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A Class of the Equations of Evolution in a Banach Space

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§ 0. Introduction

We consider in this note a special class of the equations of evolution

$$du/dt = A(t)u + f(t) \tag{0.1}$$

in a Banach space under the Hypotheses $1°\sim4°$ described in § 1. The most restrictive one of these hypotheses is 4°, and so this type of equations is modelled on parabolic differential equations. The existence of the fundamental solution $U(t, s)$ of (0.1) is known under Hypotheses $1°\sim3°$ only (T. Kato [2]). However, when Hypothesis 4° is also satisfied, we can construct the fundamental solution $U(t, s)$ by another method which makes it easy to deduce various properties of $U(t, s)$. In constructing $U(t, s)$, we use E. E. Levy's approximation method with respect to the time variable.

In § 1, we consider the general theory of the present class of equations and it is shown that $\partial U(t, s)/\partial t$ $(=A(t)U(t, s))$ is a bounded operator and its norm is bounded by $C(t-s)^{-1}$ with some constant C. In § 2, perturbation theory is considered. The perturbing operator corresponds to a lower order term in case of differential equations and satisfies less restrictive condition about smoothness in time variable than $A(t)$. In § 3, we give an example and in § 4 we consider higher derivatives of $U(t, s)$ under a more restrictive assumption about the differentiability of $A(t)$ in t. Finally, in § 5 we consider a special case which includes the case of a parabolic differential equation in the whole Euclidean space.

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§ 1. The fundamental solution

We consider the equation of evolution

$$dx(t)/dt = A(t)\,x(t) + f(t)\,, \qquad a \leqq t \leqq b \tag{1.1}$$

and the associated homogeneous equation

$$(1.1') \qquad dx(t)/dt = A(t)x(t)\,.$$

Here the unknown $x(t)$ is an element of a complex Banach space $\mathfrak{X}$ depending on a real variable t, while $f(t)$ is a given element of $\mathfrak{X}$ and $A(t)$ is a given, in general unbounded, additive operator in $\mathfrak{X}$, both depending on t.

We denote by $\exp(tA)$ the semi-group of bounded operators which has A as the infinitesimal generator, and by I the identical mapping.

Hypotheses 1°. $A(t)$ is defined for $a \leqq t \leqq b$ and is an infinitesimal generator of some semi-group of bounded operators for each t.

2°. 1) The domain $\mathfrak{D}$ of $A(t)$ is independent of t, 2) the bounded operator $B(t, s) = [I - A(t)].[I - A(s)]^{-1}$ is uniformly bounded for $a \leqq s$, $t \leqq b$, 3) $B(t, s)$ satisfies Lipschitz condition in t for every s in the uniform operator topology.

3°. $B(t, s)$ is strongly continuously differentiable in t for every s.

4° for each s and t with $a \leqq s \leqq b$ and $t > 0$, $(d/dt)\exp(tA(s))$ is a bounded operator and there are positive constants C and t_0 such that

$$\|(d/dt)\exp(tA(s))\| = \|A(s)\exp(tA(s))\| \leqq C/t\,,$$

for any s and $t \leqq t_0$.

Regarding Hypothesis 4°,

Theorem. *Under the Hypotheses* 1°, 2° *and* 3°, *Hypothesis* 4° *is satisfied if and only if there exist positive constants C_1 and τ_0 such that for every τ with $|\tau| \geqq \tau_0$ and every s with $a \leqq s \leqq b$, we have*

$$(1.2) \qquad \|((1+i\tau)I - A(s))^{-1}\| \leqq C_1/|\tau|\,.$$

The proof is quite the same as that of Theorem 1 in K. Yosida [5].

In order to simplify the notation, we shall write $A(t)$ instead of $A(t) - I$. Then there are positive constants M and K such that for every t, s and r,

$$(1.3) \qquad \|A(t)A(s)^{-1}\| \leqq M \quad \text{and} \quad \|(A(t) - A(r))A(s)^{-1}\| \leqq K|t-r|$$

We determine an operator $R(t, s)$ so that

$$(1.4) \qquad U(t, s) = \exp((t-s)A(s)) + \int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau$$

should be the fundamental solution of (1.1′). As preparation, we shall prove some lemmas.

Lemma 1.1. $\exp((t-s)A(s))$ *and* $(A(t) - A(s))\exp((t-s)A(s))$ *are uniformly bounded and strogly continuous in s and t simultaneously in $a \leqq s \leqq t \leqq b$. As $t - s \downarrow 0$, the latter tends to* 0 *strongly.*

Proof. $\exp((t-s)A(s))$ has a norm not exceeding one, so it is uniformly bounded. By (1.2) and (1.3) we have

$$\|(A(t)-A(s))\exp(t-s)A(s))\| \leqq \|(A(t)-A(s))A(s)^{-1}\| \, \|A(s)\exp((t-s)A(s))\| \leqq KC.$$

Hence, this is also uniformly bounded. Next, we must prove that for every $x \in \mathfrak{X}$

$$\exp((t'-s')A(s')x-\exp((t-s)A(s))x \to 0 \quad \text{and}$$
$$(A(t')-A(s'))\exp((t'-s')A(s'))x-(A(t)-A(s))\exp((t-s)A(s))x \to 0$$

strongly as $t' \to t$ and $s' \to s$. But, by the uniform boundedness of those operators, we have only to prove the above convergence for every $x \in \mathfrak{D}$. For $x \in \mathfrak{D}$, we have

$$\begin{aligned}
&\|\exp((t'-s')A(s'))x-\exp((t-s)A(s'))x\| \\
&= \left\| \int_{t-s}^{t'-s'} (d/d\tau)\exp(\tau A(s'))x d\tau \right\| = \left\| \int_{t-s}^{t'-s'} \exp(\tau A(s')A(s')x d\tau \right\| \\
&\leqq |t'-s'-t+s| M \|A(r)x\|
\end{aligned}$$

for any r with $a \leqq r \leqq b$, And, we have

$$\begin{aligned}
&\|\exp((t-s)A(s'))x-\exp((t-s)A(s))x\| \\
&= \left\| \int_0^{t-s} (d/d\sigma)\{\exp(\sigma A(s'))\exp((t-s-\sigma)A(s))x\} d\sigma \right\| \\
&= \left\| \int_0^{t-s} \exp(\sigma A(s'))(A(s')-A(s))\exp((t-s-\sigma)A(s))x d\sigma \right\| \\
&\leqq \int_0^{t-s} \left\| (A(s')-A(s))A(s)^{-1} \right\| \, \|A(s)x\| d\sigma \to 0 \quad \text{as} \quad s' \to s.
\end{aligned}$$

Thus, the strong continuity of $\exp((t-s)A(s))$ is proved. Next,

$$\begin{aligned}
&(A(t')-A(s'))\exp((t'-s')A(s'))x-(A(t)-A(s))\exp((t-s)A(s))x \\
&= (A(t')-A(s')-A(t)+A(s))\exp((t'-s')A(s'))x \\
&+ (A(t)-A(s))\{\exp((t'-s')A(s'))x-\exp((t-s)A(s))x\}.
\end{aligned}$$

The norm of the first term does not exceed $K(|t'-t|+|s'-s|)\|A(s')x\|$. The second term is equal to

$$\begin{aligned}
&(A(t)-A(s))\{A(s')^{-1}\exp((t'-s')A(s'))A(s')x-A(s)^{-1}\exp((t-s)A(s))A(s)x\} \\
&= (A(t)-A(s))A(s')^{-1}\{\exp((t'-s')A(s'))A(s')x-\exp((t-s)A(s))A(s)x\} \\
&+ \{(A(t)-A(s))A(s')^{-1}-(A(t)-A(s))A(s)^{-1}\}\exp((t-s)A(s))A(s)x.
\end{aligned}$$

According to the continuity of $\exp((t-s)A(s))$ proved above the first term of the right member tends to 0 as $t' \to t$ and $s' \to s$. The convergence to 0 of the second term follows from

$$\begin{aligned}&(A(t)-A(s))A(s')^{-1}-(A(t)-A(s))A(s)^{-1}\\&=(A(t)-A(s))A(s)^{-1}(A(s)-A(s'))^{-1}A(r)^{-1}A(r)A(s')^{-1}\end{aligned}$$

for any fixed r with $a \leqq r \leqq b$. The convergence of $\exp((t-s)A(s))$ to I as $t-s \downarrow 0$ is easily seen. The last part of the lemma follows from

$$(A(t)-A(s))\exp((t-s)A(s))x=(A(t)-A(s))A(s)^{-1}\exp((t-s)A(s))A(s)x$$

for any $x \in \mathfrak{D}$ and the uniform boundedness of the operator. (q. e. d.)

For the time being, we assume that $R(t, s)$ is strongly continuous in $a \leqq s \leqq t \leqq b$. Put

$$U_h(t, s)=\exp((t-s)A(s))+\int_s^{t-h}\exp((t-\tau)A(\tau))R(\tau, s)d\tau\,, \tag{1.5}$$

for sufficiently small positive number h. The integral exists as Riemann integral in the strong operator topology due to the strong continuity of the integrand. Then, for any $x \in \mathfrak{X}$, we have

$$\begin{aligned}&(\partial/\partial t)U_h(t, s)x-A(t)U_h(t, s)x=\exp(hA(t-h))R(t-h, s)x\\&-\int_s^{t-h}(A(t)-A(\tau))\exp((t-\tau)A(\tau))R(\tau, s)xd\tau-(A(t)-A(s))\exp((t-s)A(s))x.\end{aligned}$$

If we know that $(\partial/\partial t)U_h(t, s)x \to (\partial/\partial t)U(t, s)x$ and $A(t)U_h(t, s)x \to A(t)U(t, s)x$, we obtain by letting h tend to 0

$$\begin{aligned}&(\partial/\partial t)U(t, s)x-A(t)U(t, s)x\\&=R(t, s)x-\int_s^t(A(t)-A(\tau))\exp((t-\tau)A(\tau))R(\tau, s)xd\tau\\&-(A(t)-A(s))\exp((t-s)A(s))x\end{aligned}$$

using Lemma 1.1. So, in order that $U(t, s)$ should be the fundamental solution of (1.1′), $R(t, s)$ must satisfy the following integral equation:

$$\begin{aligned}&R(t, s)-\int_s^t(A(t)-A(\tau))\exp((t-\tau)A(\tau))R(\tau, s)d\tau\\&=(A(t)-A(s))\exp((t-s)A(s))\,.\end{aligned} \tag{1.6}$$

The above integral equation can be solved by the method of successive approximation. We define $R_1(t, s)=(A(t)-A(s))\exp((t-s)A(s))$ and

$$\begin{aligned}R_m(t, s)&=\int_s^t(A(t)-A(\tau))\exp((t-\tau)A(\tau))R_{m-1}(\tau, s)d\tau\\&=\int_s^t R_1(t, \tau)R_{m-1}(t, s)d\tau\,, \qquad \text{for} \quad m=2, 3, \cdots.\end{aligned}$$

In order to show that $R(t, s)=\sum_{m=1}^{\infty} R_m(t, s)$ is really the desired solution of (1. 6), we prove the following lemma.

Lemma 1. 2. *$\sum_{m=1}^{\infty} R_m(t, s)$ is strongly continuous in s and t simultaneously in $a \leqq s \leqq t \leqq b$. The series converges in the uniform operator topology.*

Proof. By Lemma 1. 1, we have $\|R_1(t, s)\| \leqq KC$. If we have proved that $\|R_{m-1}(t, s)\| \leqq (KC)^{m-1}(t-s)^{m-2}/(m-2)!$ for some $m \geqq 1$, we have

$$\begin{aligned}\|R_m(t, s)\| &\leqq \int_s^t \|R_1(t, \tau)\| \, \|R_{m-1}(\tau, s)\| \, d\tau \\ &\leqq \int_s^t (KC)^m (\tau-s)^{m-2}/(m-2)! \, d\tau \\ &= (KC)^m (t-s)^{m-1}/(m-1)!\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\|R(t, s)\| &\leqq \sum_{m=1}^{\infty} \|R_m(t, s)\| \leqq KC \sum_{m=1}^{\infty} (KC(t-s))^{m-1}/(m-1)! \\ &= KC \exp(KC(t-s))\end{aligned}$$

$R_1(t, s)$ is strongly continuous in s and t by Lemma 1. 1. We assume that $R_{m-1}(t, s)$ is also continuous in the same sense. Let s, t, s' and t' be any positive numbers in the closed interval $[a, b]$ satisfying $s<t$ and $|s'-s|<\eta$, $|t'-t|<\eta$ for sufficiently small positive η. Then we have, for any $x \in \mathfrak{X}$,

$$\begin{aligned}&R_m(t', s')x - R_m(t, s)x \\ &= \left(\int_{s'}^{s+\eta} + \int_{t-\eta}^{t'}\right) R_1(t', \tau) R_{m-1}(\tau, s') x d\tau - \left(\int_{s}^{s+\eta} + \int_{t-\eta}^{t}\right) R_1(t, \tau) R_{m-1}(\tau, s) x d\tau \\ &+ \int_{s+\eta}^{t-\eta} (R_1(t', \tau) R_{m-1}(\tau, s') - R_1(t, \tau) R_{m-1}(\tau, s)) x d\tau \, .\end{aligned}$$

The norm of the first and second term of the right member can be made arbitrarily small by making η sufficiently small. The integrand of the last term tends to 0 as $t' \to t$ and $s' \to s$. With the help of the uniform boundedness of the integrand, this implies the convergence to 0 of the last term for any fixed η. Hence, $R_m(t, s)$ is also continuous in the same sense mentioned above. The series $\sum_{m=1}^{\infty} R_m(t, s)$ converges uniformly with respect to s and t even in the uniform operator topology, so the sum is also strongly continuous in s and t simultaneously. Consequently, it follows that $R(t, s)=\sum_{m=1}^{\infty} R_m(t, s)$ is really the solution of (1. 6).

Lemma 1. 3. *For any $a \leqq s < \tau < t \leqq b$, we have*

(1. 7) $$\|R(t, s)-R(\tau, s)\| \leqq K_1(t-\tau)(t-s)^{-1}+K_2(t-s)^{\rho}(t-\tau)^{1-\rho},$$

where K_1 and K_2 are positive constants independent of s, τ and t, and ρ is any positive number less than one and K_2 depends on ρ.

Proof. $R_1(t, s)-R_1(\tau, s) = (A(t)-A(\tau))\exp((t-s)A(s))+(A(\tau)-A(s))\{\exp((t-s)A(s)-\exp((\tau-s)A(s))\}$. The norm of the first term is bounded by $KC(t-\tau)(t-s)^{-1}$. As for the second term, it is equal to

$$\begin{aligned}&(A(\tau)-A(s))\int_{\tau-s}^{t-s}(d/d\sigma)\exp(\sigma A(s))d\sigma\\ &= (A(t)-A(s))A(s)^{-1}\int_{\tau-s}^{t-s}A(s)^2\exp(\sigma A(s))d\sigma.\end{aligned}$$

And we have

$$\begin{aligned}&\left\|\int_{\tau-s}^{t-s}A(s)^2\exp(\sigma A(s))d\sigma\right\| \leqq \int_{\tau-s}^{t-s}\|(A(s)\exp(2^{-1}\sigma A(s))^2\|d\sigma\\ &\leqq 4C^2(t-\tau)(t-s)^{-1}(\tau-s)^{-1}.\end{aligned}$$

Hence, we obtain

$$\|R_1(t, s)-R_1(\tau, s)\| \leqq (KC+4C^2K)(t-\tau)(t-s)^{-1} = K_1(t-\tau)(t-s)^{-1},$$

defining K_1 by the above equation. For general m, we have

$$\begin{aligned}R_m(t, s)-R_m(\tau, s) &= \int_s^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma-\int_s^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma\\ &= \int_\tau^t R_1(t, \sigma)R_{m-1}(\sigma, s)d\sigma+\int_s^\tau (R_1(t, \sigma)-R_1(\tau, \sigma))R_{m-1}(\sigma, s)d\sigma.\end{aligned}$$

The norm of the first term is bounded by $(CK)^m(t-s)^{m-2}(t-\tau)/(m-2)!$, and hence by $(CK)^m(t-s)^{\rho+m-2}(t-\tau)^{1-\rho}/(m-2)!$ for any ρ with $0<\rho<1$. The norm of the second term is bounded by

$$\begin{aligned}&((m-2)!)^{-1}\int_s^\tau K_1(t-\tau)(t-\sigma)^{-1}(CK)^{m-1}(\sigma-s)^{m-2}d\sigma\\ &\leqq (CK)^{m-1}K_1((m-2)!)^{-1}\int_s^\tau((t-\tau)(t-\sigma))^{1-\rho}(\sigma-s)^{m-2}d\sigma,\end{aligned}$$

where we used the inequality $(t-\tau)(t-\sigma)^{-1} \leqq 1$, so by

$$K_1(KC)^{m-1}B(\rho, m-1)(t-s)^{\rho+m-2}(t-\tau)^{1-\rho}/(m-2)!.$$

Hence, we obtain

$$\begin{aligned}||R(t, s)-R(\tau, s)|| &\leqq K_1(t-\tau)(t-s)^{-1}+\sum_{m=2}^{\infty}\{(CK)^m(t-s)^{m-2}/(m-2)!\\&+K_1(KC)^{m-1}B(\rho, m-1)(t-s)^{m-2}/(m-2)!\}(t-s)^\rho(t-\tau)^{1-\rho}\\&\leqq K_1(t-\tau)(t-s)^{-1}+K_2(t-s)^\rho(t-\tau)^{1-\rho}\end{aligned}$$

for a suitable K_2.

Lemma 1.4. ext $((t-s)A(s))$ *is strongly differentiable is s and t and* $(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s))$ *is uniformly bounded in* $a\leqq s<t\leqq b$.

Proof. For any x, we have

$$\begin{aligned}&h^{-1}\{\exp((t-s-h)A(s+h))x-\exp((t-s)A(s))x\}\\&=h^{-1}\{\exp((t-s-h)A(s+h))x-\exp((t-s)A(s+h))x\}\\&+h^{-1}\{\exp((t-s)A(s+h))x-\exp((t-s)A(s))x\}\ .\end{aligned}$$

The first term is equal to

$$\begin{aligned}&-h^{-1}(\exp(hA(s+h))-I)\exp((t-s-h)A(s+h))x\\&=-h^{-1}\int_0^h\exp(\sigma A(s+h))A(s+h)\exp((t-s-h)A(s+h))xd\sigma\ .\end{aligned}$$

However,

$$\begin{aligned}&\exp(\sigma A(s+h))A(s+h)\exp((t-s-h)A(s+h))x-A(s)\exp((t-s)A(s))x\\&=\exp(\sigma A(s+h))\{A(s+h)\exp((t-s-h)A(s+h))x-A(s)\exp(t-s)A(s))x\}\\&+(\exp(\sigma A(s+h))-I)A(s)\exp((t-s)A(s))x\,,\end{aligned}$$

and $A(s+h)\exp((t-s-h)A(s+h))-A(s)\exp((t-s)A(s))$ is uniformly bounded as $h\to 0$. So, from its convergence to 0 on a dense subspace $\mathfrak{D}$, we can conclude its convergence on the whole of $\mathfrak{X}$. $\exp(\sigma A(s+h))-I$ converges strongly to 0 as $h\downarrow 0$. Hence, as $h\downarrow 0$, we have

$$\exp(\sigma A(s+h))A(s+h)\exp((t+s+h)A(s+h))x\to A(s)\exp((t-s)A(s))x\,.$$

Next, we consider the second term. First, we assume $x\in\mathfrak{D}$. Then,

$$\begin{aligned}&h^{-1}\{\exp((t-s)A(s+h))x-\exp((t-s)A(s))x\}\\&=h^{-1}\int_0^{t-s}(d/d\sigma)\{\exp(\sigma A(s+h))\exp((t-s-\sigma)A(s))x\}d\sigma\\&=\int_{t-s}^{(t-s)/2}\exp(\sigma A(s+h))h^{-1}(A(s+h)-A(s))\exp((t-s-\sigma)A(s))xd\sigma\\&+\int_0^{(t-s)/2}\cdots=I+II\,.\end{aligned}$$

$$I=\int_{(t-s)/2}^{t-s}\exp(\sigma A(s+h))h^{-1}(A(s+h)A(s)^{-1}-I)(-\partial/\partial\sigma)\exp((t-s-\sigma)A(s))xd\sigma$$
$$=-\exp((t-s)A(s+h))h^{-1}(A(s+h)A(s)^{-1}-I)x$$
$$+\exp((t-s)A(s+h)/2)h^{-1}(A(s+h)A(s)^{-1}-I)\exp((t-s)A(s)/2)x$$
$$+\int_{(t-s)/2}^{t-s}A(s+h)\exp(\sigma A(s+h))h^{-1}(A(s+h)A(s)^{-1}-I)\exp((t-s-\sigma)A(s))xd\sigma\,.$$

The norm of the integrand of the last integral is bounded by $2CK\|x\|(t-s)^{-1}$, so it is uniformly bounded with respect to h. It is easily seen that as $h\to 0$

$$A(s+h)\exp(\sigma A(s+h))h^{-1}(A(s+h)A(s)^{-1}-I)y\to$$
$$A(s)\exp(\sigma A(s))A'(s)A(s)^{-1}y$$

for any fixed y and σ. Hence, we have

$$I\to -\exp((t-s)A(s))A'(s)A(s)^{-1}x$$
$$+\exp((t-s)A(s)/2)A'(s)A(s)^{-1}\exp((t-s)A(s)/2)x$$
$$+\int_{(t-s)/2}^{t-s}A(s)\exp(\sigma A(s))A'(s)A(s)^{-1}\exp((t-s-\sigma)A(s))xd\sigma$$

as $h\downarrow 0$.
Similarly, we get

$$II\to\int_0^{(t-s)/2}\exp(\sigma A(s))A'(s)A(s)^{-1}A(s)\exp((t-s-\sigma)A(s))xd\sigma$$

as $h\downarrow 0$.
From the above proof, it is clear that

$$\|h^{-1}\{\exp((t-s-h)A(s+h))x-\exp((t-s)A(s))x\}\|\leqq(2K+2CK)\|x\|$$

for any $x\in\mathfrak{D}$. This enables us to remove the restriction $x\in\mathfrak{D}$. Summing up, we obtain

$$\begin{aligned}&(\partial/\partial s)\exp((t-s)A(s))x\\&=-A(s)\exp((t-s)A(s))x-\exp((t-s)A(s))A'(s)A(s)^{-1}x\\&+\exp(2^{-1}(t-s)A(s))A'(s)A(s)^{-1}\exp(2^{-1}(t-s)A(s))x\\&+\int_{(t-s)/2}^{t-s}A(s)\exp(\sigma A(s))A'(s)A(s)^{-1}\exp((t-s-\sigma)A(s))xd\sigma\\&+\int_0^{(t-s)/2}\exp(\sigma A(s))A(s)A'(s)^{-1}A(s)\exp((t-s-\sigma)A(s))xd\sigma\end{aligned}\tag{1.8}$$

for any $x\in\mathfrak{X}$. It is easily seen that the right member is strongly continuous in s and t simultaneously in $a\leqq s<t\leqq b$. The uniform

boundedness of $(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s))$ follows immediately, In fact, we have

$$\|(\partial/t\partial+\partial/\partial s)\exp((t-s)A(s))\|\leqq 2K(1+C) \tag{1.9}$$

By Lemma 1 and 2, it follows that $U(t, s)$ defined by (1.4) is strongly continuous in s and t simultaneously in $a\leqq s\leqq t\leqq h$. For any $x\in\mathfrak{X}$, we have

$$\begin{aligned}
A(t)U_h(t, s)x &= A(t)\exp((t-s)A(s))x\\
&+\int_s^{t-h}A(t)\exp((t-\tau)A(\tau))(R(\tau, s)x-R(t, s)x)d\tau\\
&+\int_s^{t-h}A(t)A(\tau)^{-1}(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(t, s)xd\tau\\
&+\int_s^{t-h}(\partial/\partial\tau)(A(t)A(\tau)^{-1})\exp((t-\tau)A(\tau))R(t, s)xd\tau\\
&+A(t)A(t-h)^{-1}\exp(hA(t-h))R(t, s)x\\
&+A(t)A(s)^{-1}\exp((t-s)A(s))R(t, s)\,,
\end{aligned}$$

where we used the strong continous differentiability of $A(t)A(\tau)^{-1}$ in τ which follows from the that of $A(\tau)A(t)^{-1}$. By Lemmas 3 and 4, the convergence of the third and fourth terms as $h\downarrow 0$ follows. Thus, letting h tend to 0, we obtain

$$\begin{aligned}
A(t)U(t, s)x &= A(t)\exp((t-s)A(s))x\\
&+\int_0^{t}A(t)\exp((t-\tau)A(\tau))(R(\tau, s)x-R(t, s)x)d\tau\\
&+\int_s^{t}A(t)A(\tau)^{-1}(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(t, s)xd\tau\\
&+A(t)A(s)^{-1}\exp((t-s)A(s))R(t, s)x\\
&+\int_s^{t}(\partial/\partial\tau)(A(t)A(\tau)^{-1})\exp((t-\tau)A(\tau))\cdot R(t, s)xd\tau-R(t, s)x\,.
\end{aligned} \tag{1.10}$$

Similarly, we obtain

$$\begin{aligned}
(\partial/\partial t)U(t, s)x &= A(s)\,\mathrm{ext}((t-s)A(s))\\
&+\int_s^{t}A(\tau)\exp((t-\tau)A(\tau))(R(\tau, s)-R(t, s))xd\tau\\
&+\int_s^{t}(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(t, s)xd\tau+\exp((t-s)A(s))R(t, s)x.
\end{aligned} \tag{1.11}$$

In deducing this relation, we use that $U_h(t, s)x$ and $(\partial/\partial t)U_h(t, s)x$ converge uniformly in the wider sense in $s<t\leqq b$, which can be verified easily. Using (1.10), (1.11) and (1.6), we can verify directly the equality

$(\partial/\partial t)U(t, s)x = A(t)U(t, s)x$ for any $x \in \mathfrak{X}$. This relation is also easily varified by noticing that $(\partial/\partial t)U_h(t, s)x - A(t)U_h(t, s)x$ converges to 0 as $h \downarrow 0$. From (1.10) and (1.11), it follows that $(\partial/\partial t)U(t, s)$ or $A(t)U(t, s)$ is a bounded operator for $s<t$ whose norm does not exceed $H(t-s)^{-1}$ with some positive constant H. Summing up, we obtain

Theorem 1.1. *The operator $U(t, s)$ given by* (1.4) *is strongly continuous in s and t simultaneously in $a \leqq s \leqq t \leqq b$ and strongly differentiable in t for any fixed s in $s<t\leqq b$. $(\partial/\partial t)U(t, s)$ and $A(t)U(t, s)$ are bounded operators for $s<t$ and there hold the following equality and the initial condition*:

$$(\partial/\partial t)U(t, s) = A(t)U(t, s), \quad U(s, s) = I, \tag{1.12}$$

i.e., $U(t, s)$ is a fundamental solution of (1.1). *The operator $U(t, s)$ satisfying the above conditions is determined uniquely, and for $s \leqq r \leqq t$ the following relation holds*:

$$U(t, s) = U(t, r)U(r, s). \tag{1.13}$$

There exists a positive constant H such that

$$\|(\partial/\partial t)U(t, s)\| = \|A(t)U(t, s)\| \leqq H(t-s)^{-1}$$

for $a \leqq s < t \leqq b$.

The uniqueness and the relation (1.13) follows from Theorem 1 of T. Kato [2]. Thus, $U(t, s)$ constructed above is identical to the one constructed by Kato. Hence, it follows that $A(t)U(t, s)A(s)^{-1}$ is uniformly bounded and strongly continuous in $a \leqq s \leqq t \leqq b$. This also can be verified directly using (1.10).

Next, we consider the inhomogeneous equation (1.1). If, for every t, $f(t)$ belongs to $\mathfrak{D}$ and $A(r)f(t)$ is continuous in t for some r, the solution $x(t)$ of (1.1) in $s \leqq t \leqq b$ which satisfies the initial condition $x(s) = x$ is given by

$$x(t) = U(t, s)x + \int_s^t U(t, \tau)f(\tau)d\tau, \tag{1.14}$$

(see Kato [2]). However, for Hölder continuous $f(t)$, we may drop the assumption $f(t) \in \mathfrak{D}$ under our Hypotheses $1^\circ \sim 4^\circ$.

Theorem 1.2. *If x is any element of $\mathfrak{X}$ and $f(t)$ is Hölder continous in $a \leqq t \leqq b$*:

$$\|f(t) - f(\tau)\| \leqq F|t-\tau|^\nu, \qquad F>0, \ 0<\nu\leqq 1, \tag{1.15}$$

then (1.14) *gives the solution of the inhomogeneous equation satisfying the initial condition*:

$$x(s) = x\,. \tag{1.16}$$

Proof. Here and hereafter, we denote the second term of the right member of (1.3) by $W(t, s)$:

$$W(t, s) = \int_s^t \exp\,((t-\tau)A(\tau))R(\tau, s)d\tau\,, \tag{1.17}$$

It is clear that $(\partial/\partial t)\int_s^t W(t, \tau)f(\tau)d\tau = \int_s^t (\partial/\partial t)W(t, \tau)f(\tau)d\tau$ and $A(t)\int_s^t W(t, \tau)f(\tau)d\tau = \int_s^t A(t)W(t, \tau)f(\tau)d\tau$, because both of the right members exist due to the uniform boundedness of the integrands. Next, as in deducing (1.10) and (1.11), we obtain

$$\begin{aligned}(\partial/\partial t)&\int_s^t \exp\,((t-\tau)A(\tau))f(\tau)d\tau = \int_s^t A(\tau)\exp\,((t-\tau)A(\tau))(f(\tau)-f(t))d\tau\\ &+\int_s^t (\partial/\partial t+\partial/\partial\tau)\exp\,((t-\tau)A(\tau))f(t)d\tau+\exp\,((t-s)A(s))f(t)\end{aligned}$$

and

$$\begin{aligned}A(t)&\int_s^t \exp\,((t-\tau)A(\tau))f(\tau)d\tau = \int_s^t A(t)\exp\,((t-\tau)A(\tau))(f(\tau)-f(t))d\tau\\ &+\int_s^t A(t)A(\tau)^{-1}(\partial/\partial t+\partial/\partial\tau)\exp\,((t-\tau)A(\tau))f(t)d\tau\\ &-\int_s^t (\partial/\partial\tau)(A(t)A(\tau)^{-1})\exp\,((t-\tau)A(\tau))f(t)d\tau\\ &-f(t)+A(t)A(s)^{-1}\exp\,((t-s)A(s))f(t)\,.\end{aligned}$$

Hence, both of the left members of the above equalities exist. By the relation

$$\begin{aligned}(\partial/\partial t)&\int_s^{t-h} U(t, \tau)f(\tau)d\tau-f(t)-A(t)\int_s^{t-h} U(t, \tau)f(\tau)d\tau\\ &=U(t, t-h)f(t-h)-f(t)\,,\end{aligned}$$

the left member tends to 0 as $h\downarrow 0$. Summing up the above results, we obtain the proof of the Theorem.

Theorem 1.3. *If, for some* s, $\overline{\lim}_{t\downarrow s}(t-s)\,\|(\partial/\partial t)U(t, s)\|<e^{-1}$, *then the domain* $\mathfrak{D}$ *of* $A(t)$, $a\leq t\leq b$, *is the whole of* $\mathfrak{X}$ *and* $A(t)$ *is bounded for every* t *with* $a\leq t\leq b$.

Proof. This follows immediately from

$$\overline{\lim}_{t\downarrow s}(t-s)\,\|(\partial/\partial t)U(t, s)\| = \overline{\lim}_{t\downarrow s}(t-s)\,\|(\partial/\partial t)((t-s)A(s))\|$$

and Theorem 10. 3. 6 of Hill and Philips [3] (in p. 311).

Next, we assume that similar hypotheses are satisfied by the adjoint system $\{A^*(t)\}$: Hypotheses 1*. The domain of $A^*(t)$ is dense in the adjoint space $\mathfrak{X}^*$ of $\mathfrak{X}$ (consequently, $A^*(t)$ is the infinitesinal generator of a semi-group whose norm does not exceed one for each t). 2*. The domain $\mathfrak{D}^*$ of $A^*(t)$ is independent of t and the bounded operator $B'(t, s) = [I-A^*(t)][I-A^*(s)]^{-1}$ is uniformly bounded with respect to s and t. $B'(t, s)$ satisfies Lipschitz condition in t for every s in the uniform operator topology. 3*. $B'(t, s)$ is strongly differentiable in t for each s and $(\partial/\partial t) B'(t, s)$ is strongly continuous in t.

Combining 1* and Hypotheses 1°$\sim$4° about $A(t)$, we can conclude that for each s and t $(\partial/\partial t)\exp(tA^*(s))$ is bounded and satisfies the inequality

$$(1.18)\qquad \|(\partial/\partial t)\exp(tA^*(s))\| = \|A^*(s)\exp(tA^*(s))\| \leqq Ct^{-1}.$$

By our convention of notations about $\{A(t)\}$, $A^*(t)$ is replaced by $A^*(t)-I$. Hence, with some positive constants M^* and K^*, we have

$$(1.19)\qquad \begin{aligned} &\|A^*(t)A^*(s)^{-1}\| \leqq M^* \quad \text{and} \\ &\|(A^*(t)-A^*(r))A^*(s)^{-1}\| \leqq K^*|t-r| \end{aligned}$$

for each s, t and r.

We can construct the fundamental solution $U'(t, s)$ of the adjoint equation :

$$(1.20)\qquad -(d/ds)v(s) = A^*(s)v(s)+g(s),$$

as we did for the quation (1. 1). Namely, first we define $R'(t, s)$ as the solution of integral equation

$$(1.21)\qquad \begin{aligned} &R'(t, s)+\int_s^t (A^*(\tau)-A^*(s))\exp((t-s)A^*(s))R'(t, \tau)d\tau \\ &= -(A^*(t)-A^*(s))\,\mathrm{ext}((t-s)A^*(t)). \end{aligned}$$

$R'(t, s)$ is represented as $R'(t, s) = \sum_{m=1}^{\infty} R'_m(t, s)$, where $R'_1(t, s) = -\exp((t-s)A^*(t))$ and

$$R'_m(t, s) = (-1)^m \int_s^t (A^*(\tau)-A^*(s))\exp((\tau-s)A^*(\tau))\cdot R'_{m-1}(t, \tau)d\tau,$$

for $m=2, 3, \cdots$. With the help of $R'(t, s)$, we may define $U'(t, s)$ by

$$(1.22)\qquad U'(t, s) = \exp((t-s)A^*(t))+\int_s^t \exp((\tau-s)A^*(\tau))R'(t, \tau)d\tau$$

It is easily seen that $U'(t, s)$ defined above is the fundamental solution of (1.20):

(1.23) $$-(\partial/\partial s)U'(t, s) = A^*(s)U'(t, s) \quad \text{and} \quad U'(t, t) = I.$$

As in the case of $U(t, s)$, we can show the following two relations:

(1.24)
$$\begin{aligned} A^*(s)U'(t, s) &= A^*(s)\exp((t-s)A^*(t)) \\ &+\int_s^t A^*(s)\exp((\tau-s)A^*(\tau))(R'(t,\tau)-R'(t,s))d\tau \\ &-\int_s^t (\partial/\partial\tau)(A^*(s)A^*(\tau)^{-1})\exp((\tau-s)A^*(\tau))R'(t,s)d\tau \\ &+\int_s^t A^*(s)A^*(\tau)^{-1}(-\partial/\partial s-\partial/\partial\tau)\exp((\tau-s)A^*(\tau))R'(t,s)dt-R'(t,s) \\ &+A^*(s)A^*(t)^{-1}\exp((t-s)A^*(t))R'(t,s), \end{aligned}$$

(1.25)
$$\begin{aligned} -(\partial/\partial s)U'(t, s) &= A^*(t)\exp((t-s)A^*(t)) \\ &+\int_s^t A^*(\tau)\exp((\tau-s)A^*(\tau))(R'(t,\tau)-R'(t,s))d\tau \\ &-\exp((t-s)A^*(t))R'(t,s) \\ &+\int_s^t (-\partial/\partial s-\partial/\partial\tau)\exp((\tau-s)A^*(\tau))R'(t,s)d\tau. \end{aligned}$$

With the help of the above relations, we may prove the inequality:

(1.26) $$\|(\partial/\partial s)U'(t, s)\| = \|A^*(s)U'(t, s)\| \leqq H^*(t-s)^{-1}$$

with some positive constant H^*.

For any $x \in \mathfrak{X}$ and $x^* \in \mathfrak{X}^*$, we have

$$\begin{aligned} 0 &= \int_s^t ((\partial/\partial\tau)U(\tau, s)x - A(\tau)U(\tau, s)x,\ U'(t,\tau)x^*)d\tau \\ &-\int_s^t (U(\tau, s)x,\ -(\partial/\partial\tau)U'(t,\tau)x^* - A^*(\tau)U'(t,\tau)x^*)d\tau \\ &= (x, U^*(t,s)x^*) - (x, U'(t,s)x^*). \end{aligned}$$

Hence, we obtain

(1.28) $$U'(t, s) = U^*(t, s)$$

i.e., the fundamental solution constructed above is normal in the sense of S. D. Eidelman [1], and

(1.29) $$-(\partial/\partial s)U(t, s)x = U(t, s)A(s)x \quad \text{for} \quad x \in \mathfrak{D}$$

(1.30) $$(\partial/\partial t)U^*(t, s)x = U^*(t, s)A^*(t)x^* \quad \text{for} \quad x^* \in \mathfrak{D}^*.$$

Hypothesis 4° is not necessary for deducing (1. 28)$\sim$(1. 30). By differentiating both sides of $U(t, s)=U(t, r)U(r, s)$ with respect to s and t, we obtain $\partial^2 U(t, s)/\partial t\partial s=(\partial/\partial t)U(t, r)(\partial/\partial s)U(r, s)$. Setting $r=(t+s)/2$, we get

$$\|\partial^2 U(t, s)/\partial t\partial s\| \leqq 4HH^*(t-s)^{-2}. \tag{1.31}$$

Similar inequalities holds for $A(t)/(\partial/\partial s)U(t, s)$, $A(t)U(t, s)A(s)$ and $(\partial/\partial t)U(t, s)A(s)$, the last two of which have the unique bounded extensions.

Lemma 1. 5. *Under Hypotheses* $1^\circ\sim 4^\circ$ *and* $1^*\sim 3^*$, *the unique bounded extension of* $\int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau\, A(s)$ *is uniformly bounded in* $a\leqq s\leqq t\leqq b$.

Proof. First, we notice that

$$\|(I-A(t)^{-1}A(s))x\| = \|(I-A^*(s)A^*(t)^{-1})^*x\| \leqq K^*|t-s|\,\|x\| \qquad \text{for} \quad x\in\mathfrak{D}.$$

We have with $s_1=(t+s)/2$

$$\begin{aligned}&\left\|\int_s^{s_1}\exp((t-\tau)A(\tau))(A(\tau)-A(s))\exp((t-s)A(s))d\tau\, A(s)x\right\|\\ &=\left\|\int_s^{s_1}A(\tau)\exp((t-\tau)A(\tau))(I-A(\tau)^{-1}A(s))A(s)\exp((\tau-s))xd\tau\right\|\\ &\leqq\int_s^{s_1}C(t-\tau)^{-1}K^*(\tau-s)C(\tau-s)^{-1}\|x\|d\tau\leqq C^2K^*\|x\|.\end{aligned}$$

and

$$\begin{aligned}&\left\|\int_{s_1}^t \exp((t-\tau)A(\tau))(A(\tau)-A(s))\exp((\tau-s)A(s))d\tau\, A(s)x\right\|\\ &=\left\|\int_{s_1}^t \exp((t-\tau)A(\tau))(A(\tau)-A(s))\exp(2^{-1}(\tau-s)A(s))A(s)\right.\\ &\qquad\qquad\left.\exp(2^{-1}(\tau-s)A(s))xd\tau\right\|\leqq 4C^2K\|x\|.\end{aligned}$$

Hence, we get

$$\left\|\int_s^t \exp((t-\tau)A(\tau))R_1(\tau, s)d\tau\, A(s)x\right\|\leqq C^2(4K+K^*)\|x\|.$$

Let us assume that for some $m\geqq 1$ we have

$$\begin{aligned}&\left\|\int_s^t \exp((t-\tau)A(\tau))R_m(\tau, s)d\tau A(s)x\right\|\\ &\leqq K^{*m-1}C^{m+1}(4K+K^*)(t-s)^{m-1}\|x\|/(m-1)!\,.\end{aligned}$$

Then, we have

$$\int_s^t \exp((t-\tau)A(\tau))R_{m+1}(\tau, s))d\tau A(s)x$$
$$= \int_s^t \exp((t-\tau)A(\tau)) \int_s^\tau R_m(\tau, \sigma)R_1(\sigma, s)d\sigma d\tau A(s)x$$
$$= \int_s^t \int_\sigma^t \exp((t-\tau)A(\tau))R_m(\tau, \sigma)d\tau(A(\sigma)-A(s)) \exp((\sigma-s)A(s))d\sigma A(s)x$$
$$= \int_s^t \int_\sigma^t \exp((t-\tau)A(\tau))R_m(\tau, \sigma)d\tau A(\sigma)(I-A(\sigma)^{-1}A(s))A(s) \exp((\sigma-s)A(s))xd\sigma.$$

Using the assumption of the induction we get

$$\left\| \int_s^t \exp((t-\tau)A(\tau))R_{m+1}(\tau, s)d\tau A(s)x \right\|$$
$$\leqq \int_s^t K^{*m-1}C^{m+1}(4K+K^*)(t-\sigma)^{m-1}K^*C\|x\|/(m-1)!\, d\sigma$$
$$= K^{*m}C^{m+2}(4K+K^*)(t-s)^m\|x\|/m!\,.$$

Thus, we obtain

$$\left\| \int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau A(s)x \right\|$$
$$\leqq C^2(4K+K^*) \exp(CK^*(t-s))\|x\|\,. \qquad \text{(q.e.d.)}$$

By (1. 11) and (1. 29), we have

$$(\partial/\partial t+\partial/\partial s)U(t, s) = (\partial/\partial t)W(t, s)-\int_s^t \exp((t-\tau)A(\tau))R(\tau, s)d\tau A(s)\,.$$

With the help of Lemma 1. 5, we obtain

Theorem 1. 4. *Under the Hypotheses* $1°\sim4°$ *and* $1^*\sim3^*$, $(\partial/\partial t)U(t, s)+(\partial/\partial s)U(t, s)$ *is uniformly bounded in* $a\leqq s\leqq t\leqq b$.

§ 2. **Perturbation theory.** In this article, we consider perturbation theory under rather restrictive assumptions:

Assumption 1) A closed operator $B(t)$ is defined in $a\leqq t\leqq b$, whose domain contains the domain $\mathfrak{D}$ of $A(t)$.
2) A bounded operator $B(t)A(s)^{-1}$ is continuous in $a\leqq t\leqq b$ for every s in the uniform operator topology.
3) There exist positive constants C_1, C_2, $\rho\leqq1$ and $\lambda\leqq1$ such that

$$\begin{aligned} &\|B(t) \exp(\tau A(s))\| \leqq C_1\tau^{-(1-\rho)}, \\ &\|(B(t')-B(t)) \exp(\tau A(s))\| \leqq C_2|t'-t|\tau^{-(1-\rho)} \end{aligned} \tag{2. 1}$$

for $a\leqq t,\ t',\ s\leqq b$ and $\tau>0$.

Under the above assumptions, we consider a perturbed equation

$$(d/dt)x(t) = (A(t)+B(t))x(t)+f(t)\,. \tag{2.2}$$

The fundamental solution $V(t, s)$ of the above equation is formally given by the series:

$$V(t, s) = \sum_{m=0}^{\infty} U_m(t, s)\,, \tag{2.3}$$

where $U_0(t, s)=U(t, s)$ and $U_m(t, s)=\int_s^t U(t, \sigma)B(\sigma)U_{m-1}(\sigma, s)d\sigma$, $m=1, 2, \cdots$. $U_m(t, s)$ is also written in the following form:

$$U_m(t, s) = \int_s^t U_{m-1}(t, \sigma)B(\sigma)U(\sigma, s)d\sigma\,, \tag{2.4}$$

and we use this form mainly in the sequel.

Lemma 2.1. *For any t, t' and s in the closed interval $[a, b]$, we have*

$$\begin{aligned} &||B(\tau)U(t, s)|| \leqq C_3(t-s)^{-(1-\rho)} \\ &||(B(t')-B(t))U(\sigma, s)|| \leqq C_4|t'-t|^{\lambda}(\sigma-s)^{-(1-\rho)} \end{aligned} \tag{2.5}$$

with constants C_3 and C_4 independent of t, t' τ, σ and s.

Proof. By (2, 1),

$$\begin{aligned} ||B(\tau)W(t, s)|| &= \left\|\int_s^t B(\tau)\exp((t-\tau)A(\tau))R(\tau, s)d\tau\right\| \\ &\leqq \int_s^t C_1(t-\tau)^{-(1-\rho)} \sup_{\tau, s} ||R(t, s)||d\tau = \rho^{-1}C_1 \sup ||R(\tau, s)||(t-s)^{\rho}\,. \end{aligned}$$

Similarly, we get $||(B(t')-B(t))W(t, \sigma)|| \leqq C_2|t'-t|^{\lambda}\rho^{-1}(t-\sigma)^{\rho} \sup ||R(\tau, s)||$.

Lemma 2.2. *$B(\sigma)U(t, s)$ is strongly continuous in σ, t and s simultaneously and strongly differentiable in t and the derivative satisfies:*

$$(\partial/\partial t)B(\sigma)U(t, s) = B(\sigma)(\partial/\partial t)U(t, s) \quad \text{and} \tag{2.6}$$

$$||(\partial/\partial t)B(\sigma)U(t, s)|| \leqq C_7(t-s)^{\rho-2}\,, \tag{2.7}$$

with some constant C_7 independent of t, s and σ.

Proof. We first consider the derivative of $B(\sigma)U_h(t, s)x$. For $h>0$, we have

$$\begin{aligned} &(\Delta t)^{-1}\{B(\sigma)U_h(t+\Delta t, s)x - B(\sigma)U_h(t, s)x\} \\ &= (\Delta t)^{-1}\{B(\sigma)\exp((t+\Delta t-s)A(s))x - B(\sigma)\exp((t-s)A(s))x\} \end{aligned} \tag{2.8}$$

$$+(\Delta t)^{-1}\int_{t-h}^{t+\Delta t-h} B(\sigma)\exp((t+\Delta t-\tau)A(\tau))R(\tau, s)xd\tau$$
$$+(\Delta t)^{-1}\int_{s}^{t-h}(B(\sigma)\exp((t+\Delta t-\tau)A(\tau))-B(\sigma)\exp((t-\tau)A(\tau)))R(\tau, s)xd\tau .$$

The first term is easily seen to converge to $B(\sigma)A(s)\exp((t-s)A(s))x$ as $\Delta t\to 0$. It is also easy to show that the second term converges to $B(\sigma)\exp(hA(t-h))R(t-h, s)x$, if we remark that $h>0$. Finally, we consider the last term. For $s<\tau<t-h$, we have

$$\begin{aligned}
&B(\sigma)\{\exp((t+\Delta t-\tau)A(\tau))-\exp((t-\tau)A(\tau))\}\\
&=B(\sigma)\exp(2^{-1}hA(\tau))\{\exp((t+\Delta t-\tau-h)A(\tau))\\
&-\exp((t-\tau-h)A(\tau))\exp(2^{-1}hA(\tau))\}\\
&=B(\sigma)\exp(2^{-1}hA(\tau))\int_{t-\tau-h}^{t+\Delta t-\tau-h}\exp(rA(\tau))dr\,A(\tau)\exp(2^{-1}hA(\tau)) .
\end{aligned}$$

Therefore

$$\begin{aligned}
(2.9)\quad &(\Delta t)^{-1}\{B(\sigma)\exp((t+\Delta t-\tau)A(\tau))-B(\sigma)\exp((t-\tau)A(\tau)\}R(\tau, s)x\\
&=B(\sigma)\exp(2^{-1}hA(\tau))(\Delta t)^{-1}\int_{t-\tau-h}^{t+\Delta t-\tau-h}\exp(rA(\tau))dr\,A(\tau)\exp(2^{-1}hA(\tau))R(\tau, s)x
\end{aligned}$$

tends to

$$\begin{aligned}
&B(\sigma)\exp(2^{-1}hA(\tau))\exp((t-\tau-h)A(\tau))A(\tau)\exp(2^{-1}hA(\tau))R(\tau, s)x\\
&=B(\sigma)A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)x
\end{aligned}$$

for any fixed τ in $[s, t-h]$ as $\Delta t\to 0$. Thus, we can conclude the convergence of the last term to $\int_{s}^{t-h}B(\sigma)A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)xd\tau$, if we notice the uniform boundedness of the integrand which is implied by the relation (2. 9). So, we have proved that

$$\begin{aligned}
&(\partial/\partial t)(B(\sigma)U_h(t, s)x) = B(\sigma)A(s)\exp((t-s)A(s))x\\
&+B(\sigma)\exp(hA(t-h))-R(t-h, s)x+\int_{s}^{t-h}B(\sigma)A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)xd\tau\\
&=B(\sigma)(\partial/\partial t)U_h(t, s)x .
\end{aligned}$$

The middle member of the above equality is written as

$$\begin{aligned}
&B(\sigma)A(s)\exp((t-s)A(s))x+B(\sigma)\exp(hA(t-h))R(t-h, s)x\\
&+B(\sigma)\int_{s}^{t-h}A(\tau)\exp((t-\tau)A(\tau))(R(\tau, s)-R(t, s))xd\tau\\
&+B(\sigma)\int_{s}^{t-h}(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(t, s)xd\tau\\
&-B(\sigma)\exp(hA(t-h))R(t, s)x\\
&+B(\sigma)\exp((t-s)A(s))R(t, s)x .
\end{aligned}$$

Noting that by Lemma 1.3.

$$\begin{aligned}&\|B(\sigma)\exp(hA(t-h))R(t,s)x-B(\sigma)\exp(hA(t-h))R(t-h,s)x\|\\&\leqq C_1h^{\rho-1}\{K_1h(t-s)^{-1}+K_2h^{1-\rho'}(t-s)^{\rho'}\}\\&=C_1K_1h^{\rho}(t-s)^{-1}+C_1K_2h^{\rho-\rho'}(t-s)^{1-\rho'}\end{aligned}$$

for sufficiently small ρ', we obtain the relation

$$\begin{aligned}&(\partial/\partial t)B(\sigma)U(t,s)x=B(\sigma)A(s)\exp((t-s)A(s))x\\&+B(\sigma)\int_s^t A(\tau)\exp((t-\tau)A(\tau))(R(\tau,s)-R(t,s))xd\tau\\&+B(\sigma)\int_s^t(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(t,s)xd\tau\\&+B(\sigma)\exp((t-s)A(s))R(t,s)x\\&=B(\sigma)(\partial/\partial t)U(t,s)x\end{aligned}$$

We estimate each term of the right member of the above equality:

$$\begin{aligned}&\|B(\sigma)A(s)\exp((t-s)A(s))x\|\\&\leqq\|B(\sigma)\exp(2^{-1}(t-s)A(s))\|\,\|A(s)\exp(2^{-1}(t-s)A(s))\|\,\|x\|\\&=2^{2-\rho}CC_1(t-s)^{\rho-2}\|x\|.\end{aligned}$$

$$\begin{aligned}&\|B(\sigma)\int_s^t A(\tau)\exp((t-\tau)A(\tau))(R(\tau,s)-R(t,s))xd\tau\|\\&\leqq\int_s^t\|B(\sigma)\exp(2^{-1}(t-\tau)A(\tau))A(\tau)\exp(2^{-1}(t-\tau)A(\tau))(R(\tau,s)\\&-R(t,s))x\|d\tau\leqq\int_s^t C_1(2^{-1}(t-\tau))^{\rho-1}C(2^{-1}(t-\tau))^{-1}\{K_1(t-\tau)(t-s)^{-1}\\&+K_1(t-s)^{\rho'}(t-\tau)^{1-\rho'}\}\,\|x\|d\tau\\&\leqq 2^{1-\rho}CC_1\{K_1\rho^{-1}(t-s)^{\rho-1}+K_2(\rho-\rho')^{-1}(t-s)^{\rho}\}\,\|x\|.\end{aligned}$$

As for the third term, the uniform boundedness of $(t-\tau)^{1-\rho}\|B(\sigma)(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))\|$ which is easily verified proves that the term in question is bounded by const.$(t-s)^{\rho}\|x\|$. It is immediately seen that the last term is bounded by const.$(t-s)^{\rho-1}\|x\|$. Thus, we have proved the lemma.

Lemma 2.3. *For any $m\geqq 1$, we have*

$$\|U_m(t,s)\|\leqq\{C_3\Gamma(\rho)(t-s)^{\rho}\}^m/\Gamma(1+m\rho),\tag{2.10}$$

$$\|B(\sigma)U_m(t,s)\|\leqq\{C_3\Gamma(\rho)(t-s)^{\rho}\}^{m+1}/\Gamma((m+1)\rho)(t-s).\tag{2.11}$$

The above inequalities are easily obtained by induction.

Lemma 2.4. *We have with* $s_1=2^{-1}(t+s)$

$$
\begin{aligned}
(\partial/\partial t)U_1(t, s) &= \int_{s_1}^{t} A(\sigma)\exp((t-\sigma)A(\sigma))(B(\sigma)-B(t))U(\sigma, s)d\sigma \\
&+\int_{s_1}^{t}(\partial/\partial t+\partial/\partial\sigma)\exp((t-\sigma)A(\sigma))B(t)U(\sigma, s)d\sigma \\
&+\exp(2^{-1}(t-s)A(s_1))B(t)U(s_1, s) \\
&+\int_{s_1}^{t}\exp((t-\sigma)A(\sigma))B(t)(\partial/\partial\sigma)U(\sigma, s)d\sigma \\
&+\int_{s}^{s_1}A(\sigma)\exp((t-\sigma)A(\sigma))B(\sigma)U(\sigma, s)d\sigma \\
&+\int_{s}^{t}(\partial/\partial t)W(t, \sigma)B(\sigma)U(\sigma, s)d\sigma,
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
A(t)U_1(t, s) &= \int_{s_1}^{t} A(t)\exp((t-\sigma)A(\sigma))(B(\sigma)-B(t))U(\sigma, s)d\sigma \\
&+\int_{s_1}^{t}A(t)A(\sigma)^{-1}(\partial/\partial t+\partial/\partial\sigma)\exp((t-\sigma)A(\sigma))B(t)U(\sigma, s)d\sigma \\
&-B(t)U(t, s)+A(t)A(s_1)^{-1}\exp(2^{-1}(t-s)A(s_1))B(t)U(t, s) \\
&+\int_{s_1}^{t}(\partial/\partial\sigma)(A(t)A(\sigma)^{-1})\exp((t-\sigma)A(\sigma))B(t)U(\sigma, s)d\sigma \\
&+\int_{s_1}^{t}A(t)A(\sigma)^{-1}\exp((t-\sigma)A(\sigma))B(t)(\partial/\partial\sigma)U(\sigma, s)d\sigma \\
&+\int_{s}^{s_1}A(t)\exp((t-\sigma)A(\sigma))B(\sigma)U(\sigma, s)d\sigma \\
&+\int_{s}^{t}A(t)W(t, \sigma)B(\sigma)U(\sigma, s)d\sigma.
\end{aligned}
\tag{2.13}
$$

$$
\begin{aligned}
&\|(\partial/\partial t)U_1(t, s)\| \leqq C_5(t-s)^{\rho-1}, \\
&\|A(t)U_1(t, s)\| \leqq C_6(t-s)^{\rho-1},
\end{aligned}
\tag{2.14}
$$

where C_5 *and* C_6 *are constants independent of* t *and* s.

The deduction of the above relations is tedious and troublesome, but no special technique is needed. So we omit the proof.

The following two lemmas are easily proved by induction.

Lemma 2.5. *For* $m>1$, *we have*

$$(\partial/\partial t)U_m(t, s) = \int_{s}^{t}(\partial/\partial t)U_{m-1}(t, \sigma)B(\sigma)U(\sigma, s)d\sigma, \tag{2.15}$$

$$A(t)U_m(t, s) = \int_{s}^{t}A(t)U_{m-1}(t, \sigma)B(\sigma)U(\sigma, s)d\sigma. \tag{2.16}$$

There hold the following inequalities:

$$(2.17)\qquad \|(\partial/\partial t)U_m(t, s)\| \leqq C_5(C_3\Gamma(\rho)(t-s)^{\rho})^m/C_3\Gamma(m\rho)(t-s)\,,$$

$$(2.18)\qquad \|A(t)U_m(t, s)\| \leqq C_6(C_3\Gamma(\rho)(t-s)^{\rho})^m/C_3\Gamma(m\rho)(t-s)\,.$$

Lemma 2.6. *For $m\geqq 1$, we have the following relations*:

$$(2.19)\qquad (\partial/\partial t)U_m(t, s) = A(t)\,U_m(t, s)+B(t)U_{m-1}(t, s)\,.$$

Using the lemmas proved above we can conclude that the series (2.3) converges in the uniform operator topology and that $V(t, s)$ is strongly continuous in t and s simultaneously in $a\leqq s\leqq t\leqq b$. Furthermore, we can operate $\partial/\partial t$, $A(t)$ and $B(t)$ term by term, and $V(t, s)$ is shown to be the fundamental solution of the equation (2.2). Thus, we have

Theorem 2.1. *Under the assumptions* 1), 2), 3) *on $B(t)$, the operator $V(t, s)$ defined by the formula* (2.3) *gives the unique fundamental solution of the equation* (2.2) *with the following properties*:

(2.20) *$V(t, s)$ is strongly continuous in t and s simultaneously in*
$$a\leqq s\leqq t\leqq b\,,$$

$$(2.21)\qquad V(t, r) = V(t, s)V(s, r) \qquad \textit{for}\quad r\leqq s\leqq t\,,$$

$$(2.22)\qquad \|(\partial/\partial t)V(t, s)\| \leqq H_1(t-s)^{-1},\quad \|A(t)V(t, s)\| \leqq H_2(t-s)^{-1}$$
$$\|B(t)V(t, s)\| \leqq H_3(t-s)^{\rho-1}$$

with certain positive constants H_1, H_2 and H_3 independent of t and s.

§ 3. **Example.** As an example, we consider a parabolic differential equation with real coefficients:

$$(3.1)\qquad \begin{aligned}\partial u(t, x)/\partial t &= \sum_{i,j=1}^{n} a_{ij}(t, x)\partial^2 u(t, x)/\partial x_i\partial x_j\\ &+ \sum_{i=1}^{n} b_i(t, x)\partial u(t, x)/\partial x_i + c(t, x)u(t, x)+f(t, x)\end{aligned}$$

in a bounded domain $x\in G$ and $a\leqq t\leqq b$. We assume that

I) $\partial a_{ij}(t, x)/\partial t$ $(i, j=1, \cdots, n)$ are continuous in $\bar{G}\times[a, b]$,

II) $\partial^2 a_{ij}(t, x)/\partial x_k\partial x_l$, $\partial a_{ij}(t, x)/\partial x_k$, $\partial b_i(t, x)/\partial x_k$, $b_i(t, x)$ and $c(t, x)$ are continuous in $\bar{G}\times[a, b]$ and Hölder continuous in t continuous in $\bar{G}\times[a, b]$.

We put $D_i=\partial/\partial x_i$, $i=1, \cdots, n$, $A(t) = \sum a_{ij}(t, x)D_iD_j$ and $B(t) = \sum b_i(t, x)D_i+c(t, x)$.

We consider the following functional spaces in G:

$C_0^\infty(G)$: the set of all complex-valued infinitely differentiable functions with compact support in G.

$L^2(G)$: the set of all complex-valued squarely integrable functions in G.
H_m: $\{u \mid u \in L^2(g)$; the distribution derivative $D^\alpha u \in L^2(G)$ for every α with $|\alpha| \leqq m\}$.
$\overset{\circ}{H}_1$: the closure of $C_0^\infty(G)$ in H_1.

According to Nirenberg [4], $A(t)-\lambda I$ maps $\overset{\circ}{H}_1 \cap H_2$ onto $L^2(G)$ in one-to-one way for sufficiently large real number λ for every $t \in [a, b]$. By considering $e^{-\lambda t}u$ instead of u, we may assume that $A(t)$ itself has this property.

As preparation we consider an equation with sufficiently smooth coefficients independent of t.

$$A = \sum_{i,j=1}^m a_{ij}(x) D_i D_j + \sum_{i=3}^m b_i(x) D_i + c(x).$$

Lemma 3.1. *For some real number α, we have*

$$\overline{\lim_{|\tau|\uparrow\infty}} \sqrt{|\tau|}\, \| D_i((\alpha+\sqrt{-1}\tau)I-A)^{-1} \| < \infty. \tag{3.2}$$

Proof. We use notations in K. Yosida [6], pp. 111 and 112. For a pair (τ, w), $|\tau| \geq 2m^2\beta$, satisfying

$$|\mathfrak{J}_m(((\alpha+\sqrt{-1}\tau)I-A)w, w)| \leqq \sqrt{|\tau|-m^2\beta}\, \|w\|\, \|w\|_1,$$

we have

$$(|\tau|-m^2\beta)\,\|w\|^2 - m\beta\,\|w\|_1^2 \leqq \sqrt{|\tau|-m^2\beta}\,\|w\|\,\|w\|_1,$$

which implies $\sqrt{|\tau|-m^1\beta}\,\|w\| \leqq \sqrt{2m\beta+1}\,\|w\|_1$. Thus, for such a pair (τ, w) we have

$$\begin{aligned} |\mathfrak{R}_e(((\alpha+\sqrt{-1}\,\tau)I-A)w, w)| &\geqq (\delta - m\beta\nu)\,\|w\|_1^2 \\ &\geqq (\delta-m\beta\nu)(|\tau|-m^2\beta)^{1/2}(2m\beta+1)^{-1/2}\,\|w\|\,\|w\|_1. \end{aligned}$$

Thus (3.2) is proved.

Lemma 3.2. *Let T_t be the semigroup with A as its infinitesimal generator. Then $D_i T_t$ is bounded and*

$$\overline{\lim_{t\downarrow 0}} \sqrt{t}\, \|D_i T_t\| < \infty.$$

Proof. The representation of T_t:

$$T_t u = (2\pi i)^{-1} \int e^{\lambda t} (\lambda I - A)^{-1} u\, d\lambda, \qquad u \in \mathfrak{D}(A),\ t > 0,$$

where the integration is performed along the path $\tilde{\lambda}(s) = 2^{-1}\sigma(s) + i\tau(s)$ (see K. Yosida [5], p. 339), together with the above lemma gives the proof of the present lemma.

$\mathfrak{D}(A(t))=\mathring{H}_1 \cap H_2$ is independent of t and it is clear from the above assumptions and lemma and the results given in K. Yosida's papers that the Hypotheses given in §1 are all satisfied. If we consider $B(t)$ as a perturbing operator, then $B(t)$ also satisfies the assumptions given in §2. Thus we can construct the fundamental solution $U(t, s)$ for the equation (3.1) and that of its adjoint equation and they satisfy the estinates given in the corresponding theorems. We do not discuss here whether the solution $U(t, s)\phi+\int_s^t U(t, \sigma)f(\sigma)d\sigma$ corresponding to the initial data ϕ and the right member f is a classical one of the mixed problem.

§4. Successive derivatives of $U(t, s)$. We consider here successive derivatives of the fundamental solution $U(t, s)$ under the more restrictive assumptions about the smoothness of $A(t)A(s)^{-1}$.

Lemma 4.1. $(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s))$ *converges strongly to* 0 *as* $t-s\downarrow 0$.

Proof. Because of the uniform boundedness of the operators we have only to show its convergence to 0 on $\mathfrak{D}$. This convergence follows from

$$(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s))x$$
$$=\int_0^{t-s}\exp(\sigma A(s))A'(s)A(s)^{-1}\exp((t-s-\sigma)A(s))A(s)xd\sigma .$$

We assume that $A(t)A(s)^{-1}$ is twice continuously differentiable in t in the uniform operator topology for every s.

Lemma 4.2. $(\partial/\partial t+\partial/\partial s)R_1(t, s)$ *and* $(\partial/\partial t+\partial/\partial s)^2\exp((t-s)A(s))$ *are uniformly bounded and the former tends strongly to* 0 *as* $t-s\downarrow 0$.

Proof. From the relation

$$(\partial/\partial t+\partial/\partial s)R_1(t, s) = (A'(t)-A'(s))A(s)^{-1}\exp((t-s)A(s))$$
$$+(A(t)-A(s))(\partial/\partial t+\partial/\partial s)\exp((t-s)A(s)),$$

We can prove the uniform boundedness of the former. Its convergence to 0 as $t-s\downarrow 0$ can be proved as in Lemma 4.1. The uniformbounded-ness of the latter is also proved similarly.

Lemma 4.3. $R(t, s)$ *is twice continuously differentiable strongly in* t *for every* s *and the derivatives satisfy the estimates*:

$$\|(\partial/\partial t)R(t, s)\|\leqq L_1/t-s, \quad \|(\partial/\partial t)^2R(t, s)\|\leqq L_2/(t-s)^2 .$$

We omit the tedious proof of the above statement.

Under the assumptions made in this article, we can write the derivatives of $U(t, s)$ as follows $(s_1=(t+s)/2)$:

$$(\partial/\partial t)U(t, s) = A(s)\exp((t-s)A(s))+\int_s^{s_1}A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)d\tau$$
$$+\int_{s_1}^t(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))R(\tau, s)d\tau$$
$$+\exp((t-s_1)A(s_1)R(s_1, s)+\int_{s_1}^t\exp((t-\tau)A(\tau))(\partial/\partial\tau)R(\tau, s)d\tau.$$

$$(\partial/\partial t)^2U(t, s) = A(s)^2\exp((t-s)A(s))$$
$$+\int_s^{s_1}A(\tau)^2\,\mathrm{ext}\,((t-\tau)A(\tau))R(\tau, s)d\tau$$
$$+\int_{s_1}^{s}(\partial/\partial t+\partial/\partial\tau)^2\exp((t-\tau)A(\tau))R(\tau, s)d\tau$$
$$+(\partial/\partial t+\partial/\partial s_1)\exp((t-s_1)A(s_1))R(s_1, s)$$
$$+2\int_{s_1}^t(\partial/\partial t+\partial/\partial\tau)\exp((t-\tau)A(\tau))(\partial/\partial\tau)R(\tau, s)d\tau$$
$$+A(s_1)\exp((t-s_1))R(s_1, s)+\exp((t-s_1)A(s_1))(\partial/\partial s_1)R(s_1, s)$$
$$+\int_{s_1}^t\exp((t-\tau)A(\tau))(\partial/\partial\tau)^2R(\tau, s)d\tau.$$

Similar formulas are obtained for $A(t)^2U(t, s)$, $(\partial/\partial t)(A(t)U(t, s))$ and $A(t)(\partial/\partial t)U(t, s)$. Thus we obtain

Theorem 4.1. *Under the assumptions made above* $(\partial/\partial t)^2U(t, s)$, $A(t)^2U(t, s)$, $(\partial/\partial t)(A(t)U(t, s))$ *and* $A(t)(\partial/\partial t)U(t, s)$ *exist and their norms are bounded above by* $H_4/(t-s)^2$ *for some positive constant* H_4. *If Hypothesis* $1^*\sim3^*$ *for* $\{A^*(t)\}$ *are satisfied, then*

$(\partial/\partial t)^2(\partial/\partial s)U(t, s)$, $(\partial/\partial t)^2(\partial/\partial s)^2U(t, s)$, $(\partial/\partial t)^2U(t, s)A(s)$,
$(\partial/\partial t)U(t, s)A(s)^2$, $(\partial/\partial t)^2U((t, s)A(s)^2$ *etc.*

all exist and satisfy the similar estimates as above.

When $A(t)A(s)^{-1}$ is differentiable more times, we can obtain similar results for the higher derivatives of $U(t, s)$ than two.

§ **5.** We consider a special case where each commutator $A(t)A(s)-A(s)A(t)$ satisfies the following conditions:

I) $(A(t)A(s)-A(s)A(t))A(s)^{-1}A^{-1}$ is bounded where A may be taken as one of $A(t)$, $a\leqq t\leqq b$,

II) for some positive constants N and $\lambda\leqq1$.

$$\|(A(t)A(s)-A(s)A(t))A(s)^{-1}A^{-1}\|\leqq N(t-s)^{\lambda}.$$

In this case we may replace the differentiability of $A(t)A(s)^{-1}$ in Hypotheses 2° and 3° by its Hölder continuity in t with exponent λ:

$$\|(A(t')-A(t))A(s)^{-1}\| \leqq K(t'-t)^{\lambda} \qquad \text{for} \quad a \leqq s,\ t,\ t' \leqq b\,.$$

Under these assumptions, we can construct the Green operator $U(t, s)$ and the auxiliary operator $R(t, s)$ just in the same way as in § 1.

Lemma 5.1. *For $a \leqq s \leqq t \leqq b$, we have*

$$\|R(t, s)\| \leqq C_1(t-s)^{\lambda-1}$$

with some constant C_1. Moreover, $\|A(t)R(t, s)\|$ is bounded and with some constant C_2

$$\|A(t)R(t, s)\| \leqq C_2(t-s)^{\lambda-2}\,.$$

Proof. $\|R_1(t, s)\| = \|(A(t)-A(s))A(s)^{-1}A(s)\exp((t-s)A(s))\| \leqq KC(t-s)^{\lambda-1}$. By induction, we can prove $\|R_m(t, s)\| \leqq (CK\Gamma(\lambda))^m(t-s)^{m\lambda-1}/\Gamma(m\lambda)$. And

$$\begin{aligned}
&\|A(t)R_1(t, s)\| = \|A(t)A(s)^{-1}A(s)(A(t)-A(s))\exp((t-s)A(s))\| \\
&= \|A(t)A(s)^{-1}\| \{\|(A(t)-A(s))A(s)\exp((t-s)A(s))\| \\
&+ \|(A(t)A(s)-A(s)A(t))\exp((t-s)A(s))\|\} \\
&\leqq M(4C^2K(t-s)^{\lambda-2}+4C^2MN(t-s)^{\lambda-2}) = K_1(t-s)^{\lambda-2}\,.
\end{aligned}$$

Similarly, $\|A(t)R_i(t, s)\| \leqq K_i(t-s)^{\lambda-2}$ for $2 \leqq i \leqq m_0$ where m_0 is the least natural number satisfying $m_0\lambda-2>-1$. By induction, it follows that for $m=1, 2, \cdots$

$$\|A(t)R_{m_0+m}(t, s)\| \leqq K_{m_0}\Gamma(m_0\lambda-1)(KC\Gamma(\lambda))^m(t-s)^{(m_0+m)\lambda-2}/\Gamma((m_0+m)\lambda-1)\,.$$

q. e. d.

Using the above lemma, we obtain

$$\begin{aligned}
&(\partial/\partial t)U(t, s) = A(s)\exp((t-s)A(s))+R(t, s) \\
&+\int_{s_1}^{t}\exp((t-\tau)A(\tau))A(\tau)R(\tau, s)d\tau+\int_{s}^{s_1}A(\tau)\exp((t-\tau)A(\tau))R(\tau, s)d\tau\,, \\
&A(t)U(t, s) = A(t)\exp((t-s)A(s)) \\
&+\int_{s}^{t}A(t)A(\tau)^{-1}\exp((t-\tau)A(\tau))A(\tau)R(\tau, s)d\tau \\
&+\int_{s}^{s_1}A(t)\exp((t-\tau)A(\tau))R(\tau, s)d\tau\,.
\end{aligned}$$

We have $\|(\partial/\partial t)U(t, s)\| = \|A(t)U(t, s)\| \leqq H/t-s$ with some positive constant H as in the previous case.

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Bibliography

[1] S. D. Eidelman: On fundamental solutions of parabolic system. Math. Sbor. **38** (1956), 51–92.

[2] T. Kato: Integration of the equation of evolution in a Banach space. J. Math. Soc. Japan, **5** (1953), 208–234.

[3] E. Hille and R. S. Phillips: Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ. (1957).

[4] L. Nirenberg: Remarks on strongly elliptic partial differential equations, Comm. Pure and Appl. Math. **8** (1955), 649–675.

[5] K. Yosida: On the differentiability of semi-groups of linear operators, Proc. Japan Acad. **34** (1958), 337–340.

[6] K. Yosida: An abstract analyticity in time for solutions of a diffusion equation, Proc. Japan Acad. **35** (1959), 109–113.