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ON MULTIPLY TRANSITIVE GROUPS III

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The main purpose of this paper is to improve Theorem 1 in [4].

Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, $H=G_{1,2,3,4}$ the subgroup of G consisting of all the elements fixing the four letters 1, 2, 3 and 4, and let Δ be the totality of the letters fixed by H. Then the normalizer N of H in G fixes Δ. If we denote by N^{Δ} the restriction of N on Δ, then by the theorem of Jordan ([2]) and Witt ([5]) N^{Δ} is one of the following groups: S_4, S_5, A_6 or M_{11}.

In the first section, we shall consider the number of fixed letters of an involution. We shall prove especially that if N^{Δ} is A_6 or M_{11} then the number r of the fixed letters of any involution in G satisfies the relation

$$n = r^2+2$$

and consequently all involutions have the same number of fixed letters.

Now let P be a Sylow 2-group of H, Δ' the totality of the letters fixed by P and N' the normalizer of P in G. Then, by the theorem of M. Hall ([1], Theorem 5.8.1), $(N')^{\Delta'}$ is one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. In the second section, we shall first consider the case in which $P\neq 1$ and P is transitive on $\Omega-\Delta'$ and we shall prove that if $n\geq 35$ $(N')^{\Delta'}$ must be S_4 or S_5. As a corollary we have that if G is not alternating nor symmetric group and if P $(\neq 1)$ is transitive and regular on $\Omega-\Delta'$ then G is M_{12} or M_{23}. Since a transitive group which is abelian is regular, this gives an improvement of Theorem 1 in [4].

Notation. For a set X let $|X|$ denote the number of elements of X. For a set S of permutations on Ω the totality of the letters fixed by S is denoted by $I(S)$. If a subset Δ of Ω is a fixed block, i.e. if $\Delta^S=\Delta$, then the restriction of S on Δ is denoted by S^{Δ}. For a permutation group G the subgroup of G consisting of all the elements fixing the letters $i, j, \cdots, k$ is denoted by $G_{i,j,\cdots,k}$.

1. Number of fixed letters of an involution.

Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, H the subgroup of G fixing four letters, $\Delta=I(H)$ and let N be the normalizer of H in G. Then N^{Δ} must be one of the following groups: S_4, S_5, A_6 or M_{11}.

Proposition 1. *If $N^{\Delta}=A_6$ or M_{11}, then the number r of the fixed letters of any involution in G satisfies the relation*

$$n = r^2+2.$$

Proof. (1) Suppose that $N^{\Delta}=A_6$. Let a be an arbitrary involution. Since G is 4-fold transitive, taking a conjugate of a if necessary, we may assume that

$$a = (1, 2) \cdots.$$

Let (k, l) be a 2-cycle of a different from $(1, 2)$. Then a normalizes $G_{1,2,k,l}$ and by assumption a is an even permutation on $\Delta_1=I(G_{1,2,k,l})$. Therefore we have

$$a^{\Delta_1} = (1, 2)(i)(j)(k, l).$$

Thus at least two letters are fixed by a. Now for a subset $\{i, j\}$ of $I(a)$, a normalizes $G_{1,2,i,j}$, therefore for $\Delta_2=I(G_{1,2,i,j})$ we have

$$a^{\Delta_2} = (1, 2)(i)(j)(k, l).$$

Thus $\{i, j\}$ determines uniquely a 2-cycle (k, l) of a, and then $G_{1,2,i,j}=G_{1,2,k,l}$ and $\{i, j\}=I(a)\cap I(G_{1,2,k,l})$. We consider the map φ: $\{i, j\}\rightarrow(k, l)$ from the family of all subsets of $I(a)$ consisting of two letters into the family of all 2-cycles of a different from $(1, 2)$. From above φ is onto. To show that φ is one to one, suppose that $\varphi(\{i, j\})=\varphi(\{i', j'\})=(k, l)$. Then $I(a)\cap I(G_{1,2,k,l})=\{i, j\}=\{i', j'\}$. Hence φ is one to one and the number of 2-cycles of a different from $(1, 2)$ is ${}_rC_2$. Thus we have

$$n = 2+r+2\cdot{}_rC_2 = r^2+2.$$

(2) Suppose that $N^{\Delta}=M_{11}$ and let a be an arbitrary involution. As in (1), we may assumed that $a=(1, 2)\cdots$, and we can easily see that at least two letters are fixed by a. If $\{i_1, i_2\}$ is a subset of $I(a)$, then a normalizes $G_{1,2,i_1,i_2}$ and for $\Delta_1=I(G_{1,2,i_1,i_2})$ a^{Δ_1} is, being an involution of M_{11}, of the following form:

$$a^{\Delta_1} = (1, 2)(i_1)(i_2)(i_3)(k_1, l_1)(k_2, l_2)(k_3, l_3).$$

Then $G_{1,2,i_\mu,i_\nu}=G_{1,2,k_\rho,l_\rho}$ and thus $\{i_1, i_2\}$ determines uniquely a set of three 2-cycles (k_1, l_1), (k_2, l_2), (k_3, l_3). Now consider the map

$$\varphi:\ \{i_1, i_2\}\rightarrow\{(k_1, l_1), (k_2, l_2), (k_3, l_3)\}$$

from the family of all subsets of $I(a)$ consisting of two letters into the family of the sets of three 2-cycles of a different from (1, 2). If a 2-cycle (k, l) of a different from (1, 2) is given, then a normalizes $G_{1,2,k,l}$ and for $\Delta_2=I(G_{1,2,k,l})$ a^{Δ_2} has, being an involution of M_{11}, just three fixed letters $\{i_1, i_2, i_3\}$. Then $I(a)\cap I(G_{1,2,k,l})=\{i_1, i_2, i_3\}$ and $\varphi(\{i_1, i_2\})=\varphi(\{i_1, i_3\})=\varphi(\{i_2, i_3\})\supset(k, l)$. Now, from the definition of φ, $\varphi(\{i_1, i_2\})\supset(k, l)$ if and only if $G_{1,2,i_1,i_2}=G_{1,2,k,l}$, i.e. $\{i_1, i_2\}\subset I(a)\cap I(G_{1,2,k,l})$. Hence the set of 2-cycles of a different from (1, 2) is the disjoint union of the images of φ and each inverse image of φ consists of three subsets. Therefore the number of 2-cycles of a different from (1, 2) is ${}_rC_2$ and we have

$$n = 2+r+2\cdot{}_rC_2 = r^2+2\,.$$

Proposition 2. *If $N^{\Delta}=S_5$, then the number r of the fixed letters of any involution in G satisfies the following relation*:

$$r(r-1)\equiv 0 \qquad (mod\ 3)\,.$$

Proof. We may assume that $r\geq 2$ and the given involution is $a=(1, 2)\cdots$. If $\{i_1, i_2\}$ is a subset of $I(a)$, then a normalizes $G_{1,2,i_1,i_2}$ and for $\Delta_1=I(G_{1,2,i_1,i_2})$ we have

$$a^{\Delta_1} = (1, 2)\,(i_1)\,(i_2)\,(i_3)\,,$$

and $G_{1,2,i_1,i_2}=G_{1,2,i_1,i_3}=G_{1,2,i_2,i_3}$. Now consider the map

$$\varphi:\ \{i_1, i_2\}\rightarrow G_{1,2,i_1,i_2}$$

from the family of all subsets of $I(a)$ consisting of two letters into the family of the subgroups of G. Then each inverse image of φ consists of three subsets and hence we have

$${}_rC_2 = \frac{r(r-1)}{2}\equiv 0 \qquad (\mathrm{mod}\ 3)\,.$$

2. Main theorem.

Let G be again a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$. It is known that the only 4-fold transitive (not alternating nor symmetric) groups on less than 35 letters are the four Mathieu groups M_{11}, M_{12}, M_{23} and M_{24}. Therefore in the following we may assume that $n\geq 35$.

Now let $H=G_{1,2,3,4}$, $\Delta=I(H)$ and let P be a Sylow 2-group of H, $\Delta'=I(P)$. Then $\Delta'\supset\Delta$. We denote the normalizers of H and P by N and N' respectively. Then $(N')^{\Delta'}$ is one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. We first prove the following

Proposition 3. *If P is transitive on $\Omega-\Delta'$ and $n\geq 35$, then $(N')^{\Delta'}$ must be S_4 or S_5.*

Proof. We first remark that if $i\in\Delta'-\Delta$ the length of the set of transitivity of H containing i is odd since the subgroup of H fixing i cantains a Sylow 2-group P of H.

The proof in the following is by contradiction.

(1) Suppose that $(N')^{\Delta'}=A_6$ and $\Delta'=\{1, 2, 3, 4, 5, 6\}$. Then N^{Δ} must be A_6, S_5 or S_4.

(1.1) Suppose $N^{\Delta}=A_6$ and let a be a central involution of P. Then, by Lemma 2 in [3], $|I(a)|=6$ and, by Proposition 1, we have

$$n = 6^2+2 = 38\,.$$

Consider the map $\varphi: i\rightarrow G_{1,2,3,i}$ from the set $\{4, 5, \cdots, 38\}$ into the family of subgroups of G. If $I(G_{1,2,3,i})=\{1, 2, 3, i, j, k\}$ then $\varphi^{-1}(G_{1,2,3,i})$ consists of the three letters i, j, k. Hence we have

$$38-3 = 35 \equiv 0 \qquad (\mathrm{mod}\ 3)\,,$$

which is a contradiction.

(1.2) Suppose that $N^{\Delta}=S_5$ and $\Delta=\{1, 2, 3, 4, 5\}$. By the quadruple transitivity G contains an involution $a=(1, 2)(3)(4)\cdots$. Since $H=G_{1,2,3,4}$ is normalized by a, Δ is fixed by a and hence a fixes the letter 5, i.e.

$$a = (1, 2)(3)(4)(5)\cdots\,.$$

Now the number of Sylow 2-groups of H is odd. Therefore there is a Sylow 2-group of H which is normalized by a. We may assume that it is P. Then Δ' is fixed by a and we have

$$a^{\Delta'} = (1, 2)(3)(4)(5)(6)\,.$$

But this is a contradiction since $a^{\Delta'}$ must be an even permutation.

(1.3) Suppose that $N^{\Delta}=S_4$ and let $\Gamma=\Omega-\Delta'$. Then the sets of transitivity of H on $\Omega-\Delta=\{5, 6\}\cup\Gamma$ can be assumed to be one of the following:

(i) $\{5, 6\}$ and Γ,

(ii) $\{5, 6\}\cup\Gamma$.

Since Γ is a set of transitivity of the 2-group P $|\Gamma|$ is a power of 2.

Hence in both cases the length of the set of transitivity containing the latter 5 ($\in\Delta'-\Delta$) is even. This is a contradiction by the first remark.

(2) Suppose that $(N')^{\Delta'}=A_7$ and $\Delta'=\{1, 2, \cdots, 7\}$. Then N^{Δ} must be A_6, S_5 or S_4.

(2.1) Suppose $N^{\Delta}=A_6$ and let a be a central involution of P. Then $|I(a)|=7$ and we have by Proposition 1

$$n = 7^2+2 = 51 .$$

Since P is transitive on $\Omega-\Delta'$ and $|\Omega-\Delta'|=51-7=44$ is not a power of 2 we have a contradiction.

(2.2) Suppose that $N^{\Delta}=S_5$ and $\Delta=\{1, 2, 3, 4, 5\}$. Then the sets of transitivity of H on $\Omega-\Delta=\{6, 7\}\cup\Gamma$ may be assumed to be one of the following:

(i) $\{6, 7\}$ and Γ,

(ii) $\{6, 7\}\cup\Gamma$.

But in both cases the length of the set of transitivity containing the letter 6 ($\in\Delta'-\Delta$) is even. This is a contradiction by the first remark.

(2.3) Suppose $N^{\Delta}=S_4$ and let $\Gamma=\Omega-\Delta'$. Then the sets of transitivity of H on $\Omega-\Delta=\{5, 6, 7\}\cup\Gamma$ may be assumed to be one of the following:

(i) $\{5, 6, 7\}$ and Γ,

(ii) $\{5, 6\}$ and $\{7\}\cup\Gamma$,

(iii) $\{5, 6, 7\}\cup\Gamma$.

The case (ii) does not occur since the length of the set of transitivity containing the letter 5 ($\in\Delta'-\Delta$) is even in this case. In the case (iii), H is transitive on $\Omega-\Delta$. Hence G is 5-fold transitive and then the subgroup G_1 fixing the letter 1 is 4-fold transitive on $\{2, 3, \cdots, n\}$ and satisfies the assumption in (1). Thus as in (1) we have a contradiction.

We shall now consider the case (i). Let P' be an arbitrary Sylow 2-group of H. Then there is an element x of H such that $x^{-1}Px=P'$. Since $\{5, 6, 7\}$ is a set of transitivity of H, it is fixed by x. Therefore $H_{5,6,7}\supset P$ implies $x^{-1}H_{5,6,7}x=H_{5,6,7}\supset P'$ and hence we have $I(P')=\{1, 2, \cdots, 7\}$. This shows that $I(P')$ is independent of the choice of Sylow 2-group P' of H and is uniquely determined by H. Let $a=(1, 2)\cdots$ be an involution of G which is conjugate to a central involution of P. Then $|I(a)|=7$. If $\{i_1, i_2\}$ is a subset of $I(a)$, then $G_{1,2,i_1,i_2}$ is normalized by a. Therefore there is a Sylow 2-group P'' of $G_{1,2,i_1,i_2}$ which is normalized by a. Let $I(P'')=\{1, 2, i_1, i_2, i_3, k, l\}$. Since a is an even permutation on $I(P'')$, we may assume

$$a = (1, 2)(i_1)(i_2)(i_3)(k, l)\cdots .$$

Now $I(P'')$ is uniquely determined by $G_{1,2,i_1,i_2}$, therefore $\{i_1, i_2\}$ determines uniquely a 2-cycle (k, l) of a and we have the map

$$\varphi: \{i_1, i_2\} \to (k, l)$$

from the family of all the subsets of $I(a)$ consisting of two letters into the family of 2-cycles of a different from (1, 2). By the definition of φ, it is easy to see that $\varphi(\{i_1, i_2\})=(k, l)$ if and only if $G_{1,2,i_1,i_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group in common, and φ is onto. Now suppose that $\varphi(\{i_1, i_2\})=\varphi(\{j_1, j_2\})=(k, l)$. Then $G_{1,2,i_1,i_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group P_1 in common, and $G_{1,2,j_1,j_2}$ and $G_{1,2,k,l}$ have a Sylow 2-group P_2 in common. Since both P_1 and P_2 are Sylow 2-groups of $G_{1,2,k,l}$ we have $I(P_1)=I(P_2)$. Therefore $\{j_1, j_2\} \subset I(a) \cap I(P_1) = \{i_1, i_2, i_3\}$. Thus we have that each inverse image of φ consists of three subsets of $I(a)$ and hence the number of 2-cycles of a different from (1, 2) is $\frac{1}{3}\,{}_7C_2=7$. In this way we have

$$n = 2+7+14 = 23,$$

which contradicts the assumption.

(3) Suppose that $(N')^{\Delta'}=M_{11}$ and $\Delta'=\{1, 2, \cdots, 11\}$. Then N^{Δ} must be one of the following groups: M_{11}, A_6, S_5 or S_4.

(3.1) Suppose $N^{\Delta}=M_{11}$ and let a be a central involution of P. Then $|I(a)|=11$ and by Proposition 1 we have $n=11^2+2=123$. Since P is transitive on $\Omega-\Delta'$ and $|\Omega-\Delta'|=123-11=112$ is not a power of 2, we have a contradiction.

(3.2) Suppose $N^{\Delta}=A_6$. In the same way as in (3.1) we have $n=123$, which is a contradiction.

(3.3) Suppose $N^{\Delta}=S_5$ and let a be a central involution of P. Then $|I(a)|=11$ and by Proposition 2 we have

$$11\,(11-1) = 110 \equiv 0 \qquad (\mathrm{mod}\ 3).$$

This is a contradiction.

(3.4) Suppose $N^{\Delta}=S_4$. Since the length of a set of transitivity of H containing one of the letters in $\Delta'-\Delta=\{5, 6, \cdots, 11\}$ is odd, the sets of transitivity of H may be assumed to be one of the following:

(i) $\{5, 6, 7\}$, $\{8, 9, 10\}$ and $\{11\} \cup \Gamma$,

(ii) $\{5, 6, 7, 8, 9, 10, 11\}$ and Γ,

(iii) $\{5, 6, 7, 8, 9, 10, 11\} \cup \Gamma$.

First consider the case (i). Since $(N')^{\Delta'}=M_{11}$, there is an element x in N' such that

$$x^{\Delta'} = (1, 2, 3, 4)\,(i_1, i_2, i_3, i_4)\,(k)\,(l)\,(m).$$

Then x normalizes $H=G_{1,2,3,4}$. Hence x must fix two sets of transitivity $\{5, 6, 7\}$ and $\{8, 9, 10\}$ or interchange them. But, from the form of $x^{\Delta'}$, this is impossible.

In the case (iii), G is 5-fold transitive. Hence $X=G_1$ is 4-fold transitive on $\{2, 3, \cdots, n\}$ and P is a Sylow 2-group of $X_{2,3,4,5}$. Since $|I(P)-\{1\}|=10$, we have a contradiction by the theorem of M. Hall ([1], Theorem 5.8.1).

Now consider the case (ii). If P' is an arbitrary Sylow 2-group of H there is an element x of H such that $P'=x^{-1}Px$. Since $\{5, 6, \cdots, 11\}$ is a set of transitivity of H, it is left invariant by x. Therefore $H_{5,6,\cdots,11}\supset P$ implies $H_{5,6,\cdots,11}\supset P'$ and we have $I(P')=\{1, 2, \cdots, 11\}$. This shows that $I(P')$ is independent of the choice of P' and is determined uniquely by H. We denote it by $J(H)$. Let $a=(1, 2)\cdots$ be an involution which is conjugate to a central involution of P. We consider the map

$$\varphi: \{i_1, i_2\} \to J(G_{1,2,i_1,i_2})$$

which assigns $J(G_{1,2,i_1,i_2})$ to a subset $\{i_1, i_2\}$ of $I(a)$. Since a normalizes $G_{1,2,i_1,i_2}$, there is a Sylow 2-group P'' of $G_{1,2,i_1,i_2}$ such that $a^{-1}P''a=P''$. Let $\Delta''=I(P'')=J(G_{1,2,i_1,i_2})$. Then $a^{\Delta''}$ is an involution of M_{11}. Hence we have

$$a^{\Delta''} = (1, 2)(i_1)(i_2)(i_3)(k_1, l_1)(k_2, l_2)(k_3, l_3)$$

and $I(a)\cap J(G_{1,2,i_1,i_2})=\{i_1, i_2, i_3\}$. Now it is easy to see that the inverse image $\varphi^{-1}(J(G_{1,2,i_1,i_2}))$ consists of three subsets $\{i_1, i_2\}$, $\{i_1, i_3\}$, $\{i_2, i_3\}$. Therefore we have

$${}_{11}C_2 = \frac{11\cdot 10}{2} \equiv 0 \qquad (\mathrm{mod}\ 3),$$

which is a contradiction.

From Proposition 3 we have easily an improvement of Theorem 1 in [4].

Theorem. *Let G be a 4-fold transitive group on $\Omega=\{1, 2, \cdots, n\}$, excluding S_n and A_n. If a Sylow 2-group P of the subgroup fixing four letters is not trivial, and transitive and regular on $\Omega-I(P)$, then G must be M_{12} or M_{23}.*

Proof. We use the same notations as before. For $n<35$ the theorem is trivial. Therefore we may assume $n\geq 35$, and then, by Proposition 3, we need consider only the case in which $(N')^{\Delta'}=S_4$ or S_5.

(1) Suppose $(N')^{\Delta'}=S_4$. Then G is 5-fold transitive. Hence $X=G_1$

is 4-fold transitive on $\{2, 3, \cdots, n\}$. Since $X_{2,3,4}=H$ and a Sylow 2-group P of H is regular on $\{5, 6, \cdots, n\}$, $X_{2,3,4,5}$ is of odd order. Therefore, by the theorem of M. Hall, X must be one of the following groups: S_4, S_5, A_6, A_7 or M_{11}. But this contradicts the assumption $n\geq 35$.

(2) Suppose $(N')^{\Delta'}=S_5$ and let $\Delta'=\{1, 2, 3, 4, 5\}$. Then $N^{\Delta}=S_4$ or S_5. If $N^{\Delta}=S_4$, then H is transitive on $\{5, 6, \cdots, n\}$ and hence G is 5-fold transitive. Then $X=G_1$ is 4-fold transitive on $\{2, 3, \cdots, n\}$ and satisfies the assumption in (1). Therefore as in (1) we have a contradiction. On the other hand, if $N^{\Delta}=S_5$ then by Proposition 2 we have

$$5(5-1)\equiv 0 \qquad (\text{mod } 3),$$

since for a central involution a of P $|I(a)|=5$. But this is a contradiction.

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