Pontrjagin class mod 3 of a real bundle ζ. Then $p_s(\zeta)=(-1)^s C_{2s}(\zeta\otimes C)$ mod 3 where $C_{2s}(\zeta\otimes C)$ is a $2s$-th Chern class of $\zeta\otimes C$. From (17) we obtain

$$\mathcal{P}_3^{2i}(p_s(\zeta)) = (p_i^2(\zeta) + 2\sum_{l=1}^{i} p_{i+l}(\zeta)p_{i-l}(\zeta))p_s(\zeta) + \sum_{l>s}(\cdots)p_l(\zeta)$$

and $\mathcal{P}_3^{2i+1}(p_s(\zeta)) = 2(\sum_{l=1}^{i} p_{i-l}(\zeta)p_{i+l}(\zeta))p_s(\zeta) + \sum_{l>s}(\cdots)p_l(\zeta)$. This completes the proof of Proposition 5.

4. Proofs of Theorem 3, 4 and 5. To prove Theorem 3, we discuss the following lemmas. The proofs of Lemma 1, 2 and 3 are omitted.

Lemma 1. *Let $A_0, A_1, \cdots, A_n$ be integers with $\nu_3(A_j)>0$ for all $j\neq n$ and $\nu_3(A_n)\geqq 0$. If $\nu_3(A_n)<\nu_3(A_j)$ for all $j\neq n$, then*

$$\nu_3(\sum_{j=0}^{n} A_j) = \nu_3(A_n)\,.$$

Lemma 2. *If r, s, a and u are positive integers with $s<a<3^u$ and $(r, 3)=1$ then following hold.*

$$\nu_3\left\{\binom{rs^u+s}{a}\right\} = \nu_3\left\{\binom{3^u+s}{a}\right\} \leqq \nu_3\left\{\binom{3^u}{a}\right\}. \tag{18}$$

$$\nu_3\left\{\binom{3^u}{a}\right\} = u-\nu_3(a)\,. \tag{19}$$

Lemma 3. *If u and n are positive integers, then*

$$\nu_3((3^u)!) = \frac{3^u-1}{2} \tag{20}$$

$$\nu_3((2n+1)!)\leqq n \tag{21}$$

$$\nu_3((2n)!)<n\,. \tag{22}$$

Put $A_j=(-1)^j(-3)^{i-j-1}\binom{q}{i-j}\cdot\binom{i-j}{j}$ $\left(j = 0, 1, \cdots, \left[\frac{i}{2}\right]\right)$ for some positive integer q, $i>2$ with $q>i-j$.

Lemma 4. *Let A_j be above integers. Then*

$$\nu_3(\sum_{j=0}^{[i/2]} A_j) = \nu_3(A_{[i/2]}) \quad \textit{for } i\equiv 1 \bmod 6 \textit{ and } i\equiv 0 \bmod 2\,.$$

Proof. If $i=2n$, then for each $l=1, 2, \cdots, n-1$

$$\nu_3(A_{n-l}) = (n+l-1)-\nu_3((2l)!)-\nu_3((n-l)!)+\nu_3(q)+\cdots+\nu_3(q-n-l+1)\,.$$

From Lemma 3 (22) $\nu_3(A_{n-l})-\nu_3(A_n)>l-\nu_3((2l)!)$. Then we have

$$\nu_3(A_j)>\nu_3(A_n) \text{ and } \nu_3(A_j)>0 \quad \text{for all } j\neq n\,.$$

Therefore by Lemma 1 we obtain $\nu_3(\sum_{j=0}^{[i/2]} A_j)=\nu_3(A_{[i/2]})$ for $i\equiv 0 \bmod 2$.
From Lemma 3 (21), we obtain

$$\nu_3(A_{n-l})-\nu_3(A_n)>\nu_3(n!)-\nu_3((n-l)!)>0$$

under the conditon $i\equiv 1 \bmod 6$, $\left[\frac{i}{2}\right]=n=3m$.

Now we prove the theorems.

Proof of Theorem 3. Let ζ' be the extension over $L^m(3)$ of ζ. By the structure of $\widetilde{KO}$-ring of the lens space ([4]), ζ' is stably equivalent to $q\eta_{L^m}$, for some $q\in \boldsymbol{Z}_{3^{[m/2]}}$. Since $\zeta'-t=q\bar{\sigma}_m\in\widetilde{KO}(L^m(3))$, we have

$$\zeta-t = q(i^*\eta_{L^m}-2)\in\widetilde{KO}(L^n(3)) \tag{23}$$

where $i^*\colon \widetilde{KO}(L^m(3))\to\widetilde{KO}(L^n(3))$ is the induced homomorphism of natural embedding $i\colon L^n(3)\to L^m(3)$. If $2q\leqq t$, then ζ is stably equivalent to $\eta_{L^n}\oplus\cdots(q)\cdots\oplus\eta_{L^n}$ for some integer q $(0\leqq 2q\leqq t)$. If $2q>t$, $\gamma^i(q\bar{\sigma}_m)=0$ for all $i>g.\dim(q\bar{\sigma}_m)$ ([1] Prop. 2·3). Since $t\geqq g.\dim(q\bar{\sigma}_m)$, we have

$$\gamma^i(q\bar{\sigma}_m)=0 \quad \text{for all } i>t\,. \tag{24}$$

According to the Theorem of Kambe ([4] Lemma 4.8),

$$\begin{aligned}\gamma_t(q\bar{\sigma}_m) &= (1+\bar{\sigma}_m(t-t^2))^q\\ &= \sum_{\alpha=0}^{2q}\Big(\sum_{j=0}^{[2/\alpha]}A_j\Big)\bar{\sigma}_m t^\alpha\end{aligned}$$

where $A_j=(-1)^j(-3)^{\alpha-j-1}\binom{q}{\alpha-j}\binom{\alpha-j}{j}$.

Then we have $\gamma^i(q\bar{\sigma}_m)=\sum_{j=0}^{[\alpha/2]}A_j\bar{\sigma}_m$. From (23),

$$\Big(\sum_{j=0}^{[\alpha/2]}A_j\Big)\bar{\sigma}_m = 0\in\widetilde{KO}(L^m(3)) = Z_{3^{[m/2]}} \quad\text{for all } i>t. \quad\text{Therefore}$$

$$\nu_3\Big(\sum_{j=0}^{[\alpha/2]}A_j\Big)\geqq\Big[\frac{m}{2}\Big] \quad\text{for all } i>t\,. \tag{24}$$

Now, according to Lemma 4, we have

$$\nu_3\Big(\sum_{j=0}^{[\alpha/2]}A_j\Big) = \nu_3(A_{[i/2]}) \quad\text{for } i>t \ (i\equiv 0 \bmod 2 \text{ and } i\equiv 1 \bmod 6)$$

And so we have

$$\Big(i-\Big[\frac{i}{2}\Big]-1\Big)+\nu_3\Big\{\binom{q}{i-[i/2]}\Big\}+\nu_3\Big\{\binom{i-[i/2]}{[i/2]}\Big\}\geqq\Big[\frac{m}{2}\Big] \quad\text{for } i>t,\ i\equiv 0 \bmod 2 \tag{25}$$

and $i\equiv 1 \bmod 6$.

Now the total Pontrjagin class mod 3 of $q\eta_{L^m}$ is given by the equation $p(q\eta_{L^m})=(1+x^2)^q$. Since $m>2t-1$, Theorem 1 implies that there exists an integer s such that

$$p(\zeta')=(1+x^2)^s,\quad 0\leqq 2s\leqq t\,.$$

Hence we have

$$(1+x^2)^q \equiv (1+x^2)^s \bmod 3, \text{ i.e.,}$$
$$1+\binom{q-s}{1}x^2+\cdots+\binom{q-s}{[m/2]}x^{2[m/2]} \equiv 1 \bmod 3\,.$$

This implies that there exists an integer u such that

$$q-s = 3^u r,\ (r, 3) = 1 \quad \text{and} \quad 3^u > [m/2]\,. \tag{26}$$

Then we obtain the following

$$\begin{aligned} \nu_3\left\{\binom{q}{i-[i/2]}\right\} &= \nu_3\left\{\binom{r3^u+s}{i-[i/2]}\right\} \\ &\leqq \nu_3\left\{\binom{3^u}{i-[i/2]}\right\} \quad \text{for } t<i<m \text{ (by Lemma 2)} \\ &= u-\nu_3(i-[i/2])\,. \end{aligned}$$

Hence from (25) $u+(i-[i/2]-1)-\nu_3(i-[i/2])+\nu_3\left\{\binom{i-[i/2]}{[i/2]}\right\} \geqq \left[\frac{m}{2}\right]$ for $t<i<m$ and $i\equiv 0 \bmod 2$, $i\equiv 1 \bmod 6$. By the assumption (5) of Theorem 3, we have

$$\begin{aligned} u &\geqq [m/2]-\mathrm{Min}\left[(i-[i/2]-1)-\nu_3(i-[i/2])+\nu_3\left\{\binom{i-[i/2]}{[i/2]}\right\}\right] \\ &= [m/2]-\beta_3(t, m) \geqq [n/2]\,. \end{aligned} \tag{27}$$

According to (23), (26) and (27), there exists an integer s such that

$$\begin{aligned} &0 \leqq 2s \leqq t\,, \\ &\zeta-t = (r3^u+s)\bar{\sigma}_n \\ & = s\bar{\sigma}_n\,. \end{aligned}$$

This completes the proof of Theorem 3.

Proof of Theorem 4. By the contraposition of Theorem 3 and the main theorem of Kambe ([4] Th. 2), it is clear.

Proof of Theorem 5. Since $\tau(L^n(3))\oplus 1=(n+1)\eta_{L^n}$ and $n+1>n=\left[\frac{2n+1}{2}\right]$ $=[1/2 \dim \tau(L^n(3))]$, Theorem 4 implies that the tangent bundle τ is not extendible to $L^{\phi(2n+1,n)}(3)$. For every $m>2n+1$, $\beta_3(2n+1, m)\leqq n$ whenever $n\equiv 0 \bmod 3$, $n\equiv 1 \bmod 3$ $\beta_3(2n+1, m)<n$ whenever $n\equiv 2 \bmod 3$. Then $\phi(2n+1, n)$ $=2\,(2n+1)$.
This completes the proof of Theorem 5.

REMARK. The following table shows the value of $\phi(t, n)$ where $1 \leqq t \leqq 10$ and $1 \leqq n \leqq 16$.

$t \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	3	3	3	4	4	6	6	8	8	10	10	12	12	14	14	16
2	5	5	5	6	6	8	8	10	10	12	12	14	14	16	16	18
3	6	6	6	6	6	8	8	10	10	12	12	14	14	16	16	18
4	8	8	8	8	8	8	8	10	10	12	12	14	14	16	16	18
5	10	10	10	10	10	10	10	10	10	12	12	14	14	16	16	18
6	12	12	12	12	12	12	12	14	14	16	16	18	18	20	20	22
7	14	14	14	14	14	14	14	14	14	16	16	18	18	20	20	22
8	16	16	16	16	16	16	16	16	16	18	18	20	20	22	22	24
9	18	18	18	18	18	18	18	18	18	18	18	20	20	22	22	24
10	20	20	20	20	20	20	20	20	20	20	20	20	20	22	22	24

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References

[1] M.F. Atiyah: *Immersions and embeddings of manifold,* Topology **1** (1962), 125–132.

[2] A. Borel et J.P. Serre: *Groupes de Lie et puissances réduites de steenrod,* Amer. J. Math. **75** (1953), 409–448.

[3] F. Hirzebruch: Topological Methods in Algebraic Geometry, Appendix by Schwarzenberger, Springer, Berlin, 1966.

[4] T. Kambe: *The structure of K_{Λ}-rings of lens space and their application,* J. Math. Soc. Japan. **18** (1966), 135–146.

[5] R.L.E. Schwarzenberger: *Extendible vector bundles over real projective space,* Quart. J. Math. Oxford Ser. **17** (1966), 19–21.