BLOCK IDEMPOTENTS AND NORMAL p-SUBGROUPS

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1. Introduction

In the theory of modular representations of a finite group G in an algebraically closed field Ω of characteristic p, Brauer has proved a useful reduction theorem for blocks [2, §§ 11, 12], [5, (88.8)], which can be reformulated as follows:

THEOREM 1 (Brauer). *Let P be an arbitrary p-subgroup of G; let $N = N_G(P)$ and $W = PC_G(P)$. Then there is a natural one-to-one correspondence between the set of all blocks of G which have P as a defect group and the set of orbits, under conjugation by N/P, of those blocks $\bar{B}$ of W/P such that:*

(1.1) *$\bar{B}$ has defect* 0, *and*

(1.2) $|\bar{S} : W/P| \not\equiv 0 \pmod{p}$, *where $\bar{S}$ is the stability group of $\bar{B}$ in N/P.*

The proof of this theorem can be divided into three parts: first a reduction from G to N by Brauer's first main theorem [2, (10 B)], then a reduction from N to W which is obtained by analyzing relationships between N and its normal p-subgroup P [2, (11 B)], and finally a relatively easy reduction from W to W/P. The arguments in [2] rely heavily on the theory of Brauer characters. Rosenberg [12, Theorem 5.3] has given a proof of the first reduction which operates within centers of group algebras, thus avoiding the use of character theory, and which eliminates the assumption that Ω is algebraically closed; and Conlon [4, p. 166] and Reynolds [11, Theorem 1] have generalized Rosenberg's argument to twisted group algebras, that is, to the case of projective representations.

It is the aim of the present paper to obtain a corresponding treatment of the second reduction, and so to prove by character-free methods a version

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(Theorem 5) of Theorem 1 for twisted group algebras over arbitrary Ω, which embodies the first two reductions. The generalization to the twisted case causes almost no complications, while the generalization concerning Ω forces us to replace the stability group by a certain smaller group. We also perform the third reduction to obtain another version (Theorem 6) of Theorem 1; however we have not been able to avoid using character theory at one point here (cf. the Remark preceding Theorem 2). In the twisted case, the third reduction is not possible in general unless a purely inseparable extension of Ω is allowed.

We develop our tools in sections 2 and 4, leading up to the block theory for a group and a normal subgroup (cf. [6, § 1], [10, § 10]). The heart of the paper is in § 5, where we prove Theorem 5 by placing it in the setting of this theory. Here the normal subgroup is $PC_G(P)$, P being a normal p-subgroup. This situation is summed up in Theorem 4, which is closely related to a theorem of Fong [6, (1 F)]; then Theorem 5 is obtained as a corollary. The final section is devoted to the third reduction.

2. Commutative algebras

Let Λ be a (finite-dimensional associative) commutative algebra with identity 1 over a field Ω. There are natural one-to-one correspondences between (a) the *block idempotents*, i.e. primitive idempotents, of Λ, (b) the maximal ideals of Λ, and (c) the irreducible representations of Λ in Ω, in which the idempotent e corresponds to the ideal

$$M_\Lambda(e) = \Lambda(1-e) + \operatorname{rad} \Lambda, \tag{2.1}$$

which in turn is the kernel of the corresponding representation [2, pp. 415-416], [12, pp. 210-211]. (Here rad Λ denotes the radical of Λ.) We shall find it convenient to emphasize the ideals rather than the representations, since the latter need not have degree 1. We have $1=\sum e$, summing over the set of all block idempotents e of Λ. Each idempotent d of Λ is the sum over some subset of this set; we call the elements of this subset the *summands* of d in Λ. Note that these summands are necessarily block idempotents of Λ.

Suppose that Δ is a subalgebra of Λ with $1 \in \Delta$, that d is a block idempotent of Δ, and that e is a summand of d in Λ. Then $\Delta(1-d) \subseteq \Lambda(1-d) \subseteq \Lambda(1-e)$, so that in the notation of (2.1)

$$M_{\Delta}(d) = \Delta(1-d) + \text{rad}\, \Delta \subseteq \{\Delta \cap \Lambda(1-e)\} + \{\Delta \cap \text{rad}\, \Lambda\} \subseteq \Delta \cap M_{\Lambda}(e).$$

Since $M_{\Delta}(d)$ is a maximal ideal of Δ,

$$M_{\Delta}(d) = \Delta \cap M_{\Lambda}(e). \tag{2.2}$$

Next we state some results involving field extensions, which we shall need later to reduce our considerations to the case where Ω is a splitting field.

LEMMA 1. *Let Λ be a commutative algebra with 1 over Ω. Let $\Omega^{\#}$ be a finite extension of Ω which is a splitting field for Λ, and let $\Lambda^{\#} = \Omega^{\#} \otimes_{\Omega} \Lambda$ be the $\Omega^{\#}$-algebra obtained from Λ by scalar extension. Let e be a block idempotent of Λ, and $e^{\#}$ a summand of e in $\Lambda^{\#}$. Let k be an Ω-automorphism of Λ such that $k(e) = e$, and let $k^{\#}$ be the extension of k to an $\Omega^{\#}$-automorphism of $\Lambda^{\#}$. Then*

$$k(x) \equiv x \pmod{M_{\Lambda}(e)}, \qquad x \in \Lambda, \tag{2.3}$$

if and only if $k^{\#}(e^{\#}) = e^{\#}$.

Proof. Set $h(x) = x + M_{\Lambda}(e) \in \Lambda/M_{\Lambda}(e)$. Let $h^{\#}$ be the irreducible representation of $\Lambda^{\#}$ in $\Omega^{\#}$ corresponding to $e^{\#}$. Then the kernel of $h^{\#}|\Lambda$ contains, and hence equals, $M_{\Lambda}(e)$; therefore there is an Ω-monomorphism i of $\Lambda/M_{\Lambda}(e)$ into $\Omega^{\#}$ such that $i \circ h = h^{\#}|\Lambda$. Then (2.3) is equivalent in turn to each of the statements $h \circ k = h$, $(h^{\#}|\Lambda) \circ k = h^{\#}|\Lambda$, and $h^{\#} \circ k^{\#} = h^{\#}$. $\Lambda^{\#}/\text{rad}\, \Lambda^{\#}$ is a direct sum of copies of $\Omega^{\#}$; and since the natural mapping of $\Lambda^{\#}$ onto $\Lambda^{\#}/\text{rad}\, \Lambda^{\#}$ maps block idempotents on block idempotents (cf. [1, Theorem 9.3 C]), one of these copies is $(\Lambda^{\#}/\text{rad}\, \Lambda^{\#})(e^{\#} + \text{rad}\, \Lambda^{\#})$. Then $h^{\#} \circ k^{\#} = h^{\#}$ if and only if $k^{\#}(e^{\#}) \equiv k^{\#} \pmod{\text{rad}\, \Lambda^{\#}}$; and since $k^{\#}$ permutes block idempotents, the latter holds if and only if $k^{\#}(e^{\#}) = k^{\#}$.

LEMMA 2. (Noether) [7, § 9], [13, § 124]. *Let Ω, Λ, $\Omega^{\#}$, $\Lambda^{\#}$, and e be as in Lemma 1. Assume in addition that $\Omega^{\#}$ is normal over Ω; then the Galois group $\mathcal{G}$ of $\Omega^{\#}$ over Ω has a natural action on $\Lambda^{\#}$, which permutes the block idempotents of $\Lambda^{\#}$. Then the set of all summands of e in $\Lambda^{\#}$ is an orbit under $\mathcal{G}$.*

Proof. The conclusion is equivalent to the following statement: if h is the irreducible representation of Λ in Ω corresponding to e, then the distinct irreducible constituents $h^{\#}|\Lambda$ of h in $\Omega^{\#}$ form an orbit under algebraic conjugation by $\mathcal{G}$. We can take h as in Lemma 1; as in that lemma, these constituents correspond naturally to the Ω-monomorphisms i of the field $\Lambda/M_{\Lambda}(e)$ into $\Omega^{\#}$.

The result is now clear.

COROLLARY. *Let Λ be a commutative algebra with* 1 *over Ω, let Ω^* be a finite purely inseparable extension field of Ω, and let $\Lambda^* = \Omega^* \otimes_\Omega \Lambda$. Then Λ and Λ^* have the same block idempotents.*

To prove this, apply Lemma 2 with $\Omega \subseteq \Omega^* \subseteq \Omega^\#$, using the fact that the Galois groups of $\Omega^\#$ over Ω and over Ω^* are equal.

3. Twisted group algebras

Let G be a finite group and let Ω be a field of prime characteristic p. Let $\Gamma(G)$ be a *twisted group algebra* of G over Ω: that is, an associative Ω-algebra with a basis $\{(g) : g \in G\}$ such that $(g)(h) = \varepsilon_{g,h}(gh)$, $g, h \in G$, where $0 \neq \varepsilon_{g,h} \in \Omega$ [1], [11], [14]. Here the mapping $\varepsilon : (g, h) \to \varepsilon_{g,h}$ must be a factor set (or 2-cocycle) of G in Ω. For any $x \in \Gamma(G)$ and $g \in G$, define

$$x^g = (g)^{-1} x (g). \tag{3.1}$$

The mapping $x \to x^g$ is an Ω-automorphism of the algebra $\Gamma(G)$. Furthermore $(x^g)^h = x^{gh}$ for $g, h \in G$, so that G is represented homomorphically by these automorphisms. The set $\{x \in \Gamma(G) : x^g = x$ for all $g \in G\}$ of fixed points of this action of G on $\Gamma(G)$ is clearly the center $\Lambda(G)$ of $\Gamma(G)$. For any subgroup X of G, let $\Gamma(X)$ be the subalgebra of $\Gamma(G)$ with basis $\{(g) : g \in X\}$, and let $\Lambda(X)$ be the center of $\Gamma(X)$.

An element $g \in G$ is called *ε-regular* (or, in the terminology of [4], a *u-element*) if $(g)^h = (g)$ for all $h \in G$ such that $g^h = g$, where we set $g^h = h^{-1}gh$. By [4, § 1] we can assume without loss of generality that ε has been chosen so that

$$(g)^h = (g^h), \qquad g, h \in G, \qquad g \ \varepsilon\text{-regular}. \tag{3.2}$$

Then the *class sums* $(K) = \sum_{g \in K} (g)$ for all the ε-regular conjugate classes K of G (that is, all classes consisting of ε-regular elements) form a basis of $\Lambda(G)$ [4, p. 155], [14, p. 174].

For any $g \in K$, any p-Sylow subgroup of $C_G(g)$ is called a *defect group* of K in G. Each block idempotent e of $\Lambda(G)$ can be written in the form

$$e = \sum_K \alpha[K](K), \qquad \alpha[K] \in \Omega, \tag{3.3}$$

summing over the ε-regular classes K. The largest of the defect groups of those K for which $\alpha[K] \neq 0$ are called the *defect groups* of e in G; these make up a full set of conjugate subgroups of G [4, § 3], [12, § 2].

Remark. In (3.3), $\alpha[K]=0$ unless K consists of p-regular elements. This can be proved as in [9, p. 178], working in a splitting field of $\Lambda(G)$ and using projective Brauer characters [8, p. 59]. This remark will be used in § 6.

THEOREM 2. *Let $\Omega^{\#}$ be a finite normal extension of Ω, let $\Lambda^{\#}(G) = \Omega^{\#} \otimes_{\Omega} \Lambda(G)$, and let $e^{\#}$ be a summand of e in $\Lambda^{\#}(G)$. Then the defect groups of $e^{\#}$ in G coincide with those of e.*

Proof. Observe that $\Lambda^{\#}(G)$ is the center of a twisted group algebra over $\Omega^{\#}$. We can assume without loss of generality that $\Omega^{\#}$ is a splitting field for $\Lambda(G)$: cf. the proof of the corollary to Lemma 2. By that corollary, we can then replace Ω by its purely inseparable closure in $\Omega^{\#}$, and so assume also that $\Omega^{\#}$ is separable over Ω. By Lemma 2, $e = \sum_{\sigma} e^{\#\sigma}$ where σ runs over a certain subset of $\mathscr{G}$.

Let D be a defect group of $e^{\#}$, and hence of each $e^{\#\sigma}$, in G, and let $N = N_G(D)$. There is an algebra-homomorphism s, the *Brauer homomorphism* [4, p. 156], [12, Lemma 3.3], of $\Lambda^{\#}(G)$ into $\Lambda^{\#}(N)$ such that

$$(3.4) \qquad s((K)) = \sum_{g \in K \cap C_G(D)} (g), \qquad K \ \varepsilon\text{-regular};$$

here we assume (3.2). Then $s(e) = \sum_{\sigma} s(e^{\#\sigma}) = \sum_{\sigma} (s(e^{\#}))^{\sigma}$. By the reduction from G to N [11, Theorem 1], [12, Theorem 5.3], the elements $s(e^{\#\sigma})$ are distinct block idempotents of $\Lambda^{\#}(N)$ with defect group D in N, and $\mathscr{G}$ permutes the subsums of the sum $\sum_{\sigma} e^{\#\sigma}$ and those of the sum $\sum_{\sigma} s(e^{\#\sigma})$ in corresponding ways. Any summand of $s(e)$ in $\Lambda(N)$ in the sense of § 2 is a subsum $\sum_{\tau} s(e^{\#\tau})$ of $\sum_{\sigma} s(e^{\#\sigma})$ which is invariant under $\mathscr{G}$; then $\sum_{\tau} e^{\#\tau}$ is invariant under $\mathscr{G}$, and so is in $\Lambda(G)$ (by separability), and therefore equals e. In other words, $s(e)$ is a block idempotent of $\Lambda(N)$. By [4, p. 166], [12, Proposition 4.4], $s(e^{\#\sigma})$ is a linear combination of class sums of N corresponding to classes with defect group D; then the same holds for $s(e)$, so that $s(e)$ has defect group D in N.

The expression (3.3) for e involves only class sums of classes of G with defect groups conjugate to subgroups of D in G, since the same holds for $e^{\#}$ by the definition of D. But by (3.4) and the fact that $s(e)$ has defect group

D in N, some one of these classes of G contains a class of N with defect group D in N, and hence has a defect group containing D in G. Therefore D is a defect group for e in G, and the theorem is proved.

4. Normal subgroups

Under the assumptions of § 3, let H be a given normal subgroup of G. Then $\Gamma(H)$ is stable under the action (3.1) of G on $\Gamma(G)$, and the algebra $\Delta(G, H)$ of fixed points of $\Gamma(H)$ under this action is given by

$$\Delta(G, H) = \Lambda(G) \cap \Gamma(H) = \Lambda(G) \cap \Lambda(H). \tag{4.1}$$

Clearly this action of G permutes the block idempotents f of $\Lambda(H)$, and the block idempotents of $\Delta(G, H)$ are the sums

$$d = \sum_g f^g \tag{4.2}$$

over the orbits of this permutation representation [6, p. 266], [10, p. 346]. In (4.2), f is fixed and g runs over a set of right coset representatives in G of the *stability group* (or *inertial group*)

$$I_G(f) = \{u \in G : f^u = f\} \tag{4.3}$$

of f in G. Each such d can also be expressed as a sum

$$d = \sum e \tag{4.4}$$

of some of the block idempotents e of $\Lambda(G)$, namely the summands of d in $\Lambda(G)$ in the sense of § 2. By (2.2),

$$M_{\Delta(G,H)}(d) = \Delta(G, H) \cap M_{\Lambda(G)}(e) = \Lambda(G, H) \cap M_{\Lambda(H)}(f). \tag{4.5}$$

For each $u \in I_G(f)$, the action $x \to x^u$ of (3.1) maps $M_{\Lambda(H)}(f)$ onto itself, and hence induces an Ω-automorphism

$$x + M_{\Lambda(H)}(f) \to x^u + M_{\Lambda(H)}(f)$$

of the field $\Lambda(H)/M_{\Lambda(H)}(f)$, regarded as containing Ω in the natural way. The set of those u for which this automorphism is the identity is a subgroup of $I_G(f)$ which we call the *small stability group* of f in G (relative to H). If Ω is a splitting field for $\Lambda(H)$, then $\Lambda(H)/M_{\Lambda(H)}(f) = \Omega$ and this group is simply $I_G(f)$. For general Ω, the following theorem is an immediate consequence of Lemma 1.

THEOREM 3. *Let $\Omega^{\#}$ be a finite normal extension of Ω which is a splitting field for $\Lambda(H)$, and let $\Lambda^{\#}(H) = \Omega^{\#} \otimes_{\Omega} \Lambda(H)$. If $f^{\#}$ is any summand of f in $\Lambda^{\#}(H)$, then the small stability group of f in G is equal to $I_G(f^{\#})$.*

Accordingly, we shall denote the small stability group of f in G (relative to H) by $I^{\#}_{G,H}(f)$.

5. Normal p-subgroups

Now assume that $\Gamma(G)$ is a twisted group algebra of a finite group G over a field Ω of characteristic p, and that P is a normal p-subgroup of G. We shall study the situation of § 4 for the normal subgroups $C = C_G(P)$ and $W = CP$ of G.

LEMMA 3. *The algebras $\Lambda(G)$, $\Delta(G, W)$, and $\Delta(G, C)$ have the same set of block idempotents.*

Proof. Since $\Delta(G, C) \subseteq \Delta(G, W) \subseteq \Lambda(G)$, we need only show that every block idempotent d of $\Delta(G, C)$ is a block idempotent of $\Lambda(G)$. For any such d we have $d = \sum e$, summing over certain block idempotents of $\Lambda(G)$, by (4.4). Let s be the Brauer homomorphism of $\Lambda(G)$ into $\Lambda(N_G(P)) = \Lambda(G)$; then $s((K)) = \sum_{g \in K \cap C} (g)$ (cf. (3.4)). Then s maps $\Lambda(G)$ into $\Delta(G, C)$ and fixes each element of $\Delta(G, C)$, so that

$$d = s(d) = \sum s(e). \tag{5.1}$$

By the "twisted" generalization, implicit in [4, § 3], of [12, Proposition 4.4], the defect group in G of each e contains the normal subgroup P. Hence the expression (3.3) for e contains some term with $\alpha[K] \neq 0$ and $K \subseteq C$; this implies that $s(e) \neq 0$. Thus (5.1) expresses the block idempotent d of $\Delta(G, C)$ as a sum of non-zero orthogonal idempotents of $\Delta(G, C)$. Hence the sum in (5.1) contains only one term, and the same must hold in (4.4). This proves the lemma.

Remark. Lemma 3 is of the same type as [10, Lemma 12 D], which asserts that if H is a normal subgroup of G and if G/H is a p-group, then $\Lambda(G)$ and $\Delta(G, H)$ have the same block idempotents.

LEMMA 4. *Let e be a block idempotent of $\Lambda(G)$, and hence also of $\Delta(G, W)$; let f be a summand of e in $\Lambda(W)$, and let $S = I_G(f)$. Then*

$$e = \sum_g f^g, \tag{5.2}$$

where g ranges over a set of right coset representatives of S in G. Furthermore f is a block idempotent of $\Lambda(S)$, and each defect group of f in S is a defect group of e in G.

Proof. By Lemma 3, e is a block idempotent of $\Delta(G; W)$. Then (4.2) yields (5.2), with f a block idempotent of $\Lambda(W)$. By the definition of S, f is a block idempotent of $\Delta(S, W)$, and hence also of $\Lambda(S)$ by Lemma 3 for S. This proves everything except the statement about defect groups.

We can assume that the condition (3.2) and its analogues hold simultaneously for ε and for its restrictions $\varepsilon|S$ and $\varepsilon|W$. To see this, apply Conlon's procedure [4, pp. 154–155] first to the ε-regular classes of G, then to the $(\varepsilon|S)$-regular classes of S whose elements are not ε-regular, and finally to the $(\varepsilon|W)$-regular classes of W whose elements are not $(\varepsilon|S)$-regular. Then we can write

$$e = \sum_K \alpha[K](K), \qquad f = \sum_L \beta[L](L), \tag{5.3}$$

summing over ε-regular classes K of G and $(\varepsilon|W)$-regular classes L of W respectively. (In fact we need sum only over classes contained in C, by Lemma 3.)

By (5.2) and (5.3), $e = \sum_g \sum_L \beta[L](L)^g$. Comparing this with (5.3), we see that the value of this sum is unchanged if we restrict the summation to those L which consist of ε-regular elements. Making this restriction, we have by (2.3) that $(L)^g = (L^g)$, where $L^g = \{w^g : w \in L\}$. Hence for $L \subseteq K (\subseteq W)$ and K ε-regular, we have

$$\alpha[K] = \sum_g \beta[L^{g^{-1}}], \tag{5.4}$$

with g as in (5.2).

For $u \in S$ and $y \in N_G(L)$, $\beta[L^{(ugy)^{-1}}] = \beta[L^{y^{-1}g^{-1}u^{-1}}] = \beta[L^{g^{-1}u^{-1}}] = \beta[L^{g^{-1}}]$ (cf. (5.3)); hence $\beta[L^{g^{-1}}]$ actually depends only on the double coset $SgN_G(L)$. For any $w \in L$, $N_G(L) = WC_G(w)$, so that $SgN_G(L) = SgWC_G(w) = SWgC_G(w) = SgC_G(w)$. Then if z ranges over representatives of the distinct double cosets $SzC_G(w)$, (5.4) can be rewritten as

$$\alpha[K] = \sum_z |C_G(w) : C_{S^z}(w)| \beta[L^{z^{-1}}]. \tag{5.5}$$

By the definition of defect group, we can choose K here so that $\alpha[K] \neq 0$.

while each p-Sylow subgroup of $C_G(w)$ is a defect group of e in G. Then for some z, $\beta[L^{z^{-1}}]\neq 0$ and $|C_G(w) : C_{S^z}(w)|\not\equiv 0 \pmod p$. Replacing w by w^z and L by L^z, we have $w\in L\subseteq K$ such that

$$(5.6) \qquad \beta[L]\neq 0, \qquad |C_G(w) : C_S(w)|\not\equiv 0 \pmod p.$$

Then any p-Sylow subgroup of $C_S(w)$ is a defect group of e in G, and, since $\beta[L]\neq 0$, it is contained in some defect group of f in S.

By [12, Proposition 3.2], which carries over to the twisted case, we can impose on K the further restriction that

$$(5.7) \qquad (K)\notin M_{\Delta(G)}(e).$$

By (4.5),

$$M_{\Delta(G,W)}(e) = \Delta(G, W)\cap M_{\Delta(G)}(e) = \Delta(G, W)\cap M_{\Delta(W)}(f),$$
$$M_{\Delta(S,W)}(f) = \Delta(S, W)\cap M_{\Delta(S)}(f) = \Delta(S, W)\cap M_{\Delta(W)}(f).$$

By (5.7), $(K)\notin M_{\Delta(G,W)}(e)$; since $(K)\in\Delta(G, W)$, $(K)\notin M_{\Delta(W)}(f)$, whence $(K)\notin M_{\Delta(S,W)}(f)$. But $(K)\in\Delta(S, W)$, whence $(K)\notin M_{\Delta(S)}(f)$.

Write $(K)=\sum(K')$, where each K' is a conjugacy class of S. Then we can choose K' so that $(K')\notin M_{\Delta(S)}(f)$. K' contains some conjugate w' of w in G. Since $(K')\notin M_{\Delta(S)}(f)$, the argument of [12, Proposition 3.2] shows that any p-Sylow subgroup D of $C_S(w')$ contains a defect group of f in S. On the other hand, D is contained in a conjugate in G of a p-Sylow subgroup of $C_G(w)$. Combining these facts with those following (5.6), we see that D is both a defect group of f in S and a defect group of e in G. This completes the proof of the lemma.

THEOREM 4. *Let e, f, and S be as in Lemma 4. Then f is a block idempotent of $A(S^{\#})$, where $S^{\#}=I^{\#}_{G,W}(f)$. If D is a defect group of f in $S^{\#}$, then:*

(5.8) *D is a defect group of e in G,*

(5.9) *$W\cap D$ is a defect group of f in W, and*

(5.10) *WD/W is a p-Sylow subgroup of $S^{\#}/W$.*

Proof. In the case that Ω is a splitting field for $A(W)$, $S^{\#}=S$ and (5.8) follows from Lemma 4. In the general case, let $\Omega^{\#}$ be a finite normal extension of Ω which is a splitting field for $A(W)$. Set $A^{\#}(X)=\Omega^{\#}\otimes_{\Omega}A(X)$ for each subgroup X of G. Let $f^{\#}$ be any summand of f in $A^{\#}(W)$. Then $f^{\#}$ is a summand of e in $\Lambda^{\#}(W)$, and so there is a summand $e^{\#}$ of e in $\Lambda^{\#}(G, W) = \Lambda^{\#}(G) \cap \Lambda^{\#}(W)$ such that $f^{\#}$ is a summand of $e^{\#}$ in $\Lambda^{\#}(W)$. By Lemma 3, $e^{\#}$ is a block idempotent of $\Lambda^{\#}(G)$.

By Theorem 3, $S^{\#} = I_G(f^{\#})$. Then Lemma 4 shows that $f^{\#}$ is a block idempotent of $\Lambda^{\#}(S^{\#})$, and that each defect group of $f^{\#}$ in $S^{\#}$ is a defect group of $e^{\#}$ in G. The same lemma, with $S^{\#}$ in the role of G, shows that f is a block idempotent of $\Lambda(S^{\#})$. By Theorem 2, e and $e^{\#}$ have the same defect groups in G, and f and $f^{\#}$ have the same defect groups in $S^{\#}$. Combining these results, we obtain (5.8).

In view of (5.8), we see that in proving (5.9) and (5.10) we lose no generality by supposing that $G = S^{\#}$; then also $e = f$. As in the proof of Lemma 4, we have K such that $\alpha[K] \neq 0$ and $(K) \notin M_{\Lambda(W)}(f)$, while for each $w \in K$, each p-Sylow subgroup of $C_G(w)$ is a defect group of f in G. We can choose w so that the given defect group D is a p-Sylow subgroup of $C_G(w)$. The straightforward equation

$$|C_G(w) : D| = |C_W(w) : W \cap D| \, |WC_G(w) : WD|$$

shows that $W \cap D$ is a p-Sylow subgroup of $C_W(w)$, and that $|WC_G(w) : WD| \not\equiv 0 \pmod{p}$. Let L be the conjugate class of W containing w. Then $\beta[L] = \alpha[K] \neq 0$ since $e = f$; therefore $W \cap D$ is contained in some defect group $\tilde{D}$ of f in W.

We have $(K) = \sum_v (L)^v$, summing over right coset representatives v of $N_G(L) = WC_G(w)$ in G. Since $G = S^{\#}$, $(L) \equiv (L)^v$ and then $0 \not\equiv (K) \equiv |G : WC_G(w)|(L) \pmod{M_{\Lambda(W)}(f)}$. Then $|G : WC_G(w)| \not\equiv 0 \pmod{p}$, so that $|G : WD| \not\equiv 0 \pmod{p}$, which yields (5.9); and $(L) \notin M_{\Lambda(W)}(f)$, which implies that $W \cap D \supseteq \tilde{D}$ [12, Proposition 3.2] and yields (5.10). This completes the proof of Theorem 4.

At this point we drop the assumption that P is normal in G, and apply Theorem 4 for $N_G(P)$, in conjunction with the reduction from G to $N_G(P)$ [11, Theorem 1] (cf. [2, (10 B)], [12, Theorem 5.3]), to obtain our main theorem.

THEOREM 5. *Let $\Gamma(G)$ be a twisted group algebra of a finite group G in a field Ω of characteristic p. Let P be an arbitrary p-subgroup of G; let $N = N_G(P)$ and $W = PC_G(P)$. Then there is a natural one-to-one correspondence between the set of all block idempotents of $\Lambda(G)$ with P as a defect group in G and the set of* *orbits, under the action* (3.1) *of* N, *of those block idempotents* f *of* $\Lambda(W)$ *such that:*

(5.11) f *has* P *as its unique defect group in* W,

and

(5.12) $|I^{\#}_{N,W}(f) : W| \not\equiv 0 \pmod{p}$.

Proof. By the reduction to N, the Brauer homomorphism of $\Lambda(G)$ into $\Lambda(N)$ gives a one-to-one correspondence between the block idempotents of $\Lambda(G)$ in question and the set of the block idempotents of $\Lambda(N)$ with defect group P in N. By Lemma 3 and (4.2), *all* the block idempotents e of $\Lambda(N)$ are in one-to-one correspondence with all the orbits of block idempotents f of $\Lambda(W)$ under the action of N, where the orbit of f corresponds to e if and only if f is a summand of e in $\Lambda(W)$.

We must show that if f is a summand of e, then e has defect group P in N if and only if (5.11) and (5.12) hold. Let D be a defect group of f in $I^{\#}_{N,W}(f)$; then by (5.8), e has defect group P in N if and only if $D=P$. If (5.11) and (5.12) hold, then (5.12) implies that $D \subseteq W$, and (5.9) shows that D is a defect group of f in W, so that $D=P$ by (5.11). Conversely if $D=P$, then (5.9) gives (5.11), and (5.10) gives (5.12).

6. Reduction to W/P

Suppose again that P is a normal p-subgroup of G, and that $\Gamma(G)$ is a twisted group algebra over Ω. After replacing Ω by a suitable purely inseparable finite extension field, we can change [4, p. 154] the factor set ε of $\Gamma(G)$ to achieve that the elements (z), $z \in P$, form a normal subgroup of the multiplicative group consisting of all elements $\omega(g)$, $g \in G$, $0 \neq \omega \in \Omega$. (We do not require that (3.2) still hold.) Then the Ω-subspace of $\Gamma(G)$ spanned by all elements $(g)((z)-(1))$, $g \in G$, $z \in P$, is a nilpotent ideal of $\Gamma(G)$. If t is a homomorphism of $\Gamma(G)$ with this ideal as kernel, then the image $t(\Gamma(G))$ is a twisted group algebra $\Gamma(G/P)$ [4, p. 166], [12, Lemma 4.2]; similarly $t(\Gamma(W)) = \Gamma(W/P)$ where $W = PC_G(P)$. Changing factor sets again, we can suppose that the factor sets ε and $\bar{\varepsilon}$ of $\Gamma(G)$ and $\Gamma(G/P)$ have been so chosen that the analogues of (3.2) for $\bar{\varepsilon}$ and $\bar{\varepsilon}|(W/P)$ (but not necessarily for ε) hold, while for all $g \in G$, $t((g)) = (gP)$. Then under the actions (3.1) of G and G/P, (6.1) $$t(x^g) = (t(x))^{gP},\ x \in \Gamma(G),\ g \in G.$$

LEMMA 5. (Cf. [3, (2 G)]). *The restriction of t to $\Lambda(W)$ induces a one-to-one correspondence $f \leftrightarrow t(f)$ between the block idempotents of $\Lambda(W)$ and those of $\Lambda(W/P)$. In this correspondence, orbits under the action* (3.1) *of N correspond to orbits of N/P, $I_{G/P}(t(f)) = I_G(f)/P$, and $I^{\#}_{G/P,\,W/P}(t(f)) = I^{\#}_{G,W}(f)/P$. A subgroup D of G is defect group of f in W if and only if D/P is a defect group of t(f) in W/P.*

Proof. By [1, Theorem 9.3 C], $t|\Lambda(W)$ induces a one-to-one correspondence between the block idempotents of $\Lambda(W)$ and those of $t(\Lambda(W))$. So we will obtain the desired correspondence if we show that every block idempotent $\bar{f}$ of $\Lambda(W/P)$ lies in $t(\Lambda(W))$. By the remark preceding Theorem 2, $\bar{f}$ is a linear combination of p-regular $(\bar{\varepsilon}\,|\,W/P)$-regular class sums $(\bar{L})$ of $\Lambda(W/P)$. By [2, (11 A)], it is easy to show that each such class sum can be expressed uniquely in the form $(\bar{L}) = t((L))$ for some p-regular $(\varepsilon\,|\,W)$-regular class sum (L) of $\Lambda(W)$, and that for each $w \in L$, $C_{W/P}(wP) = C_W(w)/P$. Then $\bar{f} \in t(\Lambda(W))$, so that $\bar{f}$ has the form $t(f)$.

Now (6.1) implies the statements on orbits and stability groups. Applying our results for a suitable splitting field $\Omega^{\#}$ of $\Lambda(W)$, together with Theorem 3, we obtain the statement on small stability groups. The statement on defect groups follows from a comparison of the expressions (5.3) for f and $t(f)$.

Again dropping the assumption that P is normal in G, we obtain our final result by using Lemma 5 to reformulate Theorem 5.

THEOREM 6. *Under the assumptions of Theorem* 5, *let $\Lambda(N/P)$ be defined as above, after making a suitable purely inseparable finite field extension. Then there is a natural one-to-one correspondence between the set of all block idempotents of $\Lambda(G)$ with P as a defect group in G and the set of orbits, under the action* (3.1) *of N/P, of those block idempotents $\bar{f}$ of $\Lambda(W/P)$ such that:*

(6.2) *$\bar{f}$ has defect group* $\{1\}$ *in W/P, and*

(6.3) $|I^{\#}_{N/P,\,W/P}(\bar{f}) : W/P| \not\equiv 0 \pmod{p}$.

In general it is impossible to avoid the purely inseparable extension in Theorem 6, as the following simple example shows. Let $G = N = P = \{1, a\}$ have order 2; let $\Omega = \mathbf{Z}_2(\lambda)$ be the field of rational functions over the field with 2 elements; let $\varepsilon_{1,1} = \varepsilon_{1,a} = \varepsilon_{a,1} = 1$, $\varepsilon_{a,a} = \lambda$. Then ε is a factor set, but $\Gamma(G) \doteq \Gamma(N)$ is a field, and hence has no homomorphic image which is a twisted group algebra $\Gamma(N/P)$. This example also illustrates that in the course of Conlon's proof of the first reduction theorem [4, p. 166], it is necessary to make a purely inseparable extension. This extension causes no further complications in the proof, since it does not change the block idempotents by the corollary to our Lemma 2. Of course none of these difficulties arise in the case that ε is trivial.

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