

Kernel Functions of Diffusion Equations II

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The present paper is a continuation of the author's previous paper "Kernel Functions of Diffusion Equation I", Osaka Mathematical Journal Vol. 9, 1957, pp. 201-214. Some notations which were defined in the previous paper will be used without repeating the definitions.

2. Suppose that D be regularly open and ∂D be smooth. Then Theorem 1 of the previous paper holds and $K(x, y; t)$ is a well defined continuous non-negative function, which is smaller than $E_t(x, y)$. In this paper the dimension d is assumed to be ≥ 3 . Set

$$(1) \quad G(x, y) = \lim_{h \rightarrow 0} \int_h^\infty K(x, y; t) dt = \int_{+0}^\infty K(x, y; t) dt$$

Lemma 2.1. $G(x, y)$ is the Green's function of the Laplacian over D with zero boundary.

Proof. Take a C^2 -function $\varphi(y)$ and set $\varphi_s(x) = \int_D K(x, y; s) \varphi(y) dy$ over D . Then

$$\begin{aligned} (2) \quad \Delta_x \int_D G(x, y) \varphi_s(y) dy &= \int_D \Delta_x G(x, y) \varphi_s(y) dy \\ &= \int_D \int_{+0}^\infty \Delta_x K(x, y; t+s) dt \varphi(y) dy \\ &= \int_D \int_{+0}^\infty \frac{\partial}{\partial t} K(x, y; t+s) dt \varphi(y) dy \\ &= \int_D \lim_{h \rightarrow 0} K(x, y; h+s) \varphi(y) dy \\ &= \lim \int_D K(x, y; h+s) \varphi(y) dy \\ &= \varphi_s(x), \end{aligned}$$

Therefore by making s towards 0, we have the required relation, which proves the lemma.

Now take an arbitrary bounded open set D and consider an increasing sequence of bounded open sets $\{D_k\}$ with smooth boundaries converging to D . To each D_k we can associate the kernel function $K_k(x, y; t)$ which forms an increasing sequence of non-negative functions.

Define for each k

$$(3) \quad \lim (E_{t/n} *_{D_k})^n = K_k(x, y; t).$$

Here $*_{D_k}$ should be understood as a convolution over D_k . Evidently $K_k(x, y; t)$ is an increasing function in k and

$$(4) \quad 0 \leq \int_{D_k} K_k(x, y; t) dy \leq \int_D E_t(x, y) dy \leq 1.$$

Hence everywhere in D

$$\lim_k K_k(x, y; t) = K(x, y; t)$$

exists.

Lemma 2.3. *Suppose that both x and y are in D . Then $K(x, y; t)$ is continuous in x and in y at least separately.*

Proof. Because of being the strong limit of K_k 's, $K(x, y; t)$ is non-negative and is a strong solution of a diffusion equation $\partial U / \partial t = \Delta U$. Hence K is a genuine solution¹⁾. Therefore k is continuous in x and in y at least separately.

Since

$$(5) \quad 0 \leq K(x, y; t) \leq E_t(x, y),$$

$$(6) \quad \begin{aligned} G(x, y) &= \lim_{h \rightarrow 0} \int_h^\infty K(x, y; t) dt \\ &= \int_{+0}^\infty K(x, y; t) dt \leq \int_0^\infty E_t(x, y) dt \end{aligned}$$

is a well defined function unless $x = y$.

Clearly

$$(7) \quad \begin{aligned} G(x, y) &= \lim_{k \rightarrow \infty} G_k(x, y) \\ &= \int_{+0}^\infty K(x, y; t) dt \end{aligned}$$

when both x and y are in D . This $G(x, y)$ is called as a generalized Green's function²⁾ of the Laplacian over D .

Suppose that there are given a point y on ∂D and a sequence of points $\{y_m\}$ in D convergent to y . We further assume that

$$(8) \quad \lim_{m \rightarrow \infty} G(x, y_m) = 0.$$

1) See (15) of the previous paper I. There are other papers where this result or a more generalized one is given.

2) The author does not claim at all that the introduction of such a definition is original. Indeed, Bouligand, Kellogg and de la Vallée-Poussin already had introduced such definition. However, the author does not have any decisive information as to who was the first to have done it.

Remark: Kellogg's³⁾ result says that except for y on a set of capacity 0, (8) holds.

Lemma 2.3.

$$(9) \quad \lim_{m \rightarrow \infty} \int_s^\infty K(x, y_m; t) dt = 0.$$

Proof.

$$(10) \quad \begin{aligned} 0 &\leq \lim_{m \rightarrow \infty} \int_s^\infty K(x, y_m; t) dt \\ &\leq \lim_m \int_{+0}^\infty K(x, y_m; t) dt \\ &= \lim_{m \rightarrow \infty} G(x, y_m) = 0. \end{aligned}$$

Hence the lemma is proved.

Lemma 2.4. *The sequence*

$$(11) \quad \left\{ \int_s^\infty K(x, y_m; t) dt \right\}$$

constitutes a family of equi-continuous functions over any compact set C contained in D .

Proof.

$$(12) \quad \begin{aligned} \int_s^\infty K(x, y_m; t) dt &= \int_s^\infty \int_D K(x, z; s/2) K(z, y_m; t-s/2) dz dt \\ &= \int_D \left[K(x, z; s/2) \int_{s/2}^\infty K(z, y_m; t) dt \right] dz. \end{aligned}$$

However

$$(13) \quad \begin{aligned} \int_{s/2}^\infty K(z, y_m; t) dt &\leq \int_{s/2}^\infty E_t(x, y_m; t) dt \\ &\leq \sqrt{2\pi}^{-d} \frac{d}{2} \left(\frac{s}{2}\right)^{(-d/2)+1} \\ &= \frac{sd}{4} \sqrt{s\pi}^{-d}. \end{aligned}$$

Therefore

$$(14) \quad \begin{aligned} &\left| \int_s^\infty K(x, y_m; t) dt - \int_s^\infty K(x', y_m; t) dt \right| \\ &\leq \frac{sd}{4} \sqrt{s\pi}^{-d} \left| \int_D (K(x, z; s/2) - K(x', z; s/2)) dz \right|. \end{aligned}$$

3) See (14) of the previous paper I.

The right hand side can be made arbitrarily small if x and x' are sufficiently near to each other, because the function $\int_D K(x, z; s/2) dz$ itself is a solution of the diffusion equation, and is therefore uniformly continuous over C . This proves the lemma.

Lemma 2.5.

$$(15) \quad \lim_{m \rightarrow \infty} \int_D \left(\int_s^\infty K(x, y_m; t) dt \right)^2 dx = 0.$$

Proof. Given small positive ε , there exists a compact subset C of D such that

$$(16) \quad \begin{aligned} & \int_{D-C} \left(\int_s^\infty K(x, y_m; t) dt \right)^2 dx \\ & \leq \int_{D-C} \left(\frac{sd}{4} \right)^2 (s\pi)^{-d} dx \\ & = \left(\frac{sd}{4} \right)^2 (s\pi)^{-d} \text{meas } (D-C) \\ & \leq \varepsilon. \end{aligned}$$

For this C there exists a large m_C such that if $m \geq m_C$, then

$$(17) \quad \int_s^\infty K(x, y_m; t) dt \leq \sqrt{\varepsilon}$$

uniformly over C because of Lemmas 2.3. and 2.4.

Hence

$$(18) \quad \begin{aligned} \int_D \left(\int_s^\infty K(x, y_m; t) dt \right)^2 dx & \leq \varepsilon + \varepsilon \text{meas } C \\ & \leq \varepsilon (1 + \text{meas } D), \end{aligned}$$

if $m \geq m_C$. This proves the lemma.

Lemma 2.6.

$$(19) \quad \int_D \left(\int_{s/2}^\infty K(x, y; t) dt \right)^2 dx \geq \frac{s}{2} \int_D (K(x, y; s))^2 dx.$$

Proof. In order to prove this lemma, the Fourier expansion with respect to Δ with 0 boundary condition will be employed.

Since D_k 's are bounded domains with smooth boundaries, this type of Fourier expansion is available. Namely there exist eigenvalues $-\lambda_i^{(k)}$'s and normalized eigenfunctions $\theta_i^{(k)}$ of Δ satisfying:

$$(20) \quad \Delta \theta_i^{(k)} = -\lambda_i^{(k)} \theta_i^{(k)}$$

for $i=1, 2, \dots$, and $k=1, 2, \dots$

$$(21) \quad \lambda_i^{(k)} > 0 \quad \text{and}$$

$$(21)' \quad \lim_{i \rightarrow \infty} \lambda_i^{(k)} = \infty .$$

Then, in the L^2 -sense

$$(22) \quad \int_{s/2}^{\infty} K_k(x, y; t) dt \\ = \sum_{i=1}^{\infty} \exp(-\lambda_i^{(k)} s/2) / \lambda_i^{(k)} \cdot \theta_i^{(k)}(x) \theta_i^{(k)}(y) .$$

Therefore

$$(23) \quad \int_{D_k} \left(\int_{s/2}^{\infty} K_k(x, y; t) dt \right)^2 dx \\ = \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s/2) / \lambda_i^{(k)})^2 (\theta_i^{(k)}(y))^2 \\ = \sum_{i=1}^{\infty} \left(\exp\left(\lambda_i^{(k)} \frac{s}{2}\right) / \lambda_i^{(k)} \right)^2 (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ \geq \frac{s}{2} \sum_{i=1}^{\infty} (\exp(-\lambda_i^{(k)} s))^2 (\theta_i^{(k)}(y))^2 \\ = \frac{s}{2} \int_{D_k} (K_k(x, y; s))^2 dx .$$

By making k tend to infinity we have

$$(24) \quad \int_D \left(\int_{s/2}^{\infty} K(x, y; t) dt \right)^2 dx \geq \frac{s}{2} \int_D (K(x, y; s))^2 dx$$

which proves the lemma.

Immediately from Lemma 2.5 and Lemma 2.6,

Lemma 2.7.

$$(25) \quad \lim_{m \rightarrow \infty} \int_D (K(x, y_m; s))^2 dx = 0$$

Now we are going to prove that

Theorem 2. *If $\lim_{m \rightarrow \infty} G(x, y_m) = 0$,*

then

$$(26) \quad \lim_{m \rightarrow \infty} K(x, y_m; s) = 0$$

for any positive s .

Proof.

$$\begin{aligned}
 (27) \quad 0 \leq K(x, y_m; s) &\leq \int_D K\left(x, z; \frac{s}{2}\right) K\left(z, y_m; \frac{s}{2}\right) dz \\
 &\leq \left(\int_D \left(K\left(x, z; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \left(\int_D \left(K\left(z, y_m; \frac{s}{2}\right)\right)^2 dz\right)^{1/2} \\
 &\leq \sqrt{2\pi \frac{s}{2}}^{-d} (\text{meas } D) \left(\int_D K\left(z, y_m; \frac{s}{2}\right)\right)^2 dz\right)^{1/2}
 \end{aligned}$$

where the right hand side will go to 0 as m goes to infinity because of Lemmas 2.5, 2.6 and 2.7. Hence the theorem is proved.

Remark: Throughout this paper Δ does not have to be the Laplacian, but has only to be a Laplace-Beltrami operator with respect to a C^2 -Riemannian structure which is continuous on the boundary ∂D .

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