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# DYNAMICAL SYSTEMS ON FRACTALS IN A PLANE

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## 1. Introduction

We considered in [3] ergodic properties of one-dimensional piecewise linear mappings, and solve the spectral problem of the Perron-Frobenius operator associated with these mappings. The main tool to solve it is a renewal equation on a signed symbolic dynamics. By this renewal equation, we define a matrix which we call the Fredholm matrix, and the spectral problem of the Perron-Frobenius operator and also the dynamical zeta function are characterized by this matrix.

Extending this idea to Cantor sets in the unit interval, we considered in [5] and [6] the ergodic properties of the dynamical systems on them. Also extending the idea of signed symbolic dynamics in one-dimensional dynamical system, which express the orbits of endpoints of subintervals of monotonicity, we introduced signed symbolic dynamics on a plane in [4], which corresponds to vertices and edges of polygons, and studied dynamical systems on it.

In this article, combining these ideas, we will study the Hausdorff dimension of Cantor sets in a plane. We will consider two types of Cantor sets which is generated by Koch-like mappings and Sierpinskii-like mappings (definition will be given in §3). As in 1-dimensional mappings, we construct the $\alpha$-Fredholm matrix $\Phi(z\colon \alpha)$, and take $\alpha_0$ the maximal solution of $\det(I - \Phi(1\colon \alpha)) = 0$, and put $d_\Phi = 2\alpha_0$. The theorem which we will prove in this paper is the following:

**Theorem 1.** *Let $F$ be a Koch-like mapping or a Sierpinskii-like mapping. Assume that $d_\Phi/2$ is the simple zero of* $\det(I - \Phi(1\colon \alpha))$, *and $\xi d_\Phi/2 - \nu > 0$. Then the Hausdorff dimension of the Cantor set generated by $F$ equals $d_\Phi$.*

The numbers $\xi$, which we call the lower Lyapunov number, and $\nu$ are defined by

$$\xi = \liminf_{n\to\infty} \operatorname*{ess\,inf}_{x\in I} \frac{1}{n} \log |\det D(F^n)(x)|,$$

and

$$\nu = \limsup_{n\to\infty} \sup_{l} \frac{1}{n} \log \#\{w\colon |w| = n, \langle w\rangle \cap l \neq \emptyset\},$$

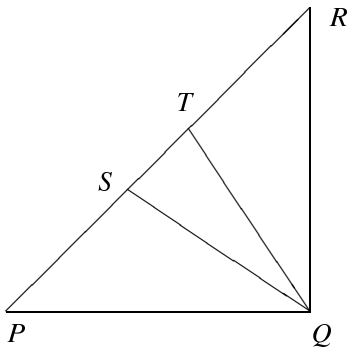


Fig. 1. Koch curve

where $D(F^n)$ is the Jacobian matrix of $F^n$, and $\sup_l$ is the supremum over all segments $l$. The definion of words $w$, $|w|$ and $\langle w \rangle$ will be given in §2. We also define

$$\overline{\xi} = \limsup_{n\to\infty} \frac{1}{n} \operatorname*{ess\,sup}_{x\in I} \log \left[\text{maximum of } \left|\text{the eigenvalue of } D(F^n)(x)\right|\right],$$

$$\underline{\xi} = \liminf_{n\to\infty} \frac{1}{n} \operatorname*{ess\,inf}_{x\in I} \log \left[\text{minimum of } \left|\text{the eigenvalue of } D(F^n)(x)\right|\right].$$

We call $F$ expanding if $\underline{\xi} > 0$.

## 2. Bernoulli type

In this section, we will consider two simple examples and explain the aim of this paper. First examples are Bernoulli Koch curves. See Fig. 1. We will denote the triangle $\triangle PQR$ by $I$, $\triangle PQS = \langle a \rangle$, $\triangle QRT = \langle b \rangle$ and $\triangle QST = X_0$. A mapping $F$ from $\langle a \rangle \cup \langle b \rangle$ to $I$ is defined as follows: the triangle $\triangle PQS$ is mapped to the triangle $\triangle PRQ$, and the triangle $\triangle RQT$ is mapped to the triangle $\triangle RPQ$. More precisely, take

$$P = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, Q = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, R = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, S = \begin{pmatrix} \lambda_a \\ \lambda_a \end{pmatrix} \text{ and } T = \begin{pmatrix} 1-\lambda_b \\ 1-\lambda_b \end{pmatrix},$$

where $\lambda_a, \lambda_b > 0$ and $\lambda_a + \lambda_b < 1$. The mapping $F$ is expressed as

$$F(x) = \begin{cases} F^a(x) = M^a x & \text{if } x \in \langle a \rangle, \\ F^b(x) = M^b \left( x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } x \in \langle b \rangle, \end{cases}$$

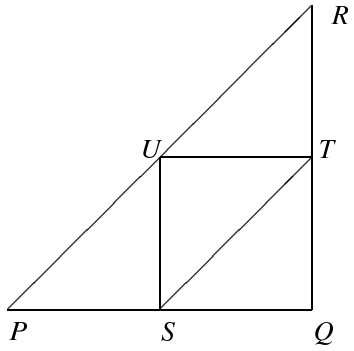


Fig. 2. Sierpinskii gasket

where

$$M^a = \begin{pmatrix} 1 & 1/\lambda_a - 1 \\ 1 & -1 \end{pmatrix},$$
$$M^b = \begin{pmatrix} -1 & 1 \\ 1/\lambda_b - 1 & 1 \end{pmatrix}.$$

Moreover, we assume the eigenvalues of $M^a M^b$ are complex conjugates. Set

$$\mathcal{C} = \{x \in I : F^n(x) \notin X_0 \text{ for all n}\},$$

where $F^n$ is the $n$-th iteration of $F$. Namely, $\mathcal{C}$ is the set of points whose orbit always stay in $\langle a \rangle$ or $\langle b \rangle$. This $\mathcal{C}$ becomes a fractal, and it becomes the usual Koch curve when $\lambda_a = \lambda_b = 1/3$, which satisfies all the above assumptions.

Another examples are Bernoulli Sierpinskiis gasket shown in Fig. 2. Here $SQ = TQ$. We will denote the triangle $\triangle PQR$ by $I$, $\triangle PSU = \langle a \rangle$, $\triangle SQT = \langle b \rangle$, $\triangle RUT = \langle c \rangle$ and $\triangle STU = X_0$. A mapping $F$ on $\langle a \rangle \cup \langle b \rangle \cup \langle c \rangle$ is defined as follows: the triangle $\triangle PUS$ is mapped to the triangle $\triangle PQR$, the triangle $\triangle QST$ is mapped to the triangle $\triangle QPR$, and the triangle $\triangle RUT$ is mapped the triangle $\triangle RQP$. More precisely, take $P, Q, R$ same as Koch curves and

$$S = \begin{pmatrix} \lambda_a \\ 0 \end{pmatrix}, U = \begin{pmatrix} \lambda_b \\ \lambda_b \end{pmatrix}, T = \begin{pmatrix} 1 \\ 1 - \lambda_a \end{pmatrix},$$

where $0 < \lambda_a, \lambda_b < 1$. Then $F$ is defined as

$$F(x) = \begin{cases} F^a(x) = M^a x & \text{if } x \in \langle a \rangle, \\ F^b(x) = M^b \left( x - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{if } x \in \langle b \rangle, \\ F^c(x) = M^c \left( x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } x \in \langle c \rangle, \end{cases}$$

where

$$\begin{aligned} M^a &= \begin{pmatrix} 1/\lambda_a & 1/\lambda_b - 1/\lambda_a \\ 1/\lambda_a & -1/\lambda_a \end{pmatrix}, \\ M^b &= \begin{pmatrix} 1/(1-\lambda_a) & 0 \\ 0 & 1/(1-\lambda_a) \end{pmatrix}, \\ M^c &= \begin{pmatrix} -1/\lambda_a & 1/\lambda_a \\ 1/(1-\lambda_b) - 1/\lambda_a & 1/\lambda_a \end{pmatrix}. \end{aligned}$$

We assume the eigenvalues of $M^a M^c$ are complex conjugates. Set also

$$\mathcal{C} = \{x \in I : F^n(x) \notin X_0 \text{ for all n}\}.$$

The usual Sierpinskii gasket is the case when $\lambda_a = \lambda_b = 1/2$, and this also satisfies all the assumptions in the above.

First note that the eigenvalues of matrices are

1. $\pm\sqrt{1/\lambda_a}$ for $M^a$, and $\pm\sqrt{1/\lambda_b}$ for $M^b$ in Koch curves,
2. $\pm\sqrt{1/(\lambda_a\lambda_b)}$ for $M^a$, $1/(1-\lambda_a)$ for $M^b$ and $\pm\sqrt{1/\{\lambda_a(1-\lambda_b)\}}$ for $M^c$ in Sierpinskii gaskets.

Namely, two eigenvalues of each matrix equal in modulus. This is one of the key point of these examples to show the argument in this section rigorously.

Moreover, note that

$$\begin{aligned} M^a M^b &= \begin{pmatrix} 1/(\lambda_a\lambda_b) - 1/\lambda_a - 1/\lambda_b & 1/\lambda_a \\ -1/\lambda_b & 0 \end{pmatrix} \quad \text{for Koch curves,} \\ M^a M^c &= \begin{pmatrix} (1-\lambda_a)/(-\lambda_a\lambda_b + \lambda_a\lambda_b^2) & 1/(\lambda_a\lambda_b) \\ -1/(\lambda_a - \lambda_a\lambda_b) & 0 \end{pmatrix} \quad \text{for Sierpinskii gaskets.} \end{aligned}$$

Therefore, for $\lambda_a$ and $\lambda_b$ which satisfies

$$\begin{aligned} &1 - 2(\lambda_a + \lambda_b) + (\lambda_a - \lambda_b)^2 < 0 \quad \text{for Koch curves,} \\ &(1-\lambda_a)^2 - 4(1-\lambda_b)\lambda_b < 0 \quad \text{for Sierpinskii gaskets,} \end{aligned}$$

the assumptions are satisfied. Thus, if $\lambda_a = \lambda_b$, then the assumptions are satisfied for Koch curves if $\lambda_a > 1/4$ and for Sierpinskii gaskets if $\lambda_a > 1/5$.

Now we will consider dynamical systems on $\mathcal{C}$ for both Koch curves and Sierpinskii gaskets. We can express them as symbolic dynamics with alphabet $\mathcal{A} = \{a, b\}$ for Koch curves and $\mathcal{A} = \{a, b, c\}$ for Sierpinskii gaskets. We define here notations about symbolic dynamics. A finite sequence of symbols $w = a_1 \cdots a_n$ ($a_i \in \mathcal{A}$) is called a word, and we define

$$
\begin{aligned}
&|w| = n \quad \text{(the length of a word)},\\
&\langle w \rangle = \cap_{i=0}^{n-1} (F^i)^{-1}(\langle a_{i+1} \rangle) \quad \text{(the region corresponding to a word)},\\
&w[k] = a_k \ \text{ for } 1 \le k \le n,\\
&w[k, l] = a_k a_{k+1} \cdots a_l \quad \text{for } 1 \le k \le l \le n,\\
&\theta w = w[2, |w|] = a_2 \cdots a_n,\\
&F^w = F^{a_n} \cdots F^{a_1},\\
&\eta(w) = \eta(a_1) \cdots \eta(a_n),
\end{aligned}
$$

where $\eta(a) = |\det M^a|^{-1}$ is the reciprocal of the Jacobian of $F$ restricted to $\langle a \rangle$. We consider the empty word $\epsilon$ with length 0. For simplicity, we consider $\langle \epsilon \rangle = I$, and $\langle \epsilon[1] \rangle = I$. We call a word $w$ admissible if $\langle w \rangle \neq \emptyset$, and define by $\mathcal{W}$ the set of all admissible words. Note that the empty word $\epsilon \in \mathcal{W}$. For an integer $M$, we denote by $\mathcal{W}_M$ the set of admissible words with length $M$. Namely, $\mathcal{W}_0 = \{\epsilon\}$, $\mathcal{W}_1 = \mathcal{A}$ and $\mathcal{W} = \cup_{M=0}^{\infty} \mathcal{W}_M$.

We can extend $F^a$ as a mapping from $R^2$ into itself. Thus, for any word $w \in \mathcal{W}$, we can extend $(F^w)^{-1}$ as a mapping from $I$ into $R^2$. For $x \in I$, we denote by $wx \in R^2$ a point which satisfies $F^w(wx) = x$. We call $wx$ exists (we sometimes denote $\exists wx$) if $wx \in \langle w \rangle$. Namely, if $y = wx$ exists, then

$$
\begin{aligned}
&F^k(y) \in \langle \theta^k w \rangle \quad (0 \le k \le |w| - 1),\\
&F^{|w|}(y) = x.
\end{aligned}
$$

Take $0 \le \alpha \le 1$. Define for a Koch curve

$$
\Phi(z\colon \alpha) = \begin{array}{c} \\ a \\ b \end{array} \begin{array}{c} \begin{array}{cc} a & b \end{array} \\ \begin{pmatrix} z\lambda_a^\alpha & z\lambda_a^\alpha \\ z\lambda_b^\alpha & z\lambda_b^\alpha \end{pmatrix} \end{array},
$$

and for a Sierpinskii gasket

$$
\Phi(z\colon \alpha) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} z(\lambda_a\lambda_b)^\alpha & z(\lambda_a\lambda_b)^\alpha & z(\lambda_a\lambda_b)^\alpha \\ z(1-\lambda_a)^{2\alpha} & z(1-\lambda_a)^{2\alpha} & z(1-\lambda_a)^{2\alpha} \\ z(\lambda_a(1-\lambda_b))^\alpha & z(\lambda_a(1-\lambda_b))^\alpha & z(\lambda_a(1-\lambda_b))^\alpha \end{pmatrix} \end{array},
$$

where $z$ is a complex number. We call these matrices $\alpha$-Fredholm matrices. For ex-

ample, $(a,a)$ and $(a,b)$ components of $\Phi(z\colon \alpha)$ of Koch curves are determined as $z$ times the reciprocal of $|\det M^a|^\alpha$, and $(b,a)$ and $(b,b)$ components of $\Phi(z\colon \alpha)$ are determined as $z$ times the reciprocal of $(\det M^b)^\alpha$. Note that $F(\langle a\rangle) \supset \langle a\rangle, \langle b\rangle$, and $F(\langle b\rangle) \supset \langle a\rangle, \langle b\rangle$. The $\alpha$-Fredholm matrices are intrinsically structure matrices with weight.

We will calculate the Hausdorff dimensions of Koch curves and Sierpinskii gaskets in terms of the $\alpha$-Fredholm matrices. In this section, we will show only a heuristic argument. So, we will consider coverings by words $\{\langle w\rangle\}_{|w|=n}$ of $\mathcal{C}$ with same length $n$ ($w \in \mathcal{W}_n$). Roughly speaking, the total Hausdoff measure with coefficient $2\alpha$ is the limit of

$$\sum_{|w|=n} \big(\text{Lebes}\langle w\rangle\big)^\alpha, \tag{1}$$

where $\text{Lebes}\langle w\rangle$ is the Lebesgue measure of a set $\langle w\rangle$. We can express (1) in terms of the $\alpha$-Fredholm matrices.

$$(1) = \begin{cases} (1,1)\Phi(1\colon \alpha)^{n-1} \begin{pmatrix} (\text{Lebes}\langle a\rangle)^\alpha \\ (\text{Lebes}\langle b\rangle)^\alpha \end{pmatrix} & \text{for Koch curves,} \\ (1,1,1)\Phi(1\colon \alpha)^{n-1} \begin{pmatrix} (\text{Lebes}\langle a\rangle)^\alpha \\ (\text{Lebes}\langle b\rangle)^\alpha \\ (\text{Lebes}\langle c\rangle)^\alpha \end{pmatrix} & \text{for Sierpinskii gaskets.} \end{cases} \tag{2}$$

As a heuristic argument, since the diameter of a set $\langle w\rangle$ is proportionate to the square root of the $\text{Lebes}\langle w\rangle$, the Hausdorff dimension is the value $2\alpha_0$ such that (1) converges to 0 for any $\alpha > \alpha_0$ and diverges for any $\alpha < \alpha_0$. Namely for $\alpha < \alpha_0$ at least one eigenvalue of $\Phi(1\colon \alpha)$ is greater than 1 in modulus, and for $\alpha > \alpha_0$ every eigenvalue of $\Phi(1\colon \alpha)$ is less than 1 in modulus. Noticing the fact that $\Phi(1\colon \alpha)$ are positive matrices and by (2), $\alpha_0$ is the maximal $\alpha$ such that $\Phi(1\colon \alpha)$ has eigenvalue 1, that is, $\alpha_0$ is the maximal solution of $\det(I - \Phi(1\colon \alpha)) = 0$. More precisely,

$$\lambda_a^{\alpha_0} + \lambda_b^{\alpha_0} = 1 \quad \text{for Koch curves,}$$
$$\frac{1}{(\lambda_a\lambda_b)^{\alpha_0}} + \frac{1}{(1-\lambda_a)^{2\alpha_0}} + \frac{1}{(\lambda_a(1-\lambda_b))^{\alpha_0}} = 1 \quad \text{for Sierpinskii gaskets.}$$

Particularly. for a Koch curve with $\lambda_a = \lambda_b = 1/3$, $\alpha_0$ is the solution of $2(1/3)^{\alpha_0} = 1$, that is, $2\alpha_0 = 2\log 2/\log 3$. And for a Sierpinskii gasket with $\lambda_a = \lambda_b = 1/2$, $\alpha_0$ is the solution of $3(1/2)^{2\alpha_0} = 1$, that is, $2\alpha_0 = \log 3/\log 2$.

We will construct another transformation $\hat{F}$ associated with $F$. See Fig. 3. Take

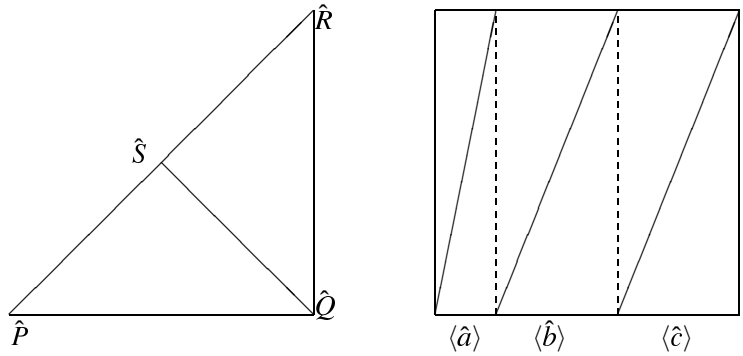


Fig. 3. Mapping $\hat{F}$ for Koch curves and Sierpinskii gaskets

$\hat{\lambda}_c = \lambda_a^{\alpha_0} = 1 - \lambda_b^{\alpha_0}$. Let $\langle\hat{a}\rangle = \triangle\hat{P}\hat{Q}\hat{S}$ and $\langle\hat{b}\rangle = \triangle\hat{Q}\hat{R}\hat{S}$, and define

$$\hat{F}(x) = \begin{cases} \hat{M}^a x & \text{if } x \in \langle\hat{a}\rangle, \\ \hat{M}^b \left( x - \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \text{if } x \in \langle\hat{b}\rangle, \end{cases}$$

where

$$\hat{M}^a = \begin{pmatrix} 1 & 1/\hat{\lambda}_c - 1 \\ 1 & -1 \end{pmatrix},$$

$$\hat{M}^b = \begin{pmatrix} -1 & 1 \\ 1/(1-\hat{\lambda}_c) - 1 & 1 \end{pmatrix}.$$

Note that the Fredholm matrix (1-Fredholm matrix) associated with $\hat{F}$ equals $\Phi(z\colon \alpha_0)$, and $\begin{pmatrix} \hat{\lambda}_c \\ 1-\hat{\lambda}_c \end{pmatrix}$ (the vector with the area corresponding to each symbol) is an eigenvector of $\Phi(1\colon \alpha_0)$ associated with eigenvalue 1. For Sierpinskii gaskets, we can not express $\hat{F}$ as a mapping on a plane. So we express it on $[0, 1]$. Define the length of each $\hat{a}, \hat{b}$ and $\hat{c}$ by

$$\begin{aligned} \text{Lebes}\langle\hat{a}\rangle &= \frac{1}{(\lambda_a\lambda_b)^{\alpha_0}}, \\ \text{Lebes}\langle\hat{b}\rangle &= \frac{1}{(1-\lambda_a)^{2\alpha_0}}, \\ \text{Lebes}\langle\hat{c}\rangle &= \frac{1}{(\lambda_a(1-\lambda_b))^{\alpha_0}}, \end{aligned}$$

and

$$\langle\hat{a}\rangle = [0, \text{Lebes}\langle\hat{a}\rangle),$$

$$\langle \hat{b} \rangle = [\text{Lebes}\langle \hat{a} \rangle, \text{Lebes}\langle \hat{a} \rangle + \text{Lebes}\langle \hat{b} \rangle),$$
$$\langle \hat{c} \rangle = [\text{Lebes}\langle \hat{a} \rangle + \text{Lebes}\langle \hat{b} \rangle, 1].$$

Then we define

$$\hat{F}(x) = \begin{cases} \dfrac{1}{\text{Lebes}\langle \hat{a} \rangle} x & \text{if } x \in \langle \hat{a} \rangle, \\ \dfrac{1}{\text{Lebes}\langle \hat{b} \rangle} (x - \text{Lebes}\langle \hat{a} \rangle) & \text{if } x \in \langle \hat{b} \rangle, \\ \dfrac{1}{\text{Lebes}\langle \hat{c} \rangle} (x - \text{Lebes}\langle \hat{a} \rangle - \text{Lebes}\langle \hat{b} \rangle) & \text{if } x \in \langle \hat{c} \rangle. \end{cases}$$

Note also the Fredholm matrix of $\hat{F}$ equals $\Phi(z\colon \alpha_0)$, and the vector $\begin{pmatrix} \text{Lebes}\langle \hat{a} \rangle \\ \text{Lebes}\langle \hat{b} \rangle \\ \text{Lebes}\langle \hat{c} \rangle \end{pmatrix}$ is an eigenvector of $\Phi(1\colon \alpha_0)$ associated with eigenvalue 1. This mapping $\hat{F}$ will help to prove the calculation of the Hausdorff dimension of $\mathcal{C}$ rigorously.

## 3. Models

We also consider two types of transformations. One is mappings essentially with two symbols like Koch curves, and the other is mappings essentially with three symbols like Sierpinskii gaskets.

We will explain Koch-like mappings first. Let $I$ be a triangle $\triangle PQR$. Take two points $S$ and $T$ on $PR$ such that $PS$, $ST$ and $TR$ are disjoint segments. Take $I_1 = \triangle PQS$, $I_2 = \triangle QRT$ and $X_0 = \triangle QST$ as in the previous section. Denote by $\mathcal{A}$ an alphabet with finite symbols. We divide $\mathcal{A}$ into two sets $\mathcal{A}_1$ and $\mathcal{A}_2$ ($\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ and $\mathcal{A}_1 \cap \mathcal{A}_2 = \emptyset$). For each $a \in \mathcal{A}_i$ ($i = 1, 2$), $\langle a \rangle$ is a convex polygon and $I_i = \cup_{a \in \mathcal{A}_i} \langle a \rangle \cup X_i$ is a partition of $I_i$ ($X_i$ is an empty set or a union of polygons). Let a transformation $F\colon I_1 \cup I_2 \to I$ satisfy

$$F(x) = \begin{cases} M_1(x - P) + P & \text{if } x \in I_1, \\ M_2(x - R) + R & \text{if } x \in I_2, \end{cases}$$

and the eigenvalues of $M_1$ and $M_2$ equal $\pm\lambda_1$ and $\pm\lambda_2$, respectively and $\lambda_1, \lambda_2 > 1$. We assume that the eigenvalues of $M_1 M_2$ are complex conjugates. Then the Cantor set which we consider is

$$\mathcal{C} = \{x \in I\colon F^n(x) \in \cup_{a \in \mathcal{A}} \langle a \rangle \text{ for all n}\}.$$

Moreover, we assume that $F$ is irreducible, that is, for any $x, y \in \mathcal{C}$ and a neighborhood $U$ of $y$ there exists $n$ such that $F^n(x) \in U$. We denote $X = \cup_{i=0}^{2} X_i$. For $a \in \mathcal{A}$,

we denote by $\widetilde{\langle a\rangle}$ the maximal connected set which contains $\langle a\rangle$ and does not intersect other $\langle b\rangle$ ($b \neq a$). Note that $\widetilde{\langle a\rangle}\backslash\langle a\rangle \subset X$.

Another examples, Sierpinskii-like mappings can be defined almost the same. Let $I$ be a triangle $\triangle PQR$. $S$, $T$ and $U$ are points on edges $PQ$, $QR$ and $PR$, respectively. Take $I_1 = \triangle PSU$, $I_2 = \triangle SQT$, $I_3 = \triangle RTU$ and $X_0 = \triangle STU$ also as in the previous section. An alphabet $\mathcal{A}$ with finite symbols is divided into three sets $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ ($\mathcal{A} = \cup_{i=1}^{3}\mathcal{A}_i$ and they are mutually disjoint). For each $a \in \mathcal{A}_i$ ($i = 1, 2, 3$), $\langle a\rangle$ is a convex polygon and $I_i = \cup_{a\in\mathcal{A}_i}\langle a\rangle \cup X_i$ is a partition of $I_i$ ($X_i$ is an empty set or a union of polygons). Let a transformation $F\colon \cup_{i=1}^{3} I_i \to I$ satisfy

$$F(x) = \begin{cases} M_1(x - P) + P & \text{if } x \in I_1, \\ M_2(x - Q) + Q & \text{if } x \in I_2, \\ M_3(x - R) + R & \text{if } x \in I_3. \end{cases}$$

The eigenvalues of $M_1$ and $M_3$ equal $\pm\lambda_1$ and $\pm\lambda_3$, respectively and $\lambda_1, \lambda_3 > 1$, and

$$M_2 = \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \lambda_2 > 1.$$

We assume the eigenvalues of $M_1M_3$ are complex conjugates. Then the Cantor set which we consider is the same as before

$$\mathcal{C} = \{x \in I\colon F^n(x) \in \cup_{a\in\mathcal{A}}\langle a\rangle \text{ for all n}\}.$$

We also assume that $F$ is irreducible. We denote $X = \cup_{i=0}^{3} X_i$, and define $\widetilde{\langle a\rangle}$ ($a \in \mathcal{A}$) as in the Koch-like mappings.

Every eigenvalues of the matrices appeared in our models is greater than 1 in modulus. Therefore, our model is expanding. When we extend the domain of $F$ to $I \cup X$ using suitable map from $X$ to $I \cup X$, the map becomes an extending map from $I \cup X$ onto itself. Thus $\{\langle a\rangle\colon a \in \mathcal{A}\} \cup \{X\}$ is a generator, that is,

$$\{F^k\langle J\rangle\colon J = (a)\ (a \in \mathcal{A}) \text{ or } J = X_i\ (\exists i), k = 0, 1, 2, \ldots\}$$

generates the usual Borel $\sigma$-algebra on $I \cup X$.

From the definition, we get

$$\overline{\xi} \leq \begin{cases} \log\max\{\lambda_1, \lambda_2\} & \text{if F is Koch-like mapping,} \\ \log\max\{\lambda_1, \lambda_2, \lambda_3\} & \text{if F is Sierpinskii-like mapping.} \end{cases}$$

$$\underline{\xi} \geq \begin{cases} \log\min\{\lambda_1, \lambda_2\} & \text{if F is Koch-like mapping,} \\ \log\min\{\lambda_1, \lambda_2, \lambda_3\} & \text{if F is Sierpinskii-like mapping,} \end{cases}$$

Moreover, from the definition,

$$2\underline{\xi} \leq \xi \leq 2\overline{\xi}.$$

Thus, if $\lambda_1 = \lambda_2 = \lambda$ for Koch-like mapping or $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ for Sierpinskii-like mapping, $\overline{\xi} = \underline{\xi} = \log \lambda$, and $\xi = 2\overline{\xi} = 2\underline{\xi} = 2 \log \lambda$.

For simplicity, we use the notation

$$M^a = M_i \quad \text{if } a \in \mathcal{A}_i.$$

## 4. Hausdorff dimensions

We will define several 'Hausdorff dimensions'.

DEFINITION 1. Let $\mathcal{C}$, $I$ and so on be as above. We will define 5 types of 'Hausdorff measures' with coefficient $\alpha$, and their difference except the last one mainly depends only on families of covers of $\mathcal{C}$.

1. (usual) Hausdoff dimension: let

$$\nu_\alpha(\mathcal{C}) = \lim_{\delta \to 0} \inf \sum_i (\text{diameter of } J_i)^\alpha,$$

where inf is taken over all covers $\{J_i\}$ of $\mathcal{C}$ by compact sets with their diameter less than $\delta$. The Hausdorff dimension $d_H$ of $\mathcal{C}$ is the critical point of $\alpha$ where $\nu_\alpha(\mathcal{C})$ converges or diverges.

2. Hausdorff dimension using only covers by words: let

$$\nu_\alpha^w(\mathcal{C}) = \lim_{\delta \to 0} \inf \sum_i (\text{Lebes}\langle w_i \rangle)^\alpha,$$

where inf is taken over all covers $\{w_i\}$ of $\mathcal{C}$ by words with their Lebesgue measure less than $\delta$. The Hausdorff dimension $d_w$ of $\mathcal{C}$ using only covers by words satisfies

$$\nu_\alpha^w(\mathcal{C}) = \begin{cases} \infty & \text{if } \alpha < \dfrac{d_w}{2}, \\ 0 & \text{if } \alpha > \dfrac{d_w}{2}. \end{cases}$$

3. Hausdorff dimension using only covers by words with same length: let

$$\nu_\alpha^{w'}(\mathcal{C}) = \lim_{n \to \infty} \inf \sum_i (\text{Lebes}\langle w_i \rangle)^\alpha,$$

where inf is taken over all covers $\{w_i\}$ of $\mathcal{C}$ by words with their length equal $n$. The

Hausdorff dimension $d'_w$ of $\mathcal{C}$ using covers by words with same length satisfies

$$\nu_\alpha^{w\prime}(\mathcal{C}) = \begin{cases} \infty & \text{if } \alpha < \dfrac{d'_w}{2}, \\ 0 & \text{if } \alpha > \dfrac{d'_w}{2}. \end{cases}$$

4. Hausdorff dimension with respect to a probability measure $\mu$ on I: Let

$$\nu_\alpha^\mu(\mathcal{C}) = \lim_{\delta\to 0} \inf \sum_i (\mu(\langle w_i\rangle))^\alpha,$$

where inf is taken over all covers $\{w_i\}$ of $\mathcal{C}$ by words with their measure $\mu(\langle w_i\rangle) < \delta$. The Hausdorff dimension $d_\mu$ of $\mathcal{C}$ with respect to $\mu$ satisfies

$$\nu_\alpha^\mu(\mathcal{C}) = \begin{cases} \infty & \text{if } \alpha < \dfrac{d_\mu}{2}, \\ 0 & \text{if } \alpha > \dfrac{d_\mu}{2}. \end{cases}$$

Note that $d_w$ equals the Hausdorff dimension with respect to the Lebesgue measure on $I$ ($d_w = d_{\text{Lebes}}$).

5. Hausdorff dimension defined by Fredholm determinant: This is a bit different from other definitions. Let $\Phi(z\colon \alpha)$ be the $\alpha$-Fredholm matrix. Then determine $\alpha_0$ as the maximal solution in modulus of

$$\det(I - \Phi(1\colon \alpha)) = 0.$$

Then we call $d_\Phi = 2\alpha_0$ the Hausdorff dimension defined by Fredholm determinant. The $\alpha$-Fredholm matrix for general mappings will be given later.

From the definition, it is trivial that $d_H \le d_w \le d'_w$. The heuristic discussions in §2 for Bernoulli cases shows $d'_w = d_\Phi$.

## 5. Markov cases

In this section, we only treat the Markov mappings.

DEFINITION 2. A Koch-like mapping or a Sierpinskii-like mapping described above is called Markov if $F(\langle a\rangle)^o \cap \langle b\rangle \neq \emptyset$ for $a, b \in \mathcal{A}$, then $\overline{F(\langle a\rangle)} \supset \langle b\rangle$, where $J^o$ and $\overline{J}$ is the interior and the closure of a set $J$, respectively.

REMARK 1. The Markov property means for any $a, b \in \mathcal{A}$

$$\overline{F(\langle a\rangle)} \setminus \bigcup_{b\colon \overline{F(\langle a\rangle)} \supset \langle b\rangle} \langle b\rangle \subset \overline{X}.$$

Now we define an $\alpha$-Fredholm matrix $\Phi(z\colon \alpha)$ by

$$\Phi(z\colon \alpha)_{a,b} = \begin{cases} z\eta(a)^{\alpha} & \text{if } \overline{F(\langle a\rangle)} \supset \langle b\rangle, \\ 0 & \text{otherwise.} \end{cases}$$

Note again that $\eta(a) = |\det M^a|^{-1}$ and $M^a = M_i$ if $a \in \mathcal{A}_i$.

We will prove Theorem 1 for Markov cases, that is, the theorem which we will prove in this section is the following:

**Theorem 2.** *Let $F$ be a Markov Koch-like mapping or a Markov Sierpinskii-like mapping. Assume that $\xi d_\Phi/2 - \nu > 0$. Then the Hausdorff dimension of the Cantor set $\mathcal{C}$ generated by $F$ equals $d_\Phi$ ($d_H = d_\Phi$).*

**Lemma 1.** *For a Markov mapping $F$, $d'_w \le d_\Phi$.*

Proof. Since $F$ is Markov, the area of $\langle w\rangle$ ($w = a_1 \cdots a_n$) is greater than or equal to

$$\eta(a_{n-1}) \cdots \eta(a_1) \times \mathrm{Lebes}\langle a_n\rangle,$$

and less than or equal to

$$\eta(a_{n-1}) \cdots \eta(a_1) \times \mathrm{Lebes}\,\widetilde{\langle a_n\rangle}.$$

Therefore, $\nu_\alpha^{w\prime}(\mathcal{C})$ is greater than or equal to

$$(1, \cdots, 1)\Phi(1\colon \alpha)^{n-1}((\mathrm{Lebes}\langle a\rangle)^{\alpha})_{a\in\mathcal{A}},$$

and less than or equal to

$$(1, \cdots, 1)\Phi(1\colon \alpha)^{n-1}((\mathrm{Lebes}\,\widetilde{\langle a\rangle})^{\alpha})_{a\in\mathcal{A}}.$$

Note that $\Phi(1\colon \alpha)$ is a non-negative matrix. Therefore, the maximal eigenvalue in modulus is non-negative real and simple. This shows that if $\alpha < \alpha_0$ then $\Phi(1\colon \alpha)^n$ diverges, and if $\alpha > \alpha_0$ then it converges to 0. Therefore for $\alpha > \alpha_0$, $\nu_\alpha^{w\prime}(\mathcal{C}) = 0$. This shows $d'_w \le d_\Phi$. □

Now, we will construct another mapping $\hat{F}$ from $[0, 1]$ into itself as we did in §2. We consider an arbitrary order in $a \in \mathcal{A}$. Let $(v(a))_{a\in\mathcal{A}}$ be an eigenvector of $\Phi(1\colon \alpha_0)$ associated with eigenvalue 1 such that $\sum_{a\in\mathcal{A}} v(a) = 1$. Note that $v(a) > 0$, because we assume $F$ irreducible. Divide the interval $[0, 1]$ by the subintervals with length $v(a)$ in an order. We ignore the endpoints, because they are unessential in our discussion. Let us denote the interval corresponding to $a$ with length $v(a)$ by $\langle\hat{a}\rangle$. From the definition

of $\Phi(1\colon \alpha_0)$, we can construct a piecewise linear mapping $\hat{F}\colon [0,1] \to [0,1]$ which maps $\langle \hat{a} \rangle$ onto $\cup_{i=1}^{k} \langle \hat{b}_i \rangle$ with slope $\eta(a)^{-\alpha_0}$, where $\langle b_i \rangle \subset \overline{F(\langle a \rangle)}$ and

$$\overline{F(\langle a \rangle)} \setminus \bigcup_{i=1}^{k} \langle b_i \rangle \subset \overline{X}$$

for each $a \in \mathcal{A}$. Note that the Fredholm matrix of $\hat{F}$ equals $\Phi(z\colon \alpha_0)$ (cf. [3]), and $\hat{F}$ is expanding and irreducible.

Now, we appeal to the following theorem:

**Theorem 3** (Billingsley[1]). *For probability measures* $\mu_1, \mu_2$,

$$\mathcal{C} \subset \left\{ x\colon \lim_{n\to\infty} \frac{\log \mu_1(\langle a^x[1,n] \rangle)}{\log \mu_2(\langle a^x[1,n] \rangle)} = \alpha \right\}$$

*for some* $0 \leq \alpha \leq \infty$, *then*

$$d_{\mu_2} = \alpha d_{\mu_1},$$

*where* $a^x[1,n] \in \mathcal{W}_n$ *is a word with length* $n$ *such that* $\langle a^x[1,n] \rangle \ni x$.

**Lemma 2.** $d_w = d'_w = d_\Phi$.

Proof. Let $\mu_1$ be the probability measure on $\mathcal{C}$ which is induced through the symbolic dynamics from the Lebesgue measure on $[0,1]$ where $\hat{F}$ acts, and $\mu_2$ be the Lebesgue measure on $I$. Then

$$\mathcal{C} \subset \left\{ x\colon \lim_{n\to\infty} \frac{\log \mu_1(\langle a^x[1,n] \rangle)}{\log \mu_2(\langle a^x[1,n] \rangle)} = \alpha_0 \right\}.$$

On the other hand, the Hausdorff dimension of $[0,1]$ where $\hat{F}$ acts equals 1, hence, from the definition, $d_{\mu_1} = 2$. Therefore, by the Billingsley theorem, $d_{\text{Lebes}} = d_{\mu_2} = 2\alpha_0 (= d_\Phi)$. As we remarked in the definition, $d_{\text{Lebes}} = d_w$, and $d'_w \geq d_w$. Therefore, by Lemma 1, we get $d_w = d'_w = 2\alpha_0$. □

It remains to prove $d_w = d_H$. To prove this, we need the following assumption.

ASSUMPTION 1. There exists a constant $K > 1$ which satisfies:

1. For any $\delta > 0$ and $x \in \mathcal{C}$, there exists a word $w = a_1 \cdots a_n$ such that $x \in \langle w \rangle$ and $\delta^2/K \leq \text{Lebes}\langle w \rangle \leq \delta^2$.
2. For any word $w$, the diameter of the circumcircle of $\langle w \rangle$ is not greater than $K$ times the diameter of the inscribed circle.

We need a very simple lemma.

**Lemma 3.** *Put* $0 < \alpha < 1$. *Then for any* $a_1, \ldots, a_n > 0$ *such that* $a_1 + \cdots + a_n = c$, *we get*

$$a_1^\alpha + \cdots + a_n^\alpha \leq n^{1-\alpha} c^\alpha.$$

The proof is easy, thus we omit it.

**Lemma 4.** *Assume that Assumption* 1 *holds for some* $K > 1$. *Then* $d_H = d_w$.

Proof. We only need to show $d_H \geq d_w$. Take any compact set $J$ whose diameter is shorter than $\delta$. We consider a rectangle with length of edges being equal to $\delta$ and containing $J$. We will cover it by words whose area are between $\delta^2$ and $\delta^2/K$. Every word which intersects the rectangle must be contained in a rectangle with length of edges equals $(2K+1)\delta$. Therefore, since the area of words are greater than $\delta^2/K$, the number of words which intersect with $J$ is at most $(2K+1)^2 K$. Take any $\alpha > d_H$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that there exists a covering of $\mathcal{C}$ by compact sets $\{J_i\}$ for which their diameters are less than $\delta$, and

$$\sum_i (\text{the diameter of } J_i)^{2\alpha} < \varepsilon.$$

For each $J_i$, we can cover it by at most $(2K+1)^2 K$ words $\{w_{ij}\}$ with their area between $\delta_i^2$ and $\delta_i^2/K$, where $\delta_i$ is the diameter of $J_i$. Thus, using Lemma 3, we get

$$\begin{aligned}
\sum_i \sum_j (\text{Lebes}\langle w_{ij}\rangle)^\alpha &\leq ((2K+1)^2 K)^{1-\alpha} \sum_i \left( \sum_j \text{Lebes}\langle w_{ij}\rangle \right)^\alpha \\
&\leq ((2K+1)^2 K)^{1-\alpha} (2K+1)^{2\alpha} \sum_i \delta_i^{2\alpha} \\
&\leq (2K+1)^2 K^{1-\alpha} \varepsilon.
\end{aligned}$$

This shows $2\alpha \geq d_w$ for any $2\alpha > d_H$. Thus $d_H \geq d_w$. □

Therefore, we only need to show our models satisfy Assumption 1.

**Lemma 5.** *Markov Koch-like mappings and Markov Sierpinskii-like mappings satisfy* Assumption 1.

Proof. For Koch-like mappings, note that $(M_1)^2$ and $(M_2)^2$ are the identity matrix times constants. Thus, for any word $w = a_1 \cdots a_n$, $M_{a_n} \cdots M_{a_1}$ can be expressed in the form $U_1 (M_1 M_2)^k U_2 \times$ constant with some $k$, where $U_1$ is either $M_2$ or the iden-

tity matrix and $U_2$ is either $M_1$ or the identity matrix. From the assumption that the eigenvalues of $M_1M_2$ are complex conjugates $p \pm qi$,

$$M_{a_n} \cdots M_{a_1} = U_1 V^{-1} N^k V U_2 \times \text{constant},$$
$$N = \begin{pmatrix} p & -q \\ q & p \end{pmatrix}$$

with some matrix $V$. Since the matrix $N$ is the matrix which only rotates and expands, it does not change forms of polygons. Therefore the matrices which deform polygons are at most three matrices $M_1$, $M_2$ and $V$. We assume that mappings are Markov, that is,

$$\widetilde{\langle a \rangle} \supset F^{|w|}(\langle w \rangle) \supset \langle a \rangle$$

with some $a \in \mathcal{A}$. This proves the lemma for Koch-like mappings, and the proof of Sierpinskii-like mappings is almost the same. □

For Markov cases, we can estimate $\nu$:

$$\nu = \limsup_{n\to\infty} \sup_{l} \frac{1}{n} \log \#\{w : |w| = n, \langle w \rangle \cap l \neq \emptyset\}.$$

Note first any word $w$ with length $n$ satisfies $F^n\langle w \rangle \supset \langle a \rangle$ with at least one $a \in \mathcal{A}$. Therefore, there exists a constant $C_1 > 0$ such that

$$\text{the diameter of } \langle w \rangle \geq C_1 e^{-\overline{\xi} n}. \tag{3}$$

Thus for any segment $l$, the number of words with length $n$ which crosses $l$ is at most $C_2 e^{\overline{\xi} n}$ with some constant $C_2 > 0$. This shows $\nu \leq \overline{\xi}$. Note that even for general mapping, $F^n\langle w \rangle \subset I \cup X$. Therefore there exists a constant $C_3 > 0$

$$\text{the diameter of } \langle w \rangle \leq C_3 e^{-\underline{\xi} n}.$$

Thus we get $\nu \geq \underline{\xi}$.

REMARK 2. For non-Markov cases, the equation (3) does not necessarily holds. This means it is not easy to calculate $\nu$ in general.

## 6. Renewal equation for Markov case

In this section, we will explain a renewal equation for Markov Koch-like mappings and Markov Sierpinskii-like mappings, and using this we define $\alpha$-Fredholm matrices, which has deep connection with the Perron-Frobenius operator associated with $F$ on $\mathcal{C}$.

Assume that $F$ is a Markov mapping on $I$. For $a \in \mathcal{A}$ and $g \in L^\infty$, define a formal power series

$$s_g^{\langle a\rangle}(z\colon \alpha) = \sum_{n=0}^{\infty} z^n \int \sum_{|w|=n} \eta(w)^\alpha 1_{\langle a\rangle}(wx) g(x)\, d\bar{x}, \tag{4}$$

where $1_J$ is the indicator function of a set $J$, and a measure $d\bar{x}$ is the Lebesgue measure restricted to $\cup_{a\in\mathcal{A}}\langle a\rangle$. Dividing into the first term and the rest of terms and then changing $F(x)$ to $x$ and $n$ to $n-1$, we get

$$\begin{aligned} s_g^{\langle a\rangle}(z\colon \alpha) &= \int 1_{\langle a\rangle}(x) g(x) d\bar{x} + \sum_{n=1}^{\infty} z^n \sum_{|w|=n} \eta(w)^\alpha \int 1_{\langle a\rangle}(wx) g(x)\, d\bar{x} \\ &= \int_{\langle a\rangle} g(x)\, d\bar{x} + z\eta(a)^\alpha \sum_{n=0}^{\infty} z^n \sum_{|w|=n} \eta(w)^\alpha \sum_{b\colon \langle b\rangle \subset \overline{F(\langle a\rangle)}} \int 1_{\langle b\rangle}(wx) g(x)\, d\bar{x} \\ &= \int_{\langle a\rangle} g(x)\, d\bar{x} + z\eta(a)^\alpha \sum_{b\colon \langle b\rangle \subset \overline{F(\langle a\rangle)}} s_g^{\langle b\rangle}(z\colon \alpha). \end{aligned}$$

Take vectors

$$\begin{aligned} s_g(z\colon \alpha) &= (s_g^a(z\colon \alpha))_{a\in\mathcal{A}}, \\ \chi_g(z\colon \alpha) &= \left( \int_{\langle a\rangle} g(x)\, d\bar{x} \right)_{a\in\mathcal{A}}. \end{aligned}$$

Then we get

$$s_g(z\colon \alpha) = \chi_g(z\colon \alpha) + \Phi(z\colon \alpha) s_g(z\colon \alpha).$$

This is the renewal equation for Markov cases. Here, $\chi_g(z\colon \alpha)$ does not depend on both $z$ and $\alpha$, but as we see later, for general cases it depends on them.

For any set $J \subset I$, we can define as before

$$s_g^J(z\colon \alpha) = \sum_{n=0}^{\infty} z^n \int \sum_{|w|=n} \eta(w)^\alpha 1_J(wx) g(x)\, d\bar{x}.$$

Note that from the definition, for a disjoint union of polygons $J = J_1 \cup J_2$,

$$s_g^J(z\colon \alpha) = s_g^{J_1}(z\colon \alpha) + s_g^{J_2}(z\colon \alpha).$$

We can construct several $\alpha$-Fredholm matrices. The smallest one can be constructed

using

$$s_g^i(z\colon \alpha) = \sum_{n=0}^{\infty} z^n \int \sum_{|w|=n} \eta(w)^\alpha \sum_{a\in\mathcal{A}_i} 1_{\langle a\rangle}(wx) g(x)\, d\bar{x}.$$

Then

$$\begin{aligned} s_g^i(z\colon \alpha) &= \sum_{a\in\mathcal{A}_i} s_g^{\langle a\rangle}(z\colon \alpha) \\ &= \sum_{a\in\mathcal{A}_i} \left( \int_{\langle a\rangle} g\, d\bar{x} + z\eta(a)^\alpha \sum_{b\colon \langle b\rangle \subset \overline{F(\langle a\rangle)}} s_g^{\langle b\rangle}(z\colon \alpha) \right) \\ &= \sum_{a\in\mathcal{A}_i} \left( \int_{\langle a\rangle} g\, d\bar{x} + z\eta(a)^\alpha s_g^{F(\langle a\rangle)}(z\colon \alpha) \right). \end{aligned}$$

Renewing the last term repeatedly until $\overline{F^n(\langle a\rangle)}$ contains $\cup_i I_i$, we get a renewal equation of the form

$$s_g^i(z\colon \alpha) = \chi_g^i(z\colon \alpha) + \sum_j \phi_{i,j}(z\colon \alpha) s_g^j(z\colon \alpha),$$

with some polynomials $\chi_g^i(z\colon \alpha)$ and $\phi_{i,j}(z\colon \alpha)$. Define $\Phi_0(z\colon \alpha) = (\phi_{i,j}(z\colon \alpha))$. We can also construct another $\alpha$-Fredholm matrix $\Phi_M(z\colon \alpha)$ using $\mathcal{W}_M$ the set of admissible words with length $M$ as an alphabet.

**Lemma 6.** *The $\alpha_0$-Fredholm determinant* $\det(I - \Phi_M(z\colon \alpha_0))$ *equals the $\alpha_0$-Fredholm determinant defined before. For any $M$, the induced mapping $\hat{F}$ is the same.*

Rough sketch of the proof. The $\alpha_0$-Fredholm matrix is the Fredholm matrix of $\hat{F}$. It expresses the eigenvalues of the Perron-Frobenius operator associated with $\hat{F}$. Using [3], we can prove the lemma (see [3] for detail).

We can express the $\alpha$-Fredholm matrix with $2 \times 2$ matrix for Koch-like mappings and $3 \times 3$ matrix for Sierpinskii-like mappings. For example, let us consider a Koch curve with $\mathcal{A}_1 = \{a, b\}$, $\mathcal{A}_2 = \{c\}$, and $F(\langle a\rangle)\backslash(\langle a\rangle \cup \langle b\rangle)$, $F(\langle b\rangle)\backslash\langle c\rangle$ and $F(\langle c\rangle)\backslash\langle a\rangle$ are contained in $X$. Then the $\alpha$-Fredholm matrix using alphabet $\{a, b, c\}$, or in other words, using $s_g^{\langle a\rangle}(z\colon \alpha)$, $s_g^{\langle b\rangle}(z\colon \alpha)$ and $s_g^{\langle c\rangle}(z\colon \alpha)$, equals

$$\begin{pmatrix} z\eta_1^\alpha & z\eta_1^\alpha & 0 \\ 0 & 0 & z\eta_1^\alpha \\ z\eta_2^\alpha & 0 & 0 \end{pmatrix}.$$

On the other hand, the $\alpha$-Fredholm matrix using $s_g^{\langle a\rangle\cup\langle b\rangle}(z\colon\alpha)$ and $s_g^{\langle c\rangle}(z\colon\alpha)$ equals

$$\begin{pmatrix} z\eta_1^\alpha & z\eta_1^\alpha \\ z^2(\eta_2\eta_1)^\alpha & 0 \end{pmatrix},$$

where $\eta_1, \eta_2$ are $|\det M_1|^{-1}, |\det M_2|^{-1}$, respectively. Note here $F^2(\langle c\rangle) \supset \langle a\rangle \cup \langle b\rangle$. Now we will consider $\det(I - \Phi(z\colon\alpha))$ for both first and second $\alpha$-Fredholm matrices. Multiplying the third row of the first one by $\eta_2$, then add to the first row. Then we can erase the third row and column, and we get the second one. In a similar way, we can show the $\alpha$-Fredholm determinant $\det(I - \Phi(z\colon\alpha))$ is invariant whatever partition we take.

REMARK 3. Note that $\hat{F}$ is expanding. We can construct an invariant probability measure $\hat{\mu}$ absolutely continuous with respect to the Lebesgue measure(cf. [3]). Since $F$ on $\mathcal{C}$ and $\hat{F}$ on $[0,1]$ can be expressed on a same symbolic dynamics, we can induce $\hat{\mu}$ to a probability measure $\mu$ on $\mathcal{C}$ which is invariant under $F$.

REMARK 4. In the definition of $s_g^J(z\colon\alpha)$, if we take integrals by the Hausdorff measure $\nu_{2\alpha}$ instead of the Lebesgue measure $d\bar{x}$ restricted to $\cup_{a\in\mathcal{A}}\langle a\rangle$, we get

$$\begin{aligned}\sum_{n=0}^{\infty} z^n \int \sum_{|w|=n} \eta(w)^\alpha 1_J(wx) g(x)\, d\nu_{2\alpha} &= \sum_{n=0}^{\infty} z^n \int P_\alpha^n 1_J(x) g(x)\, d\nu_{2\alpha} \\ &= \sum_{n=0}^{\infty} z^n \int 1_J(x) g(F^n(x))\, d\nu_{2\alpha},\end{aligned}$$

where $P_\alpha$ is the Perron-Frobenius operator associated with the mapping $F$ on $(\mathcal{C}, \nu_{2\alpha})$.

## 7. Signed Symbolic Dynamics

To construct the $\alpha$-Fredholm matrix for non-Markov cases, we need to use signed symbolic dynamics.

Let $J$ be a polygon. We denote by $D_0^J$ and $D_1^J$ the set of vertices and the set of edges of $J$, respectively. Set $D^J = D_0^J \cup D_1^J$. For each vertex $P \in D_0^J$, there exists two edges $PQ, PR \in D_1^J$. Take two half lines $PQ(\infty)$ and $PR(\infty)$ through $PQ$ or $PR$ as in Fig. 4, that is, $Q$ and $R$ lie on $PQ(\infty)$ and $PR(\infty)$, respectively. We call the union of two half lines $PQ(\infty)$ and $PR(\infty)$ a 0-dimensional screen associated with the vertex $P$. It divide the plane into two parts. We call the part which contains $J$ an interior part of this screen. For each edge $PQ \in D_1^J$, we can take a line $(\infty)PQ(\infty)$ on which both $P$ and $Q$ lie. We call this line 1-dimensional screen associated with the edge $PQ$. It also divide the plane into two parts. We call the part which contains $J$ an interior part of this screen. For $\partial \in D^J$, we denote the screen associated with it by $J^\partial$. For $a \in \mathcal{A}$ or $w \in \mathcal{W}$, abbreviating brackets, we denote screens by $a^\partial$ or $w^\partial$

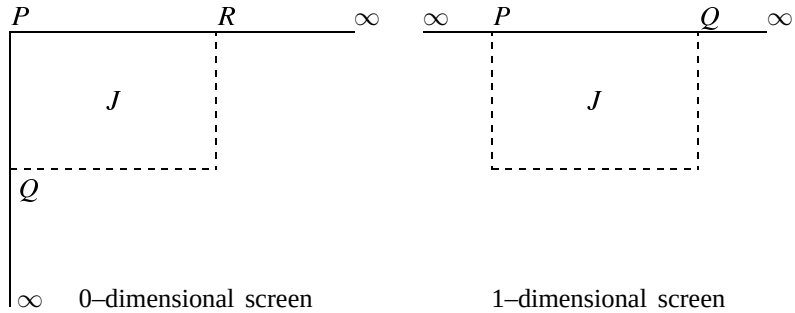


Fig. 4.

instead of $\langle a\rangle^{\partial}$ or $\langle w\rangle^{\partial}$, respectively. We denote the set of $a^{\partial}$ ($a \in \mathcal{A}, \partial \in D^{\langle a\rangle}$) and $w^{\partial}$ ($w \in \mathcal{W}, \partial \in D^{\langle w\rangle}$) by $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{W}}$, respectively.

As we described in the introduction, we will use the results in [4]. We will review its outline without proof.

Set for a screen $J^{\partial}$

$$\sigma(J^{\partial}, x) = \begin{cases} +1 & \text{if } x\text{belongs to the interior part of } J^{\partial}, \\ -1 & \text{otherwise.} \end{cases}$$

$$s(J^{\partial}) = \begin{cases} +\dfrac{1}{2} & \text{if } \partial \in D_0^J, \\ -\dfrac{1}{2} & \text{if } \partial \in D_1^J. \end{cases}$$

**Lemma 7.** *For a polygon $J$,*

$$\sum_{\partial \in D^J} s\big(J^{\partial}\big)\sigma\big(J^{\partial}, x\big)+1 = 1_J(x) \qquad a.e.\ x.$$

The proof is quite easy. But this is a key lemma to construct renewal equations.

Now we will construct renewal equations. However, it is almost impossible to construct it for $s_g^J(z\colon \alpha)$ defined in §6. So we divide them into several generating functions. Define for a polygon $J \subset \langle a\rangle$ ($a \in \mathcal{A}$) and $g \in L^{\infty}$

$$s_g^{J^{\partial}}(z\colon \alpha) = \int d\bar{x}\, g(x) \sum_{\substack{\langle w[1]\rangle \supset \langle a\rangle \\ \exists \theta wx}} z^{|w|}\eta(w)^{\alpha}\sigma(J^{\partial}, wx),$$

where the sum is over all the words $w$ either empty word, or its first symbol equals $a \in \mathcal{A}$ and there exists a point $y \in I$ such that

$$F^{|w|-1}(y) = x,$$
$$F^{i-1}(y) \in \langle w[i+1]\rangle \quad (1 \le i \le |w|-1).$$

Note that we need not assume the existence of a point $y' \in \langle a\rangle$ such that $F(y') = y$,

that is, we do not assume $wx \in \langle w \rangle$ (recall the definition of $wx$). Our aim is to construct a renewal equation of $s_g^{J^\partial}(z\colon \alpha)$ above. In [4], since we considered usual dynamical systems on a plane, we constructed it with coefficients $\eta(w)$ instead of $\eta(w)^\alpha$. Nevertheless, the construction of the renewal equation is just the same.

Using Lemma 7, we get the following lemma.

**Lemma 8.** *For a polygon $J \subset \langle a \rangle$ for some $a \in \mathcal{A}$ and $g \in L^\infty$,*

$$s_g^J(z\colon \alpha) = \int_I g(x)\, d\bar{x} + z\eta(a)^\alpha s_g^I(z\colon \alpha) + \sum_{\partial \in D^J} s(\partial) s_g^{J^\partial}(z\colon \alpha),$$

$$s_g^I(z\colon \alpha) = \#\mathcal{A} \int g\, d\bar{x} + z \left( \sum_{a \in \mathcal{A}} \eta(a)^\alpha \right) s_g^I(z\colon \alpha) + \sum_{a \in \mathcal{A}} \sum_{\partial \in D^{\langle a \rangle}} s_g^{a^\partial}(z\colon \alpha).$$

This lemma suggests that the singularities of $s_g^J(z\colon \alpha)$ are determined by the singularities of $s_g^{J^\partial}(z\colon \alpha)$.

To solve the problem of singularities of $s_g^{J^\partial}(z\colon \alpha)$, we construct a renewal equation. We need several notations which we mentioned in [4].

For $a$ and $b$ ($a, b \in \mathcal{A}$), we say that a screen $\tilde{J}$ of a polygon $J$ crosses $ab$ if $J \subset \langle a \rangle$ and $F^a(\tilde{J}) \cap \langle b \rangle \neq \emptyset$. Set

$$\langle ab, \tilde{J} \rangle = \begin{cases} \{x \in \langle b \rangle \colon \sigma(F^a(\tilde{J}), x) = +1\} & \text{if } \tilde{J} \text{ crosses ab,} \\ \emptyset & \text{otherwise.} \end{cases}$$

A screen $\langle ab, \tilde{J} \rangle^\partial$ with a face $\partial \in D^{\langle ab, \tilde{J} \rangle}$ such that $\langle ab, \tilde{J} \rangle^\partial$ and $b^{\partial'}$ are different as sets for any $\partial' \in D^{\langle b \rangle}$ is called a new screen generated by $F^a(\tilde{J})$ in $\langle b \rangle$, and we denote by $\mathrm{New}\langle ab, \tilde{J} \rangle$ the set of all the new screens of $\langle ab, \tilde{J} \rangle$.

Let $w = a_1 \cdots a_n$ be a word of length $n \geq 2$ and $b \in \mathcal{A}$. We call a screen $\tilde{J}$ of a polygon $J$ crosses $wb$ if

$$\begin{aligned} &J \subset \langle a_1 \rangle, \\ &F^{a_i}(\Delta_0\langle w[1, i], \tilde{J} \rangle) \cap \langle a_{i+1} \rangle \neq \emptyset \quad (1 \leq i \leq n-1), \\ &F^{a_n}(\tilde{K}) \cap \langle b \rangle \neq \emptyset, \end{aligned}$$

for some $\tilde{K} \in \mathrm{New}\langle w, \tilde{J} \rangle$, where we define inductively the sets $\Delta_0\langle wb, \tilde{J} \rangle$, $\langle wb, \tilde{J} \rangle$ and the new screens $\mathrm{New}\langle wb, \tilde{J} \rangle$ generated by $F^w(\tilde{J})$ in $\langle b \rangle$ as follows:

$$\langle wb, \tilde{J} \rangle = \begin{cases} \bigcap_{\tilde{K} \in \mathrm{New}\langle w, \tilde{J} \rangle} \langle a_n b, \tilde{K} \rangle & \text{if } \tilde{J} \text{ crosses } wb, \\ \emptyset & \text{otherwise,} \end{cases}$$

$$\mathrm{New}\langle wb, \tilde{J} \rangle = \begin{cases} \{\langle wb, \tilde{J} \rangle^\partial \colon \langle wb, \tilde{J} \rangle^\partial \neq b^{\partial'}, \forall \partial' \in D^{\langle b \rangle}\}, & \text{if } \tilde{J} \text{ crosses } wb, \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$\Delta_0\langle wb, \tilde{J}\rangle = \begin{cases} (\tilde{J} \cap \langle b\rangle)\backslash\Delta\langle b\rangle & \text{if } w = \epsilon, \\ (F^w(\tilde{J}) \cap \Delta\langle wb, \tilde{J}\rangle)\backslash\Delta\langle b\rangle & \text{otherwise,} \end{cases}$$

where $\Delta J$ is the boundary of a set $J$.

DEFINITION 3. We denote by $F\tilde{\mathcal{A}}$ the set of new screens generated by $F^a(a^\partial)$ in some $\langle b\rangle$ $(a, b \in \mathcal{A}, \partial \in D^{\langle a\rangle})$, that is, $F\tilde{\mathcal{A}} = \cup_{a,b\in\mathcal{A}} \cup_{\partial\in D^{\langle a\rangle}} \text{New}\langle ab, a^\partial\rangle$. For $n \geq 2$, let $F^n\tilde{\mathcal{A}}$ be the set of new screens generated by $F^a(\tilde{J})$ in some $\langle b\rangle$ $(a, b \in \mathcal{A}, \tilde{J} \in F^{n-1}\tilde{\mathcal{A}}, J \subset \langle a\rangle)$, which do not belong to $\cup_{k=0}^{n-1} F^k\tilde{\mathcal{A}}$.

DEFINITION 4. For a screen $\tilde{J}$ and $g \in L^\infty$, set

$$\chi_g^{\tilde{J}}(z\colon \alpha) = \int g(x)\sigma(\tilde{J}, x)\, d\bar{x} + z\eta(a)^\alpha \sum_{b\in\mathcal{A}} \sigma_1(F^a(\tilde{J}), b) \int g(x)\, d\bar{x},$$
$$\phi^{\tilde{J}}(\tilde{b}\colon \alpha) = s(\tilde{b})\eta(a)^\alpha \sigma_*(F^a(\tilde{J}), \tilde{b}),$$
$$\phi^{\tilde{J}}(I\colon \alpha) = \eta(a)^\alpha \sum_{b\in\mathcal{A}} \sigma_1(F^a(\tilde{J}), b)\eta(b)^\alpha,$$

and

$$\sigma_1(F^a(\tilde{J}), b) = \begin{cases} -1 & \text{if } \sigma(F^a(\tilde{J}), x) = -1 \text{ holds for a.e. } x \in \langle b\rangle, \\ +1 & \text{otherwise,} \end{cases}$$

$$\sigma_*(F^a(\tilde{J}), \tilde{b}) = \begin{cases} +1 & \text{if } \sigma(F^a(\tilde{J}), x) = +1 \text{ for all } x \in \langle b\rangle, \\ & \text{or if } \tilde{J} \text{ crosses } ab \text{ and } \tilde{b} = \langle ab, \tilde{J}\rangle^\partial \\ & \qquad \text{for some } \partial \in D^{\langle ab, \tilde{J}\rangle}, \\ -1 & \text{otherwise.} \end{cases}$$

Set for a screen $\tilde{J}, \tilde{L} \in \cup_{k=0}^\infty F^k\tilde{\mathcal{A}}$ $(J \subset \langle a\rangle$ for some $a \in \mathcal{A})$ of a polygon $J$

$$\phi(\tilde{J}, \tilde{L})(z\colon \alpha) = \begin{cases} z\phi^{\tilde{J}}(\tilde{L}\colon \alpha) & \text{if } \tilde{L} \in \tilde{\mathcal{A}}, \\ 2z\eta(a)^\alpha s(\tilde{J}) & \text{if } a, b \in \mathcal{A} \text{ such that } \tilde{J} \text{ crosses } ab \\ & \text{and } \tilde{L} \in \cup_{k=1}^\infty F^k\tilde{\mathcal{A}} \text{ is a new screen} \\ & \text{generated by } F^a(\tilde{J}) \text{ in } \langle b\rangle, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\phi(\tilde{J}, I)(z\colon \alpha) = z^2\phi^{\tilde{J}}(I\colon \alpha).$$

Then we get a renewal equation of the form using Lemma 7:

**Lemma 9.** *For all $a \in \mathcal{A}$, polygons $J \subset \langle a \rangle$ and $g \in L^\infty$, we get a formal expression*

$$s_g^{\tilde{J}}(z\colon \alpha) = \chi_g^{\tilde{J}}(z\colon \alpha) + \sum_{\tilde{L} \in \cup_{k=0}^{\infty} F^k \tilde{\mathcal{A}}} \phi(\tilde{J}, \tilde{L})(z\colon \alpha) s_g^{\tilde{L}}(z\colon \alpha) + \phi(\tilde{J}, I)(z\colon \alpha) s_g^{I}(z\colon \alpha). \tag{5}$$

This renewal equation leads an infinite dimensional renewal equation for

$$\left(s_g^{\tilde{J}}(z\colon \alpha)\right)_{\tilde{J} \in \cup_{n=0}^{\infty} F^n \tilde{\mathcal{A}} \cup I}.$$

Using this renewal equation, we want to determine the singularities of $s_g^J(z\colon \alpha)$.

Renewing all the terms corresponding to $\cup_{k=n+1}^{\infty} F^k \tilde{\mathcal{A}}$ in (5), we can construct a finite dimensional Fredholm matrix $\Phi_n(z\colon \alpha)$, which is $\cup_{k=0}^{n} F^k \tilde{\mathcal{A}} \times \cup_{k=0}^{n} F^k \tilde{\mathcal{A}}$ matrix and a $\cup_{k=0}^{n} F^k \tilde{\mathcal{A}}$ vector $\chi_{g,n}(z\colon \alpha)$. We need complicated notations and calculations to show how to construct them, thus we omit it. See [4] for detail. We get the following theorem.

**Theorem 4.** *For any $\varepsilon > 0$, there exists $n_0$ and for $n \geq n_0$ $\Phi_n(z\colon \alpha)$ and $\chi_{g,n}(z\colon \alpha)$ are analytic in $|z| < e^{\xi\alpha - \nu - \varepsilon}$ and for any $g \in L^\infty$ they satisfy*

$$(I - \Phi_n(z\colon \alpha)) s_{g,n}(z\colon \alpha) = \chi_{g,n}(z\colon \alpha),$$

*where*

$$s_{g,n}(z\colon \alpha) = \left(s_g^{\tilde{J}}(z\colon \alpha)\right)_{\tilde{J} \in \cup_{k=0}^{n} F^k \tilde{\mathcal{A}}}.$$

Using the above theorem, we can prove the singularities of $s_g^J(z\colon \alpha)$ ($J$: polygon) in $|z| < e^{\xi\alpha - \nu - \varepsilon}$ are the solutions of

$$\det(I - \Phi_n(z\colon \alpha)) = 0.$$

ASSUMPTION 2. Assume that there exists $n_0$ such that for $n \geq n_0$ there exists a solution $\alpha > \nu/\xi$ of

$$\det(I - \Phi_n(1\colon \alpha)) = 0. \tag{6}$$

We denote by $\alpha_0$ the maximal solution of (6).

Hereafter, for notational convenience, we fix $n_0$ for which $\alpha_0$ is the maximal solution of (6).

For Markov mappings, we have constructed two types of Fredholm matrices. Noticing the fact that they are constructed to express the singularities of $s_g^J(z\colon \alpha)$, the zeros of the Fredholm determinant $\det(I - \Phi(z\colon \alpha))$ are the same.

## 8. Estimate of Hausdorff dimension

**8.1. The inequality $2\alpha_0 \geq d_H$** Let $J$ be a polygon contained in some $\langle a \rangle$ ($a \in \mathcal{A}$). We constructed a renewal equation for

$$s_g^{J^\partial}(z\colon \alpha) = \int d\bar{x}\, g(x) \sum_{\substack{\langle w[1] \rangle \supset \langle a \rangle \\ \exists \theta wx}} z^{|w|} \eta(w)^\alpha \sigma(J^\partial, wx)$$

in §7. We define another generating function:

$$\underline{s}_g^{J^\partial}(z\colon \alpha) = \int d\bar{x}\, g(x) \sum_{\substack{\langle w[1] \rangle \supset \langle a \rangle \\ w=\epsilon \text{ or } \exists \theta wx}} z^{|w|} \eta(w)^\alpha \underline{\sigma}(J^\partial, wx),$$

where

$$\underline{\sigma}(J^\partial, wx) = \begin{cases} +1 & \text{if } w = \epsilon \text{ and } x[1] \cap \text{ interior of } J^\partial \neq \emptyset, \\ +1 & \text{if } w \neq \epsilon \text{ and } \langle (\theta w) x[1] \rangle \cap \text{ interior of } F^a(J^\partial) \neq \emptyset, \\ -1 & \text{otherwise,} \end{cases}$$

where $x[1] = b$ if $x \in \langle b \rangle$ ($b \in \mathcal{A}$). Then, as in Lemma 8

$$\sum_{\partial \in D^{\langle a \rangle}} s(\partial) \underline{s}_g^{a^\partial}(z\colon \alpha) + \int_I g(x)\, d\bar{x} + z\eta(a)^\alpha \underline{s}_g^I(z\colon \alpha) = \underline{s}_g^{\langle a \rangle}(z\colon \alpha),$$

where

$$\underline{s}_g^{\langle a \rangle}(z\colon \alpha) = \sum_{n=0}^{\infty} z^n \sum_{\substack{|w|=n \\ w[1]=a}} \eta(w)^\alpha \sum_{b\colon \langle wb \rangle \neq \emptyset} \int_{\langle b \rangle} g(x)\, d\bar{x}.$$

Thus, taking $g \equiv 1$, the $n$-th coefficient of $\sum_{a \in \mathcal{A}} \underline{s}_1^{\langle a \rangle}(z\colon \alpha)$ equals

$$\begin{aligned} \sum_{|w|=n} \sum_{b\colon \langle wb \rangle \neq \emptyset} \eta(w)^\alpha \operatorname{Lebes}(\langle b \rangle) &\geq \sum_{|w|=n} \sum_{b\colon \langle wb \rangle \neq \emptyset} \eta(w)^\alpha K_\alpha (\operatorname{Lebes}(\langle b \rangle))^\alpha \\ &\geq K_\alpha \sum_{|w|=n+1} (\operatorname{Lebes}(\langle w \rangle))^\alpha, \end{aligned}$$

where

$$K_\alpha = \min_{b \in \mathcal{A}} \frac{\operatorname{Lebes}(\langle b \rangle)}{(\operatorname{Lebes}(\langle b \rangle))^\alpha}.$$

Thus $\sum_{a \in \mathcal{A}} \underline{s}_1^{\langle a \rangle}(1\colon \alpha)$ diverges as $n \to \infty$ for $2\alpha < d'_w$.

Define

$$\underline{\chi}_g^{\tilde{J}}(z\colon \alpha) = \int \underline{\sigma}(\tilde{J}, x)\, d\bar{x} + z\eta(a)^{\alpha} \sum_{b\in\mathcal{A}} \underline{\sigma}_1(F^a\tilde{J}, b) \int g(x)\, d\bar{x},$$

where

$$\underline{\sigma}_1(F^a\tilde{J}, b) = \underline{\sigma}(F^a\tilde{J}, x) \quad (x \in \langle b\rangle).$$

Then we get as (5)

$$(7)\qquad \underline{s}_g^{\tilde{J}}(z\colon \alpha) = \underline{\chi}_g^{\tilde{J}}(z\colon \alpha) + \sum_{\tilde{L}\in\cup_{k=0}^{\infty}F^k\tilde{\mathcal{A}}} \phi(\tilde{J}, \tilde{L})(z\colon \alpha)\underline{s}_g^{\tilde{L}}(z\colon \alpha) + \phi(\tilde{J}, I)(z\colon \alpha)\underline{s}_g^{I}(z\colon \alpha).$$

Thus, the singularities of $\underline{s}_g^{\tilde{J}}(z\colon \alpha)$ are the same as $s_g^{\tilde{J}}(z\colon \alpha)$. Namely, take $n_0$ which is defined in Assumption 2 and reduce the infinite dimensional Fredholm matrix $\Phi(z\colon \alpha)$ above to $\cup_{k=0}^{n_0}F^k\mathcal{A}$ dimensional Fredholm matrix $\Phi_{n_0}(z\colon \alpha)$ as in (6). Then $\underline{s}_g(z\colon \alpha)$ satisfies the similar renewal equation

$$\underline{s}_{g,n_0}(z\colon \alpha) = \underline{\chi}_{g,n_0}(z\colon \alpha) + \Phi_{n_0}(z\colon \alpha)\underline{s}_{g,n_0}(z\colon \alpha).$$

Thus, the maximal singularity of $\underline{s}_g^{\tilde{J}}(z\colon \alpha)$ equals $\alpha_0$. This shows $2\alpha_0 \geq d'_w$, that is, $2\alpha_0 \geq d_H$.

**8.2. The inequality $2\alpha_0 \leq d_H$** For any non-Markov transformation $F$, we will construct Markov transformations $F_N$ for which

1. $\langle a\rangle_N \subset \langle a\rangle$, where $\langle a\rangle_N$ and $\langle a\rangle$ are the polygons corresponding to $F_N$ and $F$ ($a \in \mathcal{A}$),
2. $\langle a\rangle_N$ is monotone increasing in $N$ and $\cup_N\langle a\rangle_N = \langle a\rangle$,
3. $F_N$ is the restriction of $F$ to $\cup_{a\in\mathcal{A}}\langle a\rangle_N$,
4. the components of the Fredholm matrix $\Phi(z\colon \alpha, F_N)$ almost coincides until the coefficients $z^N$ with $\Phi(z\colon \alpha)$, where $\Phi(z\colon \alpha, F_N)$ the $\alpha$-Fredholm matrix associated with $F_N$.

Then we get $\mathcal{C}_N \subset \mathcal{C}$, where $\mathcal{C}_N$ is the Cantor set generated by $F_N$. Therefore, the Hausdorff dimension of $\mathcal{C}_N$ is less than or equal to $d_H$.

**Lemma 10.** *Assume $\xi\alpha > \nu$.*

1. $\det(I - \Phi(1\colon \alpha, F_N))$ *converges to* $\det(I - \Phi(1\colon \alpha, F))$ *as* $N \to \infty$,
2. *for sufficiently large $N$ $\xi_N\alpha > \nu_N$, where $\xi_N$ is the lower Lyapunov number associated with $F_N$.*

Proof. Let us denote by $\langle w\rangle_N$ the polygon corresponding to a word $w$ determined by $F_N$. Note first $F_N$ is the restriction of $F$. Therefore, the matrices which determine $F$ and $F_N$ are the same, that is, for a word $w$ ($|w| = n$), the Jacobian on $\langle w\rangle$

of $F^n$ and the Jacobian on $\langle w\rangle_N$ of $F_N^n$ concide if $w$ is admissible with respect to both $F$ and $F_N$. From the definition, for any word $w$, $\langle w\rangle_N \subset \langle w\rangle$. This means if $w$ is admissible with respect to $F_N$, then it is also admissible with respect to $F$. Thus, $\xi \geq \xi_N$. On the other hand, $\langle w\rangle_N \to \langle w\rangle$ as $N \to \infty$. Therefore, for an admissible word $w$ with respect to $F$, there exists $N_0$ such that for $N \geq N_0$ $\langle w\rangle_N$ is admissible. This shows $\lim_{N\to\infty} \xi_N = \xi$. Again, since $\langle w\rangle_N \subset \langle w\rangle$, we get $\nu_N \leq \nu$. From the assumption that $\xi\alpha > \nu$, $1 < e^{\xi\alpha-\nu}$. Moreover, since $\Phi(z\colon \alpha, F_N)$ almost coincides until the coefficients $z^N$ with $\Phi(z\colon \alpha)$, $\det(I - \Phi(1\colon \alpha, F_N))$ converges to $\det(I - \Phi(1\colon \alpha))$. This proves the lemma. □

Let $\alpha_{0,N}$ be the maximal zero of $\det(I - \Phi(1\colon \alpha, F_N))$. Because $F_N$ is Markov, $2\alpha_{0,N}$ equals the Hausdorff dimension of $\mathcal{C}_N$ which is less than or equal to $d_H$. This shows $d_H \geq 2\alpha_{0,N}$.

Markov transformations $F_N$ satisfying the above conditions are constructed as follows. Arrange screens $\tilde{a} \in \tilde{\mathcal{A}}$ which are generated by edges in an order $\tilde{a}_1, \tilde{a}_2, \ldots$. Let

$$O_N^0 = \{F^k(\tilde{a})\colon \tilde{a} \in \tilde{\mathcal{A}}, 1 \leq k \leq N\}.$$

Consider $e_1 = \tilde{a}_1 \cap \Delta\langle a_1\rangle$, that is, $e_1$ is an edge of $\langle a_1\rangle$ which generates $\tilde{a}_1$. Choose words $\{w_1^i\}$ which satisfy:

1. $|w_1^i| > N$.
2. $\langle w_1^i\rangle \subset \langle a_1\rangle$.
3. $(\Delta\langle w_1^i\rangle) \cap e_1 \neq \emptyset$, that is, $\langle w_1^i\rangle$ has an edge contained in $e_1$.
4. $\Delta(\cup_i \langle w_1^i\rangle) \supset e_1$.
5. If $F^k(\tilde{b})$ intersects $e_1$ $(1 \leq k \leq N, \tilde{b} \in \tilde{\mathcal{A}})$, there exists only one $w_1^i$ which intersects it.
6. Let $e_1' = \left(\Delta(\cup_i \langle w_1^i\rangle)\right) \setminus e_1$. Then, for any edge $e$ of $\langle b\rangle$ $(b \in \mathcal{A})$, $F^{w_1^i}(e_1' \cap \Delta(\langle w_1^i\rangle))$ intersects $e$ at most once for any $1 \leq k \leq N$.

We can choose such $w_1^i$ taking $|w_1^i|$ sufficiently large. Now take $\langle a_1\rangle \setminus (\cup\langle w_1^i\rangle)$ as a new polygon associated with $a_1$ and $(\tilde{a}_1 \setminus e_1) \cup e_1'$ as a new screen $\tilde{a}_1$. We can naturally define new screens generated by vertices of $\langle a_1\rangle$ using new screens generated by edges. Now define $O_N^1$ using new $\tilde{a}_1$ as we define $O_N^0$. Next consider $\tilde{a}_2$ and choose words $\{w_2^i\}_i$ same as before and do the same thing. Continue this procedure, and we get new polygons and screens. When we want to emphasize new or old polygons or screens, we will write such as $\langle a\rangle_N$ or $\langle a\rangle_{\text{old}}$ etc. Note that

$$\langle a\rangle_N = \cap_{\partial \in D\langle a\rangle} \text{ interior of new screens of } a^{\partial} \subset \langle a\rangle_{\text{old}}.$$

Take $Y_N$ the union of new $\langle a\rangle$ $(Y_N = \sum_{a\in\mathcal{A}} \langle a\rangle_N)$, and define $F_N$ the restriction of $F$ to $Y_N$. We denote by $\mathcal{C}_N$ the Cantor set generated by $F_N$. Then from the definition $C_N \subset \mathcal{C}$, and $F_N$ is a Markov transformation (cf. Remark 1). Hence, by Lemma 5, $F_N$

satisfies Assumption 1. Now consider for a polygon $J \subset \langle a \rangle_N$ ($a \in \mathcal{A}$) and $\partial \in D^J$

$$s_g^{J^\partial}(z\colon \alpha, F_N) = \sum_{n=0}^{\infty} z^n \sum_{w \in \mathcal{W}\colon |w|=n} \eta(w)^\alpha \int \sigma_N(J^\partial, wx)\delta[\langle w[1]\rangle_N \supset \langle a\rangle_N, \langle \theta w\rangle_N \neq \emptyset] g(x)\, d\bar{\bar{x}},$$

where $d\bar{\bar{x}}$ is the Lebesgue measure restricted to $Y_N$. Let us denote by $\Phi(z\colon \alpha, F_N)$ the $\alpha$-Fredholm determinant. Since $F_N$ is Markov, the Hausdorff dimension of $\mathcal{C}_N$ equals $2\alpha_{0,N}$, where $\alpha_{0,N}$ is the maximal solution of

$$\det(I - \Phi(1\colon \alpha, F_N)) = 0. \tag{8}$$

Note also, from the construction of the $\alpha$-Fredholm matrix $\Phi(z\colon \alpha, F_N)$, the singularities of $s_g^J(z\colon \alpha, F_N)$ ($J\colon$ a polygon) is determined by the equation

$$\det(I - \Phi(z\colon \alpha, F_N)) = 0. \tag{9}$$

Without loss of generality, we can assume $N \geq n_0$, where $n_0$ is determined in Assumption 2.

**Lemma 11.** *Fix any $\varepsilon > 0$. Then, there exists a constant $K > 0$ such that for a word $w$ ($|w| > N$) and $n \leq N$*

$$|\textit{the coefficient of } z^n \textit{ of } s_g^{\langle w\rangle}(z\colon \alpha)| \leq K||g||_\infty e^{-(\xi-\varepsilon)N} \tag{10}$$

Proof. We have

$$|\text{the coefficient of } z^n \text{ of } s_g^{\langle w\rangle}(z\colon \alpha)| = \left| \int \sum_{|u|=n} \eta(u)^\alpha 1_{\langle w\rangle}(ux) g(x)\, d\bar{\bar{x}} \right|$$

$$\leq K' e^{-(\xi\alpha-\varepsilon)n}||g||_\infty \sum_{|u|=n} \int 1_{\langle w\rangle}(ux)\, dx \tag{11}$$

with some constant $K'$. From the assumption $n \leq N$, only one $u$ satisfies $1_{\langle w\rangle}(ux) = 1$. Therefore,

$$\text{rhs. of (11)} \leq K' e^{-(\xi\alpha-\varepsilon)n}||g||_\infty \,\text{Lebes}\langle w\rangle,$$

and $\text{Lebes}\langle w\rangle \leq K'' e^{-(\xi-\varepsilon)N}$ with some constant $K''$. This proves the lemma. □

Recall that the renewal equation of $s^{\tilde{J}}(z\colon \alpha)$ is constructed by looking the positions of $F^k(\tilde{J})$ and $\langle b\rangle$. We will review the construction of the renewal equation. Take

$\tilde{J} \in \cup_{k=0}^{n_0} F^k\tilde{\mathcal{A}}$ ($J \subset (a)$). Then dividing $n = 0$ term and others in

$$s_g^{\tilde{J}}(z\colon \alpha) = \int \sum_{n=0}^{\infty} z^n \sum_{|w|=n} \sigma(\tilde{J}, wx)\eta(w)^\alpha g(x)\, d\bar{x},$$

we get

$$(12)\quad s_g^{\tilde{J}}(z\colon \alpha) = \int \sigma(\tilde{J}, x)g(x)\, d\bar{x} + z\eta(a)^\alpha \left[\sum_{b\in\mathcal{A}} \sigma_1(F^a(\tilde{J}), b)s_g^{\langle b\rangle}(z\colon \alpha) + \sum_{\langle b\rangle \cap F^a(\tilde{J})\neq\emptyset} s_g^{\langle ab,\tilde{J}\rangle}(z\colon \alpha)\right].$$

When we consider new $\tilde{a}$'s and $F_N$, there may happen $\sigma(F^a(\tilde{J}), b)$ differs or $\langle b\rangle \cap F^a(\tilde{J}) = \emptyset$ or not for new and old ones. However, such things may happen when at least one of new or old $F^a(\tilde{J})$ crosses $\tilde{b} \in \tilde{\mathcal{A}}$. When we consider

$$(13)\quad s_g^{\tilde{J}}(z\colon \alpha, F_N) - \int \sigma_{\text{old}}(\tilde{J}, x)g(x)\, d\bar{x} - z\eta(a)^\alpha \left[\sum_{b\in\mathcal{A}} \sigma_{1,\text{old}}(F^a(\tilde{J}), b)s_g^{\langle b\rangle}(z\colon \alpha, F_N) - \sum_{\langle b\rangle \cap F^a(\tilde{J})\neq\emptyset} s_g^{\langle ab,\tilde{J}\rangle_{\text{old}}}(z\colon \alpha, F_N)\right]$$

using old $\sigma(F^a(\tilde{J}), b)$ etc., we get

$$(14)\quad |(14)| \le \sum_{\substack{l,i \\ \langle w_l^i\rangle \text{ crosses } F^a(\tilde{J})}} |z|\eta(a)^\alpha s_g^{\langle w_l^i\rangle}(z\colon \alpha, F_N) \le |z|\eta(a)^\alpha K||g||_\infty e^{-(\xi-\varepsilon)N}.$$

We get a renewal equation dividing $s_g^{\langle b\rangle}(z\colon \alpha)$ and $s_g^{\langle ab,\tilde{J}\rangle}(z\colon \alpha)$ in (13) using Lemma 7, and continuing this procedure. So also for $s_g^{\tilde{J}}(z\colon \alpha, F_N)$, we approximate it using old $\sigma_{1,\text{old}}(F^{\tilde{J}[1,k]}(\tilde{J}), b)$ and $\langle \tilde{J}[1,k]b, \tilde{J}\rangle_{\text{old}}$ until $N$, and for $n > N$ we use new $\sigma_{1,N}(F^{\tilde{J}[1,k]}(\tilde{J}), b)$ and $\langle \tilde{J}[1,k], \tilde{J}\rangle_N$. Then the difference

$$(15)\quad s_g^{\tilde{J}}(z\colon \alpha, F_N) - \chi_g^{\tilde{J}}(z\colon \alpha, F_N) - \sum_{\tilde{b}\in\cup_{k=0}^{n_0} F^k\mathcal{A}} \Phi(z\colon \alpha, F_N)_{\tilde{J},\tilde{b}} s_g^{\tilde{b}}(z\colon \alpha, F_N)$$

is at most

$$|(15)| \le \sum_{k=0}^{N} |z|^k \eta(\tilde{J}[1,k])^\alpha K||g||_\infty e^{-(\xi-\varepsilon)N} \times \left(\text{the number of } \langle w_l^i\rangle \text{ which intersect with } \cup_{k=0}^{N} F^k\tilde{\mathcal{A}}\right)$$

$$\leq K' \sum_{k=0}^{n_0} |z|^k e^{-(\xi\alpha-\varepsilon)k} ||g||_\infty e^{-(\xi-\varepsilon)N} e^{\nu N}$$
$$= K' \frac{||g||_\infty}{1-|z|e^{-(\xi\alpha-\varepsilon)}} e^{-(\xi-\nu-\varepsilon)N}$$

with some constant $K'$. The singularities of $s_g^{\tilde{J}}(z\colon \alpha, F_N)$ in $|z| < e^{\xi\alpha-\nu}$ are determined by the zeros of $\det(I - \Phi(z\colon \alpha, F_N))$, and from the construction they converge to the zeros of $\det(I - \Phi(z\colon \alpha))$. On the other hand, the maximal zero of $\det(I - \Phi(1\colon \alpha, F_N))$ equals $\alpha_{0,N}$. This shows, if $\xi\alpha_0 > \nu$, by Lemma 10 and the assumption that $\alpha_0$ is the simple zero, $\lim_{N\to\infty} \alpha_{0,N} = \alpha_0$. Therefore,

$$d_H \geq \lim_{N\to\infty} d_H(\mathcal{C}_N) = \lim_{N\to\infty} 2\alpha_{0,N} = 2\alpha_0.$$

Combining the results, we get $d_H = 2\alpha_0 = d_\Phi$, and complete the proof of Theorem 1.

REMARK 5. It is not always possible to construct a Markov transformation $F_N$ for which $\mathcal{C}_N \supset \mathcal{C}$. Because $\langle a \rangle_N$ may intersect.

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