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On Cauchy Problem for Linear Partial Differential Equations with Variable Coefficients

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1. Introduction.

This paper is concerned with Cauchy problem for the general system of operators

$$\frac{\partial}{\partial t} U - AU, \tag{1.1}$$

where A is an (m, m)-matrix of differential operators of arbitrary order p_{ij} independent of $\frac{\partial}{\partial t}$ such that

$$\left.\begin{aligned} &A=(a_{ij}) \\ &a_{ij}=\sum_{\lambda_1+\cdots+\lambda_N=0}^{p_{ij}} a_{ij}^{(\lambda_1\cdots\lambda_N)} \frac{\partial^{\lambda_1+\cdots+\lambda_N}}{\partial x_1^{\lambda_1}\cdots\partial x_N^{\lambda_N}} \qquad (i, j=1, 2, \cdots, m)\,. \end{aligned}\right\} \tag{2.1}$$

The coefficients $a_{ij}^{(\lambda_1\cdots\lambda_N)}$ may depend on the time variable $t\in R_t^1$ as well as on the space variables $x=(x_1, x_2, \cdots, x_N)\in R_x^N$.

We shall prove in this note that this problem has a unique solution in a certain Hilbert space under the general condition about A: the hermitian part of A is semi-bounded in the sense of the norm induced by other hermitian matrices B of operators. This is a generalization of Leray's condition [7] which is used to solve Cauchy problem for regular hyperbolic equations. We owe our proof of this assertion in my former paper [12] essentially to Yosida's method on semigroup [13], but in the present note we prove this by another direct method using the duality of Hilbert spaces of functions defined on the product space $R_t^1\times R_x^N$. The idea originates from Nagumo, who considered more general abstract form of our theorem [10].

In Section 2 we consider a general uniformly strongly elliptic operator and introduce Hilbert spaces used in the later sections. In Section 3 we give our main theorem which is applicable to reversible systems which contain hyperbolic systems in the sense of Leray and to parabolic systems in more general sense than Petrowsky's [11]. Furthermore we show that our main theorem implies the hypoellipticity of such parabolic systems (Section 4).

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2. Uniformly strongly elliptic operators.

We consider first of all a smooth system of complex linear differential operators in N-Euclidean vector space R_x^N or in $N+1$-Euclidean vector space $R_t^1\times R_x^N$: that is, the coefficients of operators are defined over R_x^N or $R_t^1\times R_x^N$, where they have bounded derivatives up to sufficiently large order for the purpose at hand.

Let $B\left(x,\frac{\partial}{\partial x}\right)=\left(b_{ij}\left(x,\frac{\partial}{\partial x}\right)\right)$ $(i, j=1, 2, \cdots, m)$ be an (m, m)-smooth system of differential operators in R_x^N. Let $O(B)$ be the set $s=\{s(i)\,|\,i=1, 2, \cdots, m\}$ of positive integers such that the order $o(b_{ij})$ of b_{ij} does not exceed $s(i)+s(j)$ and let us denote by $h_{ij}\left(x,\frac{\partial}{\partial x}\right)$ the part of order $s(i)+s(j)$ of $b_{ij}\left(x,\frac{\partial}{\partial x}\right)$.

Definition 1. We call that $B\left(x,\frac{\partial}{\partial x}\right)$ is *uniformly strongly elliptic*, if the hermitian part of $H(x, i\xi)=(h_{ij}(x, i\xi))$ is uniformly positive definite for all $x\in R_x^N$ and all real vector $\xi:|\xi|=1$, i.e., there is a positive ρ such that

$$\sum_{ij}(h_{ij}(x, i\xi)+\overline{h_{ji}(x, i\xi)})\, v_i v_j>\rho \tag{1.2}$$

for any real vector v: $|v|=(\sum_{i=1}^{m}|v_i|^2)^{\frac{1}{2}}=1$.

Then since $H(x, i\xi)+H(x, i\xi)^*$ is hermitian, using (1.2) with $-\xi$, we see that

$$\sum_{ij}(h_{ij}(x, i\xi)+\overline{h_{ji}(x, i\xi)})\, v_i\overline{v_j}\geqq\rho\sum_{i=1}^{m}|\xi|^{2s(i)}|v_i|^2 \tag{2.2}$$

for all complexes $v_1\cdots v_m$. We write the above relation as follows:

$$H(x, i\xi)+H(x, i\xi)^*\geqq\rho Q'^s(i\xi),$$

where $q'(i\xi)(=|\xi|^2)=\xi_1^2+\xi_2^2+\cdots+\xi_N^2$,

$$Q'^s(i\xi)=\begin{pmatrix} q'(i\xi)^{s(1)} & & \\ & \ddots & \\ & & q'(i\xi)^{s(m)}\end{pmatrix}$$

Now let$H(x, i\xi)$ satisfy the following:

$$\sum_{ij} h_{ij}(x, i\xi)\, v_i\overline{v_j} > \rho > 0 \tag{1'.2}$$

for all $x \in R_x^N$ and all real vector $\xi: |\xi| = 1$ and all complex $v: |v| = 1$. Then $H(x, i\xi)$ is hermitian. Let $h_{ij}(x, \xi) = h'_{ij}(x, \xi) + ih''_{ij}(x, \xi)$ where the $h'_{ij}(x, \xi)$ and $h''_{ij}(x, \xi)$ are real. Then the matrix

$$\begin{pmatrix} H'(x, i\xi), & -H''(x, i\xi) \\ H''(x, i\xi), & H'(x, i\xi) \end{pmatrix} \tag{3.2}$$

is also hermitian positive definite, where $H'(x, \xi) = (h'_{ij}(x, \xi))$ and $H''(x, \xi) = (h''_{ij}(x, \xi))$.

By Leray's completion ([7]) of Gårding's lemma and from (2.2) we see the following

Lemma 1. *If* $B\left(x, \dfrac{\partial}{\partial x}\right)$ *is uniformly strongly elliptic, then there is a positive* $k(=k_\varepsilon)$ *such that for any sufficiently small* $\varepsilon > 0$

$$B\left(x, \frac{\partial}{\partial x}\right) + B\left(x, \frac{\partial}{\partial x}\right)^* + k \geqq (\rho - \varepsilon)\, Q_x^s\left(\frac{\partial}{\partial x}\right),$$

where $q(i\xi) = q'(i\xi) + 1$

$$Q_x^s(i\xi) = \begin{pmatrix} q(i\xi)^{s(1)} & & \\ & \ddots & \\ & & q(i\xi)^{s(m)} \end{pmatrix}$$

and where ρ *is the same as* ρ *in* (1.2). *Here* $B \geqq C$ *means that*

$$(Bu, u) = \sum_{ij} \int_{R_N^l} b_{ij}\left(x, \frac{\partial}{\partial x}\right) u_i \cdot \bar{u}_j dx,$$

$$(Bu, u) \geqq (Cu, u)$$

for $u = (u_1, u_2, \cdots, u_m) \in \mathfrak{D}_x$ *i.e.,* u_i *is an infinitely differentiable function defined on* R_x^N *with compact carrier.*

From Lemma 1 we see the following

Lemma 1'. *If* $B\left(x, \dfrac{\partial}{\partial x}\right)$ *is hermitian i.e.,* $(Bu, v) = (u, Bv)$ *for any* $u, v \in \mathfrak{D}_x$ *and uniformly strongly elliptic* ($H(x, i\xi)$ *satisfies* (1'.2)), *and if we set*

$$\tilde{B}\left(x, \frac{\partial}{\partial x}\right) = \begin{pmatrix} B'\left(x, \dfrac{\partial}{\partial x}\right), & -B''\left(x, \dfrac{\partial}{\partial x}\right) \\ B''\left(x, \dfrac{\partial}{\partial x}\right), & B'\left(x, \dfrac{\partial}{\partial x}\right) \end{pmatrix}$$

where $B\left(x, \frac{\partial}{\partial x}\right)=B'\left(x, \frac{\partial}{\partial x}\right)+iB''\left(x, \frac{\partial}{\partial x}\right)$, *the coefficients of* B' *and* B'' *being real. Then there are positive* k *and* ρ *such that*

$$\tilde{B}\left(x, \frac{\partial}{\partial x}\right)+k \geqq \rho Q_x^{\tilde{s}}\left(\frac{\partial}{\partial x}\right),$$

where $\tilde{s}(i)=s(i)$, $\tilde{s}(i+m)=s(i)$ *for* $i=1, 2, \cdots, m$.

For $\tilde{B}\left(x, \frac{\partial}{\partial x}\right)$ associates with (3. 2).

Let q_t be the differential operator $1-\frac{\partial^2}{\partial t^2}$ and let $B\left(t, x, \frac{\partial}{\partial x}\right)$ be a smooth system, in $R_t^1 \times R_x^N$, which does not contain $\frac{\partial}{\partial t}$ and which is hermitian and uniformly strongly elliptic with the same s and ρ for all $t \in R_t^1$. This operator is simply denoted by $B_t^{(s)}$ (when $t \in R_t^1$ is fixed), $B^{(s)}$ or B. Furthermore we introduce Hilbert spaces as follows:

Definition 2. For any integer n and any set s of integers $s(1), \cdots, s(m)$, we denote by H_x^s and $H_{t,x}^{n,s}$ the completions of spaces $\mathfrak{D}_x$ and $\mathfrak{D}_{t,x}$ with the following inner products respectively:

$$((u, v))_s=\sum_{i=1}^{m} \int_{R_x^N} q_x\left(\frac{\partial}{\partial x}\right)^{s(i)} u_i(x) \cdot \overline{v_i(x)}\, dx,$$

$$((u, v))_{n,s}=\sum_{i=1}^{m} \int_{R_t^1 \times R_x^N} q_t\left(\frac{\partial}{\partial t}\right)^n q_x\left(\frac{\partial}{\partial x}\right)^{s(i)} u_i(t, x) \cdot \overline{v_i(t, x)}\, dtdx,$$

where (for $n<0$ or $s(i)<0$) we set

$$q_x\left(\frac{\partial}{\partial x}\right)^{s(i)} f(x)=(2\pi)^{-N} \int \exp(2\pi i x \cdot \xi)\, q_x(i\xi)^{s(i)} \mathfrak{F}_x(f)\, d\xi_1 d\xi_2 \cdots d\xi_N,$$

$$q_t\left(\frac{\partial}{\partial t}\right)^n f(t)=(2\pi)^{-1} \int \exp(2\pi i t \cdot \tau)\, q(i\tau)^n \mathfrak{F}_t(f)(\tau)\, d\tau,$$

where $\mathfrak{F}_x(f)$ and $\mathfrak{F}_t(f)$ are Fourier transform of f with respect to x and t respectively.

Lemma 2. *Let* $B\left(t, x, \frac{\partial}{\partial x}\right)$ *be an* (m, m)*-smooth hermitian (symmetric) system defined on* $R_t^1 \times R_x^N$ *mentioned above. Then the weak extension of* $B\left(t, x, \frac{\partial}{\partial x}\right)+k$ *is an isomorphic opearator on* $H_{t,x}^{n,l+s}$ *onto* $H_{t,x}^{n,l-s}$ *for any integers* n *and* l, *where* $l+s=\{l+s(i) \mid i=1, 2, \cdots, m\}$, *and* k *depends only on* l *and* n.

Proof. By Lemma 1′ we may assume that the coefficients of B and the Hilbert spaces H_x^{l+s}, H_x^{l-s} are real. Then by Lemma 1 and by the smoothness of $B\left(t, x, \frac{\partial}{\partial x}\right)$ for any positive l there are positive $k(l)$ and $\rho(l)$ independent from t such that

$$(Q_x^l(B_t+k(l))\, u, u) \geqq \rho(l)(Q_x^{l+s}u, u) \quad \text{for} \quad u \in \mathfrak{D}_x . \tag{4.2}$$

Let

$(Q_x^l(B+k(l))\, u, u)$ for $u \in \mathfrak{D}_{t,x}$ be the following:

$$(Q_x^l(B+k(l))\, u, u) = \int_{R_t^1} (Q_x^l(B_t+k(l))\, u, u)\, dt \quad \text{for} \quad u \in \mathfrak{D}_{t,x} .$$

Then, since $B+k(l)$ is symmetric, it follows from (4.2) that

$$((B+k(l))\, Q_x^l u, u) \geqq \rho(l)(Q_x^{l+s}u, u) \quad \text{for} \quad u \in \mathfrak{D}_{t,x} .$$

Setting $u = Q_x^{-l}w$, using a limit process the above inequality implies that for any negative integer l

$$(Q_x^l(B+k(l))\, u, u) \geqq \rho(l)(Q_x^{l+s}u, u) \quad \text{for} \quad u \in \mathfrak{D}_{t,x} .$$

Since $\left(Q_x^l(B+k(l))\frac{\partial}{\partial t}u, \frac{\partial}{\partial t}u\right) - \left(-\frac{\partial^2}{\partial t^2}Q_x^l(B+k(l))\, u, u\right) = -\left(Q_x^l B'u, \frac{\partial}{\partial t}u\right)$ $= \left(Q_x^l B' \frac{\partial}{\partial t}\, u, u\right) - (Q_x^l B''u, u)$, where $O(B')$, $O(B'') \leqq s$, i.e., $o(b'_{ij})$, $o(b''_{ij})$ $\leqq s(i)+s(j)$, we see that

$$\begin{aligned}(q_t Q_x^l(B+k(l))\, u, u) &\geqq \rho(l)(q_t Q_x^{l+s}u, u) - \rho'(l)(Q_x^{l+s}u, u) \\ &\quad - \frac{1}{2}\left|\left((Q_x^l B' - B' Q_x^l)\frac{\partial}{\partial t}\, u, u\right)\right|\end{aligned}$$

for some positive $\rho'(l)$.

Thus by the smoothness of B' in $R_t^1 \times R_x^N$ for any integer l we see the following:

$$(q_t Q_x^l(B+k(1, l))\, u, u) \geqq \rho(1, l)(q_t Q_x^{l+s}u, u) \quad \text{for} \quad u \in \mathfrak{D}_{t,x} .$$

In the same way as above we see that for any $n \geqq 0$ and any l

$$(q_t^n(Q_x^l(B+k(n, l))\, u, u) \geqq \rho(n, l)(q_t^n Q_x^{l+s}u, u) . \tag{5.2}$$

Furthermore (5.2) implies that for any $n \geqq 0$

$$(Q_x^l(B+k(n, l))\, q_t^n u, u) \geqq \rho(n, l)(q_t^n Q_x^{l+s}u, u) ,$$

hence we see that (5.2) is valid for any n and l.

Finally from (5.2) it follows that for any n and l

$$\begin{aligned}&\|(B+k(n, l))\, u\|_{n, l-s} \geqq \rho(n, l)\|u\|_{n, l+s} \\ &\|q_t^{-n}Q^{-(l+s)}(B+k(n, l))\, q_t^n Q_x^{l-s}u\|_{n, l+s} \geqq \rho(n, l)\|u\|_{n, l-s} \quad \text{for} \quad u \in \mathfrak{D}_{t,x} .\end{aligned}$$

Accordingly we see that the duality of Hilbert space implies our assertion. (See § 3, the proof of Theorem 1 and also [5] [6] [7]). Q.E.D.

Here we remark that our extension of $B\left(t, x, \frac{\partial}{\partial x}\right)$ is not only the weak extension, but the strong one, i.e., $Bu=v$ means that there is a sequence $\{u_n\}$ in $\mathfrak{D}_{t,x}$ such that $u_n \to u$ in $H^{n,l+s}_{t,x}$ and $Bu_n \to v$ in $H^{n,l-s}_{t,x}$, and that Lemma 2 implies that if v is in $H^{n,l-s}_{t,x}$ and u is in $H^{n',l'+s}_{t,x}$ such that $n'<n$, $l'<l$ and $Bu=v$, then u is in $H^{n,l+s}_{t,x}$, but that the assertion of Lemma 2 is about differentiability of global solutions, but not about that of local solutions. (See Theorem 7).

3. Main theorem

DEFINITION 3. We say that *the hermitian part of a system* $A=A\left(t, x, \frac{\partial}{\partial x}\right)$ *of type* (2.1) *is semi-bounded in the sense of the norm induced by a system* B *in* § 2 if the following inequality holds for any $t \in R^1_t$:

$$B\left(t, x, \frac{\partial}{\partial x}\right) A\left(t, x, \frac{\partial}{\partial x}\right) + A^*\left(t, x, \frac{\partial}{\partial x}\right) B\left(t, x, \frac{\partial}{\partial x}\right) \leqq \alpha(t)\, B\left(t, x, \frac{\partial}{\partial x}\right) \tag{1.3}$$

for some continuous positive function $\alpha(t)$ on R^1_t.

For the sake of simplicity we consider only the case where $\alpha(t)$ is a constant and where the coefficients of operators and functions are all real, as in § 2.

By Leray it was shown that for a regular hyperbolic system of operators (1.1), there are B's such that they satisfy (1.3) with $-\alpha B \leqq BA + A^*B$. Then using the energy inequality followed from (1.3) and applying Cauchy-Kowalewski's theorem he solve Cauchy problem for regular hyperbolic equations. Then by Friedrichs [2], Lex [6] and Yosida [14] and others it was shown that without using Cauchy-Kowalewski's Theorem that problem can be solved for normal hyperbolic equations. Recently it was shown by S. Mizohata that applying Leray's method to regular parabolic operators (1.1) in the sense of Petrowsky [11] (See also S. D. Eidelman [1]), there are B's such that for some positive α $B_1A + A^*B_1 \leqq -\alpha B_2$, where $O(B_2)=O(B_1A)$, and that using T. Kato's Theorem for semigroup Cauchy problem can be solved for such equations. We show in this section that the weaker condition (1.3) is sufficient to solve Cauchy problem for (1.1) when initial functions and solutions u_t are considered as elements in H^s_x.

Throughout this section we assume that the hermitian part of A is semi-bounded in the sense of the norm induced by a system B in § 2 where B is considered as $B+k$ for a sufficiently large k. Then setting $\bar{A}_t = A_t - \beta I$ ($\beta > 0$) we see the following.

Lemma 3. *For any* $u \in \mathfrak{D}_{t,x}$ (1.3) *implies the following inequalities*:

$$\int_{-\infty}^{\infty} \left(\left(\frac{\partial}{\partial t} - \bar{A}_t\right) u_t,\ u_t\right)\Big)_{B_t} dt \geq \delta \int_{-\infty}^{\infty} ((u_t,\ u_t))_{B_t}\, dt \tag{2.3}$$

$$\|e^{\delta t} u_t\|_{B_t} \leq \|e^{\delta t_0} u_{t_0}\|_{B_0} + \left\{ \int_{t_0}^{t} \left\|e^{\delta\tau}\left(\frac{\partial}{\partial \tau} - \bar{A}_\tau\right) u_\tau\right\|_{B_\tau} d\tau \right\} \tag{3.3}$$

where $((u, v))_{B_t} = (B_t u, v)$ *for* $u, v \in \mathfrak{D}_x$, $t > t_0$, $\delta > 0$, $\beta > 0$.

Proof. Since $\int \left(B_t \frac{\partial}{\partial t} u_t,\ u_t\right) dt = -\int \left(B_t u_t,\ \frac{\partial}{\partial t} u_t\right) dt - \int (B_t' u_t,\ u_t)\, dt$,

where $B'(t, x, \xi) = \frac{\partial}{\partial t} B(t, x, \xi)$,

$$\int \left(B_t\left(\frac{\partial}{\partial t} - (A_t - \beta)\right) u_t,\ u_t\right) dt = -\frac{1}{2} \int (B_t'\ u_t,\ u_t)\, dt - \int (B_t A_t u_t u_t)\, dt$$
$$+\beta \int (B_t u_t,\ u_t)\, dt.$$

Since $O(B_t') = O(B_t)$ there is a $\alpha_1 > 0$ such that

$$|(B_t' u_t,\ u_t)| \leq \alpha_1 (B_t u_t,\ u_t),$$

therefore $\int \left(B_t\left(\frac{\partial}{\partial t} - \bar{A}_t\right) u_t,\ u_t\right) dt \geq \left(\beta - \frac{1}{2}(\alpha + \alpha_1)\right) \int (B_t u_t,\ u_t)\, dt.$

Furthermore

$$\begin{aligned}
\frac{d}{dt}(B_t u_t,\ u_t) &= \left(B_t \frac{\partial}{\partial t} u_t,\ u_t\right) + \left(B_t u_t,\ \frac{\partial}{\partial t} u_t\right) + (B_t' u_t,\ u_t) \\
&= ((B_t \bar{A}_t + \bar{A}_t^* B_t) u_t,\ u_t) + (B_t' u_t,\ u_t) \\
&\quad + 2\left(B\left(\frac{\partial}{\partial t} - \bar{A}_t\right) u_t,\ u_t\right) \\
&\leq (\alpha + \alpha_1 - 2\beta)(B_t u_t,\ u_t) + 2\left(B_t\left(\frac{\partial}{\partial t} - \bar{A}_t\right) u_t,\ u_t\right)
\end{aligned}$$

Therefore let β be $\gneqq \frac{1}{2}(\alpha + \alpha_1)$ and $2\delta = 2\beta - (\alpha + \alpha_1)$, then we see that

$\frac{d}{dt}[\|u_t\|_{B_t} \exp(\delta t)] \leq \{\|\left(\frac{\partial}{\partial t} - \bar{A}_t\right) u_t\|_{B_t} \exp(\delta t)\}$.

Lemma 4. *For any* $v \in H_{t,x}^{0,s}(s = O(B))$, *there is a* $u \in H_{t,x}^{0,s}$ *such that* $\left(\frac{\partial}{\partial t} - \bar{A}\right) u = v$ *in the sense of the weak solution.*

Proof. Let $((u, v))_B$ be the inner product such that

$$((u, v))_B = \int_{-\infty}^{\infty} ((u_t, v_t))_{B_t} dt \quad \text{for} \quad u, v \in \mathfrak{D}_{t,x}.$$

Then there are positive numbers α and β such that

$$((u, u))_B \geqq \alpha ||u||^2_{o,s} \tag{4.3}$$
$$\beta ||u||_{o,s} ||v||_{o,s} \geqq |((u, v))_B| \quad \text{for} \quad u, v \in \mathfrak{D}_{t,x}.$$

Therefore $((u, v))_B$ define the Hilbert space which is isomorphic to $H^{o,s}_{t,x}$. The inequality (2.3) implies the following:

$$||\left(\frac{\partial}{\partial t} - \bar{A}\right) u||_B \geqq \delta ||u||_B \tag{5.3}$$

$$||B^{-1}\left(-\frac{\partial}{\partial t} - \bar{A}^*\right) Bu||_B \geqq \delta ||u||_B \quad \text{for} \quad u \in \mathfrak{D}_{t,x}, \tag{6.3}$$

where the operator B^{-1} is considered as in Lemma 2. The B^{-1} in (6.3) is a mapping from $\mathfrak{D}_{t,x}$ into $H^{n,l+s}_{t,x}$ for sufficiently large n and l. Then from (6.3) it follows that for any $v \in H^{o,s}_{t,x}$ there is a $u \in H^{o,s}_{t,x}$ such that

$$((u, B^{-1}\left(-\frac{\partial}{\partial t} - \bar{A}^*\right) Bf))_B = ((v, f))_B \quad \text{for} \quad f \in \mathfrak{D}_{t,x}.$$

Then by a limit process

$$(u, \left(-\frac{\partial}{\partial t} - \bar{A}^*\right) Bf) = (v, Bf)$$

Furthermore by Lemma 2, we see that

$$(u, \left(-\frac{\partial}{\partial t} - \bar{A}^*\right) f) = (v, f) \quad \text{for} \quad f \in \mathfrak{D}_{t,x}.$$

This means that u is a weak solution. Q.E.D.

From now on we assume that the hermitian part of A is semi-bounded in the sense of the norm induced by two smooth systems B and B' in § 2 such that for $s = O(B)$ and $s' = O(B')$

$$s'(i) \geqq s(i) + p \quad \text{for any} \quad i = 1, 2, \cdots, m$$

where $p = \max p_{ij} \quad (i, j = 1, 2, \cdots, m)$.

Theorem 1. *For any $v \in H^{o,s}_{t,x}$, there is a unique $u \in H^{o,s}_{t,x}$ such that*

$$\left(\frac{\partial}{\partial t} - \bar{A}\right) u = v \tag{7.3}$$

in the sense of the strong solution in $H^{o,s}_{t,x}$, i.e., there is a sequence $\{u_n\} \in \mathfrak{D}_{t,x}$ such that

$$u_n \to u \qquad in \quad H^{o,s}_{t,x},$$
$$\left(\frac{\partial}{\partial t} - \bar{A}\right) u_n \to v \qquad in \quad H^{o,s}_{t,x}.$$

Proof. Let v_n be in $\mathfrak{D}_{t,x}$ such that $\{v_n\}$ converges to v in $H^{o,s}_{t,x}$. Then by Lemma 4 there are $u_n \in H^{o,s'}_{t,x}$ such that $\left(\frac{\partial}{\partial t} - \bar{A}\right) u_n = v_n$ in the sense of distribution. Then using convolution and multiplication we see that u_n is a strong solution in $H^{o,s}_{t,x}$. Thus we can apply (5. 3) to $u_n - u_m$, accordingly $\|u_n - u_m\|_{o,s} \leqq \delta' \|v_n - v_m\|_{o,s}$, which implies that $\{u_n\}$ is a Cauchy sequence in $H^{o,s}_{t,x}$. Let u be a limit element of $\{u_n\}$. Then $u \in H^{o,s}_{t,x}$, u_n converges to u in $H^{o,s}_{t,x}$ and v_n converges to v in $H^{o,s}_{t,x}$. Since u_n is a strong solution we may assume that u_n belongs to $\mathfrak{D}_{t,x}$, thus we see that u is a strong solution. The uniqueness of such solutions follows from (5. 3).

Here we remark that in the condition (1. 3) of Theorem 1 we must take $\alpha(t)$ as a constant, but that in that of the following theorem $\alpha(t)$ may be considered as a continuous functions, and A and B may be considered as smooth systems defined on $[-n, n] \times R^N_x$ for any n.

Theorem 2. *Let v be a given element in $H^{o,s}_{t,x}$ and let g be a given element in H^s_x. Then for any $t_0 > 0$ there is a unique solution $u \in C([0, t_0], H^s_x)$ such that there exists a sequence $\{u_n\}$ in $\mathfrak{D}_{t,x}$ with following properties*:

i) $u_n \to u \qquad in \quad H^{o,s}_{t,x}[0, t_0]$

ii) $\left(\frac{\partial}{\partial t} - A\right) u_n \to v \qquad in \quad H^{o,s}_{t,x}[0, t_0]$

iii) $u_n(0) \to g \qquad in \quad H^s_x$.

Proof. We may assume that $A = \bar{A}$ and $\alpha(t)$ is constant. Furthermore suppose first that $v \in \mathfrak{D}_{t,x}$. Now let g_n be in $\mathfrak{D}_x$ such that $g_n \to g$ in H^s_x. Then there exists a sequence $\{u_n'\}$ such that for sufficiently large m'

$$u_n' \in \mathfrak{D}^{m'}_{t,x}$$
$$u_n'(0) = g_n$$
$$v - \left(\frac{\partial}{\partial t} - A\right) u_n' = v_n'$$
$$v_n'(0) = 0 .$$

Then setting $v_n'(t) = 0$ for $t \leqq 0$, there is by Theorem 1 and from (3. 3) a solution u_n'' such that

$$u_n'' \in C(R_t^1, H_x^s): \text{continuous map on } R_t^1 \text{ into } H_x^s$$
$$\left(\frac{\partial}{\partial t} - A\right) u_n'' = v_n'$$
$$u_n(0) = g_n .$$

in the sense of the strong solution. Then from (3.3) $u_n''(t)=0$ for $t \leqq 0$. Therefore letting $u_n = u_n' + u_n''$, $u_n \in C(R_t^1, H_x^s)$. Furthermore $\left(\frac{\partial}{\partial t} - A\right) u_n = v + v_n'$ in the sense of distribution where $v_n'(t) = 0$ for $t \geqq 0$. Then since u_n is also a strong solution, (3.3) is applicable to u_n, hence for any $t \geqq 0$, $u_n(t)$ is a Cauchy sequence in H_x^s, therefore there exists a $u(t)$ such that $u(0) = g$ and such that $u_n(t) \to u(t)$ in H_x^s, therefore for any $t_0 > 0$ $u_n \to u$ in $H_{t,x}^{0,s}[0, t_0]$.

If $v \in H_{t,x}^{0,s}$ there is $v_n \in \mathfrak{D}_{t,x}$ such that $v_n \to v$ in $H_{t,x}^{0,s}$. Then by the above consideration there is $u_n \in C([0, t_0], H_x^s)$ such that

$$\left(\frac{\partial}{\partial t} - A\right) u_n = v_n$$

in the sense of the strong solution in $H_{t,x}^{0,s}[0, t_0]$ and

$$u_n(0) = g .$$

Therefore (3.3) is also applicable to $u_n - u_m$. Hence there is u such that $u_n \to u$ in $C([0, t_0], H_x^s)$. Here we can replace u_n by an element of $\mathfrak{D}_{t,x}$. The uniqueness of such u follows from the inequality induced by (3.3). Q.E.D.

Furthermore from now on we assume that the hermitian part of A is semi-bounded in the sense of the norm induced by three smooth systems B, B' and B'' in §2 such that for $s = O(B)$, $s' = O(B')$ and $s'' = O(B')$

$$s'(i) = s(i) + p$$
$$s''(i) = s'(i) + p \quad \text{for any} \quad i = 1, 2, \cdots, m .$$

Theorem 3. *Under the above assumption with constant $\alpha(t)$, for any $v \in H_{t,x}^{0,s'}$ the weak solution u in $H_{t,x}^{0,s'}$ such that*

$$\left(\frac{\partial}{\partial t} - \bar{A}\right) u = v ,$$

is also the strong solution in $H_{t,x}^{0,s'}$.

Proof. By Lemma 4 there is a weak solution u in $H_{t,x}^{0,s'}$. Then it is a strong solution in $H_{t,x}^{0,s}$. Furthermore by the above assumption there is a strong solution $u' \in H^{0,s'}_{t,x}$, hence it is also a strong solution in $H^{0,s}_{t,x}$. Then by Theorem 1 $u=u'$ in $H^{0,s}_{t,x}$. Therefore $u=u'$ in $H^{0,s'}_{t,x}$.

Theorem 4. *Let v be in $C([0, \infty), H^s_x) \cap H^{0,s'}_{t,x}$ and let $g \in H^{s'}_x$. Then there exists uniquely $u \in C([0, \infty), H^{s'}_x)$ such that*

$$u(0)=g,$$
$$\frac{d}{dt}u(t)=A_t u(t)+v(t) \quad in \quad H^s_x, (t \geqq 0),$$

where $\frac{d}{dt}u(t)$ is the strong derivative in H^s_x.

Proof. Let g_n be in $\mathfrak{D}_x$ such that $g_n \to g$ in $H^{s'}_x$. Then as in the proof of Theorem 2, there exists by Theorem 3 a sequence $\{u_n\}$ such that

$$u_n \in C([0, t_0], H^{s'}_x)$$
$$\left(\frac{\partial}{\partial t}-A\right)u_n=v,$$
$$u_n(0)=g_n.$$

in the sense of the strong solution in $H^{0,s'}_{t,x}[0, t_0]$ for any $t_0>0$.
Then by a limit process we see that for $(t_0 \geqq t \geqq 0)$

$$((u_n(t), f))_s=\int_0^t [((A_\tau u_n(\tau), f))_s+((v(\tau), f))_s]\, d\tau+((g_n, f))_s$$

for $f \ni \mathfrak{D}_x$.
Since using (3.3) we can find $u(t) \in C([0, t_0], H^{s'}_x)$ such that for any $t \in [0, t_0]$

$$u_n(t) \to u(t) \quad \text{in} \quad H^{s'}_x,$$

we see that

$$((u(t), f))_s=\int_0^t [((A_\tau u(\tau), f))_s+((v(\tau), f))_s]\, dt+((g, f))_s$$

Therefore $u(t)$ has a derivative in the weak sense in H^s_x with respect to t $A_t u(t)+v(t) \in C([0, t_0], H^s_x)$. Accordingly $u(t)$ has a strong derivative in H^s_x. The uniqueness of such solutions follows from the inequality induced by an inequality as (3.3). Q.E.D.

Furthermore we consider the relation between the condition (1.3) and the semigroup with the infinitesimal generator A_t.

Theorem 5. *Let $\tilde{A}_t$ be the smallest closed extension of A_t in $H^{s'}_x$ into itself. Then $\tilde{A}_t$ has the following properties*:

(*a*) *any real $\lambda>\alpha(t)$ belongs to the resolvant set of $\tilde{A}_t$,*

(*b*) $\left\|\left(1-\frac{1}{\lambda}\tilde{A}_t\right)^{-1}\right\|_{B_{t'}} \leqq 1+2\alpha(t)\lambda^{-1}$ *for $\lambda>2\alpha(t)$, and*

(c) $\left(1-\frac{1}{\lambda}\tilde{A}_t\right)^{-1}$ *is strongly continuous with respect to t in* $H_x^{s'}$ *for* $\lambda > \alpha(t)$.

Proof. The conditions (a) and (b) follows from (1. 3) with respect to B' by the same method as used in Theorem 3. Also (c) follows from the fact that the weak solution and strong solution for $\left(1-\frac{1}{\lambda}\tilde{A}_t\right) u=v$ in $H_x^{s'}$ coincide in this case. Q.E.D.

By Theorem 5 and using some modification of Yosida's method on semigroup we also obtain Theorem 4 with $v\in C_t(H_x^{s'})$.

Here we remark that in (b) of Theorem 5 we can not in general replace $\| \ \|_{B_t'}$ by $\| \ \|_{s'}$ and we assert that our advantage is to start from the relation (1. 3) without using the relation as (4. 3), so that we can treat Cauchy problem with initial function defined on whole space R_x^N for differential equations of different types at the same time.

Furthermore we remark that the inequality (b) in Theorem 5 implies that the hermitian part of A is semi-bounded by the norm induced by B'.

4. Application.

In this section we give applications of Theorem in § 3 First we consider a generalization of Leray's regular hyperbolic equations. (see [3], [4] and [7])

Let a differential operator a:

$$\frac{\partial^m}{\partial t^m}+\alpha_0\left(t, x, \frac{\partial}{\partial x}\right)\frac{\partial^{m-1}}{\partial t^{m-1}}+\cdots+\alpha_{m-1}\left(t, x, \frac{\partial}{\partial x}\right) \tag{1. 4}$$

be a smooth operator defined on $R_t^1\times R_x^N$ such that for some *odd* positive integer $p\geqq 1$

i) $o(\alpha_i)\leqq(i+1)p$

ii) $o(\alpha_i-(-1)^{(i+1)p}\alpha_i^*)\leqq ip$ for any $i=0, 1, \cdots, m-1$

iii) $o(\alpha_i\alpha_j-\alpha_j\alpha_i)\leqq(i+j+1)p$ for any $i, j,=0, 1, \cdots, m-1$.

Furthermore let α_j' be the part of order $(i+1)p$ of α_i and let $a'(t, x, \xi)$ be

$$\xi_0^m+\alpha_0'(t, x, \xi^*)\,\xi_0^{m-1}+\cdots+\alpha_{m-1}'(t, x, \xi^*)\,,$$

where $\xi^*=(0, \xi_1, \xi_2\cdots, \xi_N)$. Then we assume that a satisfies the following condition: the equation of λ

$$a'(t, x, \lambda(1, 0, \cdots, 0)+(0, \eta_1 \cdots \eta_N))=0$$

has real roots $v_i(t, x, \eta^*)$ such that

$$|v_i(t, x, \eta^*)-v_j(t, x, \eta^*)| \geq b \tag{2.4}$$

for some positive b, for different i, j $(i, j=1, 2, \cdots, m)$, any $(x, t) \in R_t^1 \times R_x^l$ and any $\eta^*: |\eta^*|=1$.

Now let A be the matrix;

$$A=\left(\begin{array}{ccc:cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1 \\ -\alpha_{m-1}, & -\alpha_{m-2}, & -\alpha_{m-3}, & -\alpha_1, & -\alpha_0 \end{array}\right)$$

and let A' be the matrix obtained by replacing α_i by α_i' in A. Then from (2.4) there is a matrix $B''=(b''_{ij}(t, x, \xi^*))$ such that

1′) B'' is positive definite and symmetric for any $(t, x) \in R_t^1 \times R_x^N$, and in fact $B'' > \rho I$ for ξ^*: $|\xi^*|=1$ and for any $(t, x) \ni R_t^1 \times R_x^N$ where ρ is determined by b in (2.4) and the maximum of absolute values of coefficients in $a'(t, x, \xi^*)$,

2′) $b''_{ij}(t, x, \xi^*)$ is a real polynomial $b''_{ij}(\alpha_0', \alpha_1', \cdots, \alpha_{m-1})$ of $\alpha_i'(t, x, \xi^*)$ such that each term of b''_{ij} is of order $((m(m-1)+2-i-j)p$ with respect to ξ^* for any t, x.

3′) $b''_{ij}(\alpha_0', \alpha_1', \cdots, \alpha'_{m-1})=b''_{ji}(\alpha_0', \alpha_1', \cdots, \alpha'_{m-1})$ for any $i, j=1, 2, \cdots, m$,

4′) $B''(\alpha_0', \alpha_1', \cdots, \alpha'_{m-1}) A'(\alpha_0', \alpha_1', \cdots, \alpha'_{m-1})$
$=A'^t(\alpha_0', \cdots, \alpha'_{m-1}) B''(\alpha_0', \cdots, \alpha'_{m-1})$

for any real vector $\alpha_0', \cdots, \alpha'_{m-1}$.

Let $b'_{ij}(\alpha_0', \alpha_1', \cdots, \alpha'_{m-1})=(-1)^{\left(j+\frac{m(m-1)}{2}+1\right)p} b''_{ij}(\alpha_0', \cdots, \alpha'_{m-1})$. Then from 2′)

$$b'_{ij}(\alpha_0'(i\xi^*), \cdots, \alpha_{m-1}(i\xi^*))=(i)^{(m-i)p}(-i)^{(m-j)p} b''_{ij}(\alpha_0'(\xi^*), \cdots, \alpha'_{m-1}(\xi^*)) .$$

Therefore we see that

1) $B'=B'(b'_{ij}(\alpha_k(t, x, i\xi^*))$ is positive definite hermitian for any $(t, x) \in R_t^1 \times R_x^N$, and in fact $B' > \rho I$ for ξ^*: $|\xi^*|=1$ and for any $(t, x) \in R_t^1 \times R_x^N$,

2) $b'_{ij}(t, x, i\xi^*)=b'_{ij}(\alpha_k'(t, x, i\xi^*))$ is a real polynomial of $\alpha_k'(t, x, i\xi^*)$,

3) $b'_{ij}(\alpha_k'(t, x, i\xi^*))=b'_{ji}(\alpha_k'(t, x, -i\xi^*))$

4) $B'(\alpha_i') A'(\alpha_i')-(-i)^p A'^t((-1)^{(i+1)p} \alpha_i') B'(\alpha_i')=0$ for any complex scalar $\alpha_i(t, x, i\xi^*)$ where ξ^* is a real vector.

Since 2′), 3′) and 4′) are satisfied for arbitrary real $\alpha_i'(t, x, \xi^*)$ of

(1.4) without (2.4), 2) 3) 4) are satisfied for α_i with i). Therefore in particular 4) is satisfied for any complex vector $\{\alpha_i'\}$ such that $\alpha_k' \in (i)^{(k+1)p} R$. Accordingly 4) is a trivial relation for any commutative free algebra over real field with m generators.

Furthermore let D be the diagonal matrix whose diagonal elements are all $b'_{11}\left(\alpha_i'\left(t, x, \frac{\partial}{\partial x}\right)\right)$. Then since $b'_{11}(\alpha_i'(t, x, i\xi^*)) > \rho'$, for some ρ' and for any $\xi^* : |\xi^*| = 1$ we see by Lemma 1 that

$$DB' + B'^*D^* + k \geqq \alpha Q^{s'}$$

where $s(i) = \left(\frac{m(m-1)}{2} + 1 - i\right)p$, $s'(i) = s(i) + \frac{1}{2} m(m-1)\, p$, $k > 0$ and $\alpha > 0$. Furthermore we may assume that $D = D(b'_{11}(\alpha_i))$ and $B' = B'(b'_{ij}(\alpha_k))$.

From ii), iii) and 4) and by the above remarks we see that

$$O(DB'A + A^*DB') \leqq s' ,$$

accordingly

$$O(B'^*D^*A + A^*B'^*D^*) \leqq s' .$$

Therefore for some positive β

$$|(DB' + B'^*D^* + k)\, A + A^*(DB' + B'^*D^* + k)| \leqq \beta(DB' + B'^*D + k) .$$

Let $B = DB' + B'^*D^* + k$. Thus we see that the hermitian part of A is semi-bounded in the sense of the norm induced by B. Such B is made by the same way as above for sufficiently large $O(B)$. Therefore our theorems in § 3 are applicable to $\left(\frac{\partial}{\partial t} - A\right)$ and we see the following

Theorem 6. *Under the assymption i), ii), iii) and* (2.4) *Cauchy problem for a is well posed in the following sense: for any* $v \in \mathcal{E}_{[-1,1]}(\mathfrak{D}_{L^2(x)})$ *and any* $g_i \in \mathfrak{D}_{L^2(x)}$ *there is a unique solution* $u \in \mathcal{E}_{[-1,1]}(\mathfrak{D}_{L^2(x)})$ *such that*

$$au = v$$
$$\left(\frac{\partial^j}{\partial t^i} u\right)(0) = g_i \qquad \textit{for} \quad i = 0, 1, 2, \cdots, m-1 .$$

Here we remark that for $p=1$ our equation with condition (2.4) is regular hyperbolic in the sense of Leray. In this case we can find a more direct proof of the existence and unigueness theorem for Cauchy problem using only the consideration of chapter VI in [6] and refine his result (see also [4]). Furthermore we remark that for $m=2$ there exist other reversible equations (See [12]. Ex. 3).

Next we consider systems of operators which stand against reversible systems within systems (1.1) satisfing the condition (1.3). These are parabolic systems, which are investigated in [1] and [9]. Here we give a more general definition of parabolic system from our abstract point of view. Then we show that our theorems in § 3 are applicable to such a system and furthermore that the hypoellipticity is a direct consequence of Theorem 1.

We say that $\left(\frac{\partial}{\partial t}-A\right)$ is parabolic if it satisfies the following conditions:

i) $A=(a_{ij})$ is an (m, m)-smooth system defined over $R_t^1\times R_x^N$ such that $o(a_{ij}\left(t, x, \frac{\partial}{\partial x}\right))\leqq p(i)-p(j)+2p$ where p and $p(i)$ are positive integers.

ii) there are (m, m)-smooth systems B_{l_0} and B_{l_0+p} described in § 2 such that

$O(B_{l_0})=\{l-p(i)\}$ for some positive integer $l_0>\max_{i=1,2,\cdots,m}(p(i))+p$,

$$O(B_{l_0+p})=\{l_0+p-p(i)\}, \text{ and}$$

$$B_{l_0}A+A^*B_{l_0}\leqq-\alpha B_{l_0+p} \text{ for some positive } \alpha. \tag{3.4}$$

Now we denote $H_{t,x}^{n,O(B_{l_0})}$ by $H(n, l_0)$. Then from (3.4) in the same way as in the proof of Theorem 6 and Lemma 2 it follows that replacing A by $A-\beta_l$ for some positive β_l, (3.4) is valid for any $l\geqq l_0$, i.e., that for any $l\geqq l_0$ there is B_l such that denoting $A_l=A-\beta_l I$

$$B_lA_l+A_l^*B_l\leqq-\alpha_lB_{l+p} \tag{4.4}$$

for some positive α_l and for positive $\beta_l\geqq k_l>0$.

From (4.4) we see that

$$(\left(-\frac{\partial}{\partial t}-A_{l+p}^*\right)B_{l+p}u,\ u)\geqq\gamma(B_{l+2p}u,\ u)$$

for some positive γ and for any $u\in\mathfrak{D}_{t,x}$.

Therefore

$$(B_{l+2p}^{-1}\left(-\frac{\partial}{\partial t}-A_{l+p}^*\right)B_{l+p}u,\ \left(-\frac{\partial}{\partial t}-A_{l+p}^*\right)B_{l+p}u)\geqq\gamma^2(B_{l+2p}u,\ u)\,.$$

Setting $B_lu'=B_{l+p}u$, by Lemma 2 the above inequality implies that

$$(B_{l+2p}^{-1}\left(-\frac{\partial}{\partial t}-A_{l+p}^*\right)B_lu',\ \left(-\frac{\partial}{\partial t}-A_{l+p}^*\right)B_lu')\geqq\gamma_1^2(B_lu',\ u')$$

$$\text{for some } \gamma_1>0 \text{ and for any } u'\in\mathfrak{D}_{t,x}\,,$$

since B_{l+p} and B_l are isomorphic transformations from $H(n, l+2p)$ onto $H_{t,x}^{n,\{-l+p(i)\}}$ and from $H(n, l)$ onto $H_{t,x}^{n,\{-l+p(i)\}}$ respectively.

Thus we see by Theorem 1 and using convolution operator that for any

$v\in H(0, l)$ there is a weak solution $u\in H(0, l+2p)$ such that

$$\left(\frac{\partial}{\partial t}-A_{l+p}\right)u=v\,.$$

Using the above fact by Theorem 3 and by Lax's method ([6]) we see the following.

Theorem 7. *Any parabolic system is hypoelliptic where we assume that coefficients of A are infinitely differentiale.*

Proof. First we show that if $v\in H(\infty, \infty)(\Omega)$, $u\in H(0, l+2p)(\Omega)$ for $l\geqq l_0$ and $\left(\frac{\partial}{\partial t}-A\right)u=v$ in Ω, then $u\in H(\infty, \infty)(\Omega)$.

For let φ be in $\mathfrak{D}_{t,x}$ such that $\varphi(x)=1$ for $x\in\Omega'(\Omega'\subset\Omega)$ and $\varphi(x)=0$ for $x\notin\Omega''$, where Ω' and Ω'' are open subsets of the open set Ω of $R^1_t\times R^l_x$. Then

$$\left(\frac{\partial}{\partial t}-A\right)\varphi u=\varphi v+v'\,,\quad v'\in H(0, l+1)\,,$$

Therefore $u\in H(0, l+2p+1)(\Omega')$ by the above consideration in particular by Theorem 3. By repeating the above process, we see that $u\in H(0, \infty)(\Omega')$. Then since $\frac{\partial}{\partial t}u=Au+v$, $u\in H(\infty, \infty)(\Omega')$. (See[12])

Next we assume that $u\in H(0, -l)$. Then

$$\left(\frac{\partial}{\partial t}-A\right)\varphi u=\varphi v+v'\,,\quad v'\in H(0, -l-2p+1)$$

Therefore for any integer $s>0$

$$\left(\frac{\partial}{\partial t}-A\right)Q_x^{-s}\varphi u=Q_x^{-s}\varphi v+Q_x^{-s}v'+v''\,,\quad v''\in H(0, -l-2p+2s+1)$$

Thus $Q_x^{-s}\varphi u\in H(0, -l+2s+1)$, and hence $u\in H(0, -l+1)(\Omega')$. Therefore we see that $u\in H(0; l+2p)(\Omega')$ for $l\geqq l_0$

Finally we assume that $u\in H(-n, -l)(\Omega)$. Then

$$\left(\frac{\partial}{\partial t}-A\right)\varphi u=\varphi v+v'\,,\quad v'\in H(-n, -l+1-2p)\,.$$

Therefore

$$\left(\frac{\partial}{\partial t}-A\right)Q_t^{-n}Q_x^{-s}\varphi u=Q_t^{-n}Q_x^{-s}\varphi v+Q_t^{-n}Q_x^{-s}v'+v''\,,$$

$v''\in H(n, -l-2p+2s)$. Therefore $Q_t^{-n}Q_x^{-s}\varphi u\in H(n+1, -l-2p+2s)$ and so $u\in H(-n+1, -l-2p)(\Omega')$. Thus we see that $u\in H(0, -l-n\cdot 2p)(\Omega')$. Accordingly using the second step described above we see that $u\in H(\infty, \infty)(\Omega')$. Q.E.D.

Finally we add that for such generaliged parabolic system mixed problem of Dirichlet type can be solved but, in general, in the weak sense.

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