FUNDAMENTAL EXACT SEQUENCES IN (CO)-HOMOLOGY FOR NON-NORMAL SUBGROUPS

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In this paper we prove the fundamental exact sequence in (co)homology for non-normal subgroups announced in our previous note [8]: Let H be a subgroup of a group G. If M is a G-module and if, for a natural number n, $H_m(U, M)=0$ for $m=1, \ldots, n-1$ and for every subgroup U of H which is an intersection of conjugates of H in G,[1] then we have an exact sequence

$$\begin{aligned}0 \leftarrow H_n(G, H, M) \xleftarrow{\varphi} H_n(G, M) \xleftarrow{\iota} H_n(H, M)_I \\ \xleftarrow{\tau} H_{n+1}(G, H, M) \xleftarrow{\varphi} H_{n+1}(G, M);\end{aligned}$$

the significance of the maps and the group $H_n(H, M)_I$ will be explained below. (We have a dual result for cohomology groups). This generalizes, on one hand, the Hochschild-Serre [6] (cf. also [2], [3]) fundamental exact sequence for the case of a normal subgroup H and extends, on the other hand, the partial sequence given by Adamson [1].

Our result for group (co)homology has in the meantime been extended to a more general case of ring (co)homology by Hattori [4]. His proof is both quite general and quite ingenious. In comparison with Hattori's [4] proof the writer's one is quite clumsy, not only that merely the group case is considered. Inspite of this (and notwithstanding that it might also be possible to prove our exact sequences by spectral sequence method[2] as in [5], [2]) the writer dares to assume that his original artless approach still retains some use, if small, in the sense that for example it serves to make explicit and elementary what are defined in [4] in highly theoretical constructions of advanced nature, particularly with respect to the residue group $H_n(H, M)_I$ and the transgression map in homology case.

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1) The assumption can be weakened a little, with respect to the number of conjugates of H to express U. See our Theorem in No. 7 for the exact form of assumption.

2) For this, however, a rather new sort of spectral sequences would perhaps be needed.

In describing the relative homology groups $H_n(G, H, M)$ (see [1], [5]) we use the standard complex $X(G, H)$ of G relative to H; its n-component $X_n(G, H)$ is the free module generated by the totality of $(n+1)$-tuples $(\sigma_0 H, \ldots, \sigma_n H)$ of right H-cosets in G and has the G-left-module structure defined by $\sigma(\sigma_0 H, \ldots, \sigma_n H) = (\sigma\sigma_0 H, \ldots, \sigma\sigma_n H)$. The differentiation in $X(G, H)$ is defined as usual, and by $(\sigma_0, \ldots, \sigma_n) \to (\sigma_0 H, \ldots, \sigma_n H)$ we obtain a homomorphism (of G-complexes) to $X(G, H)$ of the standard complex $X(G)$ of G. Turning to the complexes $M \otimes_G X(G, H)$, $M \otimes_G X(G)$ of chains over M and passing to homology, this homomorphism induces the deflation (or residuation) map φ of homology groups, which appears in the above sequence as the second and the fifth arrows. Denoting the said homomorphism of chain complexes also by φ, and denoting by ι the homomorphism $M \otimes_H X(H) \to M \otimes_G X(G)$ induced by the monomorphism $(\sigma_0, \ldots, \sigma_n) \to (\sigma_0, \ldots, \sigma_n)$ $(\sigma_i \in H)$ of $X(H)$ into $X(G)$, consider the correspondence between an n-cycle f in $M \otimes_H X(H)$ and an $n+1$-cycle h in $M \otimes_G X(G, H)$ such that there exists an $n+1$-element (chain) g in $M \otimes_G X(G)$ satisfying

$$\iota f = (-1)^n \partial g, \qquad h = \varphi g.$$

The totality of f corresponding to $h = 0$ is evidently a subgroup of the group of n-cycles in $M \otimes_H X(H)$. The residue group of the group of n-cycles in $M \otimes_H X(H)$ by the subgroup generated by this subgroup and the group of n-boundaries is denoted by $H_n(H, M)_I$. The ordinary injection map ι: $H_n(H, M) \to H_n(G, M)$, defined by the above map ι of chain complexes, induces evidently a homomorphism of $H_n(H, M)_I$ into $H_n(G, M)$ which is denoted also by the same letter ι in our sequence. Under our assumption, $H_m(U, M) = 0$ for $m = 1, \ldots, n-1$, for every $n+1$-cycle h in $M \otimes_G X(G, H)$ there is an n-cycle f in $M \otimes_H X(H)$ corresponding to h by our above correspondence, and by associating the homology class of h with the residue class in $H_n(H, M)_I$ of the homology class of f we obtain a homomorphism $H_{n+1}(G, H, M) \to H_n(H, M)$, transgression, denoted above by τ. Further, under the same assumption, our sequence is exact and, moreover, the kernel of the canonical homomorphism $H_n(H, M) \to H_n(H, M)_I$ is the subgroup of $H_n(H, M)$ generated by the homology classes of form $\iota_{H \cap \xi H \xi^{-1}, H}\kappa - \iota_{H \cap \xi^{-1} H \xi, H} T_\xi \kappa$ $(\kappa \in H_n(H \cap \xi H \xi^{-1}, M)$, where ξ runs over G, T_ξ is the transformation isomorphism of $H_n(H \cap \xi H \xi^{-1}, M)$ to

$H_n(\xi^{-1}(H \cap \xi H \xi^{-1})\xi, M) = H_n(H \cap \xi^{-1} H \xi, M)$, and the injection maps are specified with suffices indicating the groups of references. This all is the content of our fundamental exact sequence, and its proof consists of two parts. While the second part is the induction part and is straight forward, the first part is the verification of the case $n=1$ and is made by rather complicated computations. Thus, in proving the case $n=1$, we first give our principal lemma which shows, in explicit computation, that for every 2-cycle h in $M \otimes_G X(G, H)$ there are always a 1-cycle f in $M \otimes_H X(H)$ and a 2-chain g in $M \otimes_G X(G)$ satisfying $h = \varphi g$, $\iota f = -\partial g$. While merely the existence of f, g satisfying the relation is used in establishing the case $n=1$ of our fundamental exact sequence in its weak form, i.e. in the form where the residue group $H_n(H, M)_I$ is understood in its first definition, our explicit description of f, g is used in order further to show that the residue group thus defined can also be characterized by its second definition as given above (by the property dual to stability).

We are thus mainly concerned with homology group case and cohomology is ignored in the present exposition, since in our approach the homology case is more troublesome to handle and demands some new technics while the cohomology case is easier and can be handled by habitual means.[3]

In an appendix we consider the case of finite groups and discuss the relationship between our exact sequences and another similar series of exact sequences by means of the result in our previous note [9].

The writer is grateful to A. Hattori for his friendly cooperation during the preparation of the present paper. His communications have been useful to improve essentially the writer's original form of result in several points.

1. Normal form of a relative chain

Let G be a group, and H a subgroup. The n-component $X_n(G, H)$ of the (relative) standard complex $X(G, H)$ of G relative to H is, as explained in the introduction, a free module generated by the totality of $(n+1)$-tuples $(\sigma_0 H, \sigma_1 H, \ldots, \sigma_n H)$ of right H-cosets and has the G-left-module structure defined by $\sigma(\sigma_0 H, \ldots, \sigma_n H) = (\sigma\sigma_0 H, \ldots, \sigma\sigma_n H)$. The set of all $(n+1)$-tuples of form $(H, \sigma_1 H, \ldots, \sigma_n H)$ forms thus its generating system with respect to G. We

[3] The writer is communicated that in Hattori's theory, which is exposed in case of cohomology in [4], the treatments of homology and cohomology are rather, if not quite, parallel.

can further choose a subset

(1) $$s=(H,\ \sigma_1 H,\ \ldots,\ \sigma_n H),\quad s'=(H,\ \sigma_1' H,\ \ldots,\ \sigma_n' H),\ \ldots$$

of the set such that for every $(n+1)$-tuple t of form $(H,\ \tau_1 H,\ \ldots,\ \tau_n H)$ there is in the set one, and only one, element from which t is obtained by the operation of an element of H, and therefore, for every $(n+1)$-tuple $t=(\tau_0 H,\ \tau_1 H,\ \ldots,\ \tau_n H)$ there is in the set one, and only one, element, say $s^{(\mu)}$, from which t is obtained by the operation of an element of G, thus $t=\tau s^{(\mu)}\ (\tau\in G)$; here $\tau\equiv\tau_0$ r.mod H, $\equiv\tau_1\sigma_1^{(\mu)^{-1}}$ r.mod $\sigma_1^{(\mu)}H\sigma_1^{(\mu)^{-1}}$, $\ldots$, and τ is determined uniqualy up to r.mod $H\cap\sigma_1^{(\mu)}H\sigma_1^{(\mu)^{-1}}\cap\cdots\cap\sigma_n^{(\mu)}H\sigma_n^{(\mu)^{-1}}$. Such a set (1) forms thus a G-basis (or, more precisely, $Z[G]$-basis, $Z[G]$ denoting the group ring of G over the integers Z) of $X_n(G,\ H)$.

For $n=1$, in particular, such a G-basis (1) of $X_1(G,\ H)$ is a set $(H,\ \sigma H)$, $(H,\ \sigma' H)$, $\ldots$ where $\sigma,\ \sigma',\ \ldots$ form a complete representative system of double coset decomposition of G mod H. For $n\geqq 1$, generally, the system $\sigma_1,\ \sigma_1',\ \ldots$ may be chosen as such double coset representatives each taken with certain iteration.

Let M be a G-right-module. The group of (relative) n-chains of G over M relative to H is the tensor product

$$M\otimes_G X_n(G,\ H)$$

over G (or, more precisely, over the group ring $Z[G]$). Each element of $M\otimes_G X_n(G,\ H)$ is expressed in a form

(2) $$u\otimes_G s+u'\otimes_G s'+\cdots\qquad (u,\ u',\ \ldots\in E).$$

Set $K=H\cap\sigma_1 H\sigma_1^{-1}\cap\cdots\cap\sigma_n H\sigma_n^{-1}$, $K'=H\cap\sigma_1' H\sigma_1'^{-1}\cap\cdots\cap\sigma_n' H\sigma_n'^{-1}$, $\ldots$. Let M_K be, as usual, the residue-module of M modulo its submodule generated by elements of form $u-u\kappa\ (u\in M,\ \kappa\in K)$, and similarly with $M_{K'}$, etc. Then the classes in the residue-modules $M_K,\ M_{K'},\ \ldots$ of the coefficients $u,\ u',\ \ldots$ in (2) are uniquely determined. With some fixed choice of the system (1) we shall consider (2) as a normal form of n-chain in $M\otimes_G X(G,\ H)$.

2. Principal lemma

First we fix some notations to be used below. Let $\mathfrak{S}$ be a complete representative system of right H-cosets in G. Deviating somewhat from the notation in the preceding number, let $\mathfrak{G}_1=\{\alpha_1\}$ be a complete representative system of double H-cosets in G, and let for each $\alpha_1\in\mathfrak{G}_1$ a complete representative system of right $H\cap\alpha_1H\alpha_1^{-1}$-cosets in H be denoted by $\mathfrak{H}(\alpha_1)$. Thus

$$\{(H,\ \alpha_1H)\mid\alpha_1\in\mathfrak{G}_1\}\tag{3}$$

is a G-basis of $X_1(G,\ H)$, and

$$\{h\alpha_1\mid\alpha_1\in\mathfrak{G}_1,\ h\in\mathfrak{H}(\alpha_1)\}\tag{4}$$

is a complete representative system of right H-cosets in G; later we shall have occasions to specify $\mathfrak{G}$ to be this representative system (4), but for the moment such a specification is not needed.

Further, let $\mathfrak{G}_2=\{(\alpha_2,\ \beta_2)\}$ be a system of pairs of elements of G such that the system

$$\{(H,\ \alpha_2H,\ \beta_2H)\mid(\alpha_2,\ \beta_2)\in\mathfrak{G}_2\}\tag{5}$$

is, as the system (1) for the case $n=2$, a G-basis of $M\otimes_G X_2(G,\ H)$. (One way of constructing such a system $\mathfrak{G}_2$ is to start with $\mathfrak{G}_1$, take for each $\alpha_1\in\mathfrak{G}_1$ a complete representative system $\{\beta(\alpha_1)\}$ of double $(H\cap\alpha_1H\alpha_1^{-1})$-$H$-cosets in G, and to set $\mathfrak{G}_2=\{(\alpha_1,\ \beta(\alpha_1))\mid\alpha_1\in G_1,\ \beta(\alpha_1)\in\{\beta(\alpha_1)\}\}$. But we do not need to take our $\mathfrak{G}_2$ in this special form.) Let, further, $\mathfrak{H}(\alpha_2,\ \beta_2)$ be a complete representative system of right $H\cap\alpha_2H\alpha_2^{-1}\cap\beta_2H\beta_2^{-1}$-cosets in H.

Now, consider 2-chain

$$\sum_{\xi,\ \eta\in G}v(\xi,\ \eta)\otimes_G(1,\ \xi,\ \eta)\in M\otimes_G X_2(G)\quad(v(\xi,\ \eta)\in M)\tag{6}$$

of G over M. Its boundary is given by

$$\begin{aligned}(7)\qquad &\sum_{\xi,\ \eta\in G}v(\xi,\ \eta)\otimes_G((\xi,\ \eta)-(1,\ \eta)+(1,\ \xi))\\ &=\sum_{\xi,\ \eta\in G}(v(\eta,\ \eta\xi)\eta-v(\eta,\ \xi)+v(\xi,\ \eta))\otimes_G(1,\ \xi)\\ &=\sum_{a\in H;\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);\eta\in G}(v(\eta,\ \eta h\alpha_1 a)\eta-v(\eta,\ h\alpha_1 a)\\ &\quad+v(h\alpha_1 a,\ \eta))h\otimes_G(h^{-1},\ \alpha_1 a).\end{aligned}$$

The image of this by the canonical deflation (or residuation) map φ: $M\otimes_G X(G)\to M\otimes_G X(G,\ H)$ is

$$\sum_{\alpha_1\in\mathfrak{G}_1}\ \sum_{a\in H;h\in\mathfrak{H}(\alpha_1);\eta\in G}(v(\eta,\ \eta h\alpha_1 a)\eta-v(\eta,\ h\alpha_1 a)+v(h\alpha_1 a,\ \eta))h\otimes_G(H,\ \alpha_1H).\tag{8}$$

If here this relative chain (8) is 0, then there exist, by our observation in the preceding number, functions $q(\alpha_1, c)$ in M of $c \in H \cap \alpha_1 H \alpha_1^{-1}$, depending on $\alpha_1 \in \mathfrak{G}_1$, such that we have

$$\sum_{a \in H; h \in \mathfrak{H}(\alpha_1); \eta \in G} (v(\eta, \eta h \alpha_1 a) - v(\eta, h \alpha_1 a) + v(h \alpha_1 a, \eta))h = \sum_{c \in H \cap \alpha_1 H \alpha_1^{-1}} (q(\alpha_1, c)c - q(\alpha_1, c)) \tag{9}$$

for each $\alpha_1 \in \mathfrak{G}_1$.

An element h of $M \otimes_G X_2(G, H)$ can always be expressed as the image by φ of some element, say (6), of $M \otimes_G X_2(G)$. If in particular $\partial h = 0$, then, since φ and ∂ commute, the above consideration applies to the case. Now we prove our principal

LEMMA 1. *For every (relative) 2-cycle h in $M \otimes_G X(G, H)$ there are a 2-chain g in $M \otimes_G X(G)$ and a 1-cycle f in $M \otimes_H X(H)$ satisfying*

$$h = \varphi g, \qquad \iota f = -\partial g \tag{10}$$

(where ι is the canonical injection map of $M \otimes_H X(H)$ into $M \otimes_G X(G)$). More precisely, if h is the image of (6) *by φ, then we can take $q(\alpha_1, c) \in M (\alpha_1 \in \mathfrak{G}_1, c \in H \cap \alpha_1 H \alpha_1^{-1})$ so as* (9) *holds for every $\alpha_1 \in \mathfrak{G}_1$ and with*

$$\begin{aligned} g = & \sum_{\xi, \eta \in G} v(\xi, \eta) \otimes_G (1, \xi, \eta) \\ & + \sum_{\alpha_1 \in \mathfrak{G}_1; h \in \mathfrak{H}(\alpha_1); a \in H; \eta \in G} (v(\eta, \eta h \alpha_1 a)\eta - v(\eta, h \alpha_1 a) \\ & \quad + v(h \alpha_1 a, \eta))h \otimes_G ((h^{-1}, \alpha_1, \alpha_1 a) + (h^{-1}, 1, \alpha_1)) \\ & + \sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} q(\alpha_1, c) \otimes_G ((1, \alpha_1, c\alpha_1) - (1, \alpha_1, \alpha_1) \\ & \quad + (1, 1, \alpha_1) - (1, c, c\alpha_1)) \end{aligned} \tag{11}$$

and

$$\begin{aligned} f = & -\sum_{a \in H} \sum_{\alpha_1 \in \mathfrak{G}_1; h \in \mathfrak{H}(\alpha_1); \eta \in G} (v(\eta, \eta h \alpha_1 a)\eta - v(\eta, h \alpha_1 a) \\ & \quad + v(h \alpha_1 a, \eta))h \alpha_1 \otimes_H (1, a) \\ & - \sum_{\alpha_1 \in \mathfrak{G}_1; h \in \mathfrak{H}(\alpha_1)} \sum_{a \in H; \eta \in G} (v(\eta, \eta h \alpha_1 a) - v(\eta, h \alpha_1 a) + v(h \alpha_1 a, \eta)) \otimes_H (1, h) \\ & + \sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} (q(\alpha_1, c) \otimes_H (1, c) - q(\alpha_1, c)\alpha_1 \otimes_H (1, \alpha_1^{-1} c \alpha_1)) \\ & - \sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} (q(\alpha_1, c) - q(\alpha_1, c)\alpha_1) \otimes_H (1, 1) \end{aligned} \tag{12}$$

we have $\partial f = 0$ and (10).

Proof. Though somewhat abruptly, we define g as in (11). Its image φg

by the map φ is h, since the first sum in (11) is simply (6) (which is mapped to h by φ) while the image of the second sum is, by (9),

$$\sum_{\alpha_1\in\mathfrak{G}_1}\sum_{c\in H\cap\alpha_1H\alpha_1^{-1}}(q(\alpha_1,\ c)c-q(\alpha_1,\ c))\otimes_G((H,\ \alpha_1H,\ \alpha_1H)-(H,\ H,\ \alpha_1H))=0$$

and the image of the last two sums is

$$\begin{aligned}\sum_{\alpha_1\in\mathfrak{G}_1}\sum_{c\in H\cap\alpha_1H\alpha_1^{-1}}&q(\alpha_1,\ c)\otimes_G((H,\ \alpha_1H,\ \alpha_1H)-(H,\ \alpha_1H,\ \alpha_1H)\\&+(H,\ H,\ \alpha_1H)-(H,\ H,\ \alpha_1H))=0.\end{aligned}$$

On the other hand, we have (cf. (7))

$$\begin{aligned}\partial g=&\sum_{a\in H;\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1)\eta;\in G}(v(\eta,\ \eta h\alpha_1a)\eta-v(\eta,\ h\alpha_1a)+v(h\alpha_1a,\ \eta))h\otimes_G(h^{-1},\ \alpha_1a)\\
&+\sum_{a\in H;\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1)\eta;\in G}(v(\eta,\ \eta h\alpha_1a)-v(\eta,\ h\alpha_1a)+v(h\alpha_1a,\ \eta))h\otimes_G((\alpha_1,\ \alpha_1a)\\
&-(h^{-1},\ \alpha_1a)+(h^{-1},\ \alpha_1)+(1,\ \alpha_1)-(h^{-1},\ \alpha_1)+(h^{-1},\ 1))\\
&+\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}q(\alpha_1,\ c)\otimes_G((\alpha_1,\ c\alpha_1)-(1,\ c\alpha_1)+(1,\ \alpha_1)-(\alpha_1,\ \alpha_1)\\
&+(1,\ \alpha_1)-(1,\ \alpha_1)+(1,\ \alpha_1)-(1,\ \alpha_1)+(1,\ 1)-(c,\ c\alpha_1)+(1,\ c\alpha_1)-(1,\ c))\\
=&\sum_{a\in H;\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);\eta\in G}(v(\eta,\ \eta h\alpha_1a)\eta-v(\eta,\ h\alpha_1a)+v(h\alpha_1a,\ \eta))h\otimes_G((\alpha_1,\ \alpha_1a)\\
&+(1,\ \alpha_1)+(h^{-1},\ 1))\\
&+\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}q(\alpha_1,\ c)\otimes_G((\alpha_1,\ c\alpha_1)+(1,\ \alpha_1)-(\alpha_1,\ \alpha_1)+(1,\ 1)\\
&-(c,\ c\alpha_1)-(1,\ c))\\
=&\sum_{a\in H;\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);\eta\in G}(v(\eta,\ \eta h\alpha_1a)\eta-v(\eta,\ h\alpha_1a)\\
&+v(h\alpha_1a,\ \eta))h\otimes_G((\alpha_1,\ \alpha_1a)+(h^{-1},\ 1))\\
&+\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}(q(\alpha_1,\ c)c-q(\alpha_1,\ c))\otimes_G(1,\ \alpha_1)\\
&+\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}q(\alpha_1,\ c)\otimes_G((\alpha_1,\ c\alpha_1)+(1,\ \alpha_1)-(\alpha_1,\ \alpha_1)+(1,\ 1)\\
&-(c,\ c\alpha_1)-(1,\ c))\end{aligned}$$

by (9), and this is equal to $-f$, with f as in (12), since the sum of the latter two sums is

$$\begin{aligned}\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}&q(\alpha_1,\ c)\otimes_G((c,\ c\alpha_1)-(1,\ \alpha_1)+(\alpha_1,\ c\alpha_1)+(1,\ \alpha_1)\\
&-(\alpha_1,\ \alpha_1)+(1,\ 1)-(c,\ c\alpha_1)-(1,\ c))\\
&=\sum_{\alpha_1\in G_1;c\in H\cap\alpha_1H\alpha_1^{-1}}q(\alpha_1,\ c)\otimes_G((\alpha_1,\ c\alpha_1)-(1,\ c)+(1,\ 1)-(\alpha_1,\ \alpha_1)).\end{aligned}$$

Thus the relations in (10) have both been verified. From the latter of them we have $\iota\partial f = \partial\iota f = 0$ and this implies $\partial f = 0$; it is also easy to verify this by computing ∂f explicitly from (12), using (9).

3. Fundamental exact sequence in weak form for the case $n=1$

In this number we use merely the first half of our fundamental lemma; its latter half, which gives not only the existence of f, g (satisfying (10)) but the explicit forms of f, g (at least in their suitable choice), will be used in the next number.

For any given 2-cycle h in $M\otimes_G X(G, H)$ there exists, by our fundamental lemma, a 1-cycle f in $M\otimes_H X(H)$ such that there is a 2-chain g in $M\otimes_G X(G)$ satisfying (10), and here f is determined by h uniquely modulo the group formed by those 1-cycles f_0 in $M\otimes_H X(H)$ which satisfy $\iota f_0 = -\partial g_0$ with some $g_0 \in M\otimes_G X_2(G)$ such that $\varphi g_0 = 0$. We denote by $\tau^{(0)}h$ the residue class of f modulo this last group, to obtain a homomorphism $\tau^{(0)}$ of the group of 2-cycles in $M\otimes_G X(G, H)$ into the residue group of the group of 1-cycles in $M\otimes_H X(H)$ modulo the said subgroup. The kernel of this homomorphism $\tau^{(0)}$ contains the group of 2-boundaries in $M\otimes_G X(G, H)$, since if $h = \partial k$ ($k \in M\otimes_G X_3(G, H)$), then $h = \varphi\partial k_1$ with $k_1 \in M\otimes_G X_3(G)$ satisfying $\varphi k_1 = k$, and $-\partial\partial k_1 = 0 = \varphi 0$. Hence $\tau^{(0)}$ is actually a homomorphism of $H_2(G, H; M)$ into the above residue group.

Further we denote by $H_1(H, M)_I$ the residue group of $H_1(H, M)$ modulo its subgroup formed by those homology classes represented by cycles in the above subgroup of the group of cycles, i.e. cycles f_0 which satisfy $\iota f_0 = -\partial g_0$ with some $g_0 \in M\otimes_G X_2(G)$ such that $\varphi g_0 = 0$; thus $H_1(H, M)_I$ is the residue group of the group of cycles in $M\otimes_H X_1(H)$ modulo its subgroup generated by boundary cycles and cycles f_0 as above. We denote by τ the homomorphism of $H_2(G, H, M)$ into $H_1(H, M)_I$ induced by $\tau^{(0)}$.

We observe the injection ι induces a homomorphism of our group $H_1(H, M)_I$ into $H_1(G, M)$. For, if f_0 is as above, then $\iota f_0 (= -\partial g_0)$ is evidently a boundary (and, further, ι commutes with ∂). We denote the induced homomorphism of $H_1(H, M)_I$ into $H_1(G, M)$ also by ι. Now we have

Proposition 1. *For any G-module M the sequence*

$$\text{(13)} \qquad 0 \longleftarrow H_1(G, H, M) \xleftarrow{\varphi} H_1(G, M) \xleftarrow{\iota} H_1(H, M)_I$$
$$\xleftarrow{\tau} H_2(G, H, M) \xleftarrow{\varphi} H_2(G, M)$$

is exact.

Proof. Let

$$g = \sum_{\xi \in G} u(\xi) \otimes_G (H, \xi H)$$

be a 1-cycle in $M \otimes_G X(G, H)$; we have thus $0 = \partial g = \sum_{\xi \in G} (u(\xi)\xi - u(\xi)) \otimes_G (H)$, which means the existence of elements $q(a)$ in M depending on $a \in H$ such that $\sum_{\xi \in G} (u(\xi)\xi - u(\xi)) = \sum_{a \in H} (q(a)a - q(a))$. Setting

$$f = \sum_{\xi \in G} u(\xi) \otimes_G (1, \xi) - \sum_{a \in H} q(a) \otimes_G (1, a), \tag{14}$$

we obtain a 1-cycle f in $M \otimes_G X(G)$. As $(H,H) = \partial(H, H, H)$, φf is homologous to g in $M \otimes_G X(G, H)$. This proves the exactness of (13) at $H_1(G, H, M)$.

Consider next a 1-cycle

$$f = \sum_{a \in H} u(a) \otimes_H (1, a)$$

in $M \otimes_H X(H)$. We have $\varphi_* f = \sum_{a \in H} u(a) \otimes_G (H, H) = \partial(\sum_{a \in H} u(a) \otimes_G (H, H, H))$ ~ 0. Conversely, if

$$g = \sum_{\xi \in G} u(\xi) \otimes_G (1, \xi)$$

is a 1-cycle in $M \otimes_G X(G)$ such that $\varphi g \sim 0$ in $M \otimes_G X(G, H)$; thus $\varphi g = \sum_{\xi \in G} u(\xi) \otimes_G (H, \xi H) = \partial(\sum_{\xi, \eta \in G} u(\xi, \eta) \otimes_G (H, \xi H, \eta H))$ with some 2-chain (6) in $M \otimes_G X(G)$. Putting

$$g_1 = \sum_{\xi \in G} u_1(\xi) \otimes_G (1, \xi) = g - \partial(\sum_{\xi, \eta \in G} v(\xi, \eta) \otimes_G (1, \xi, \eta)), \tag{15}$$

we have $\varphi g_1 = 0$. Hence there are elements $p(\alpha_1, c)$ of M depending on $\alpha_1 \in \mathfrak{G}_1$, $c \in H \cap \alpha_1 H \alpha_1^{-1}$ such that

$$\sum_{a \in H; h \in H(\alpha_1)} u_1(h\alpha_1 a)h = \sum_{c \in H \cap \alpha_1 H \alpha_1^{-1}} (p(\alpha_1, c)c - p(\alpha_1, c))$$

for each $\alpha_1 \in \mathfrak{G}_1$. So

$$\begin{aligned} g_1 &= \sum_{\alpha_1 \in \mathfrak{G}_1; h \in \mathfrak{H}(\alpha_1); a \in H} u_1(h\alpha_1 a) \otimes_G ((h, h\alpha_1) + (1, h\alpha_1 a) - (h, h\alpha_1)) \\ &= \sum_{\alpha_1 \in \mathfrak{G}_1} \sum_{c \in H \cap \alpha_1 H \alpha_1^{-1}} (p(\alpha_1, c)c - p(\alpha_1, c)) \otimes_G (1, \alpha_1) \\ &+ \sum_{\alpha_1 \in \mathfrak{G}_1; h \in \mathfrak{H}(\alpha_1); a \in H} u_1(h\alpha_1 a) \otimes_G ((1, h\alpha_1 a) - (h, h\alpha_1)) \\ &= \sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} p(\alpha_1, c) \otimes_G (c(1, \alpha_1) - (1, \alpha_1)) \end{aligned}$$

$$+\sum_{\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);a\in H} u_1(h\alpha_1 a)\otimes_G((1,\ h\alpha_1 a)-(h,\ h\alpha_1)).$$

As $\partial((1,\ c,\ c\alpha_1)-(1,\ \alpha_1,\ c\alpha_1))=(c,\ c\alpha_1)-(1,\ c\alpha_1)+(1,\ c)-(\alpha_1,\ c\alpha_1)+(1,\ c\alpha_1)-(1,\ \alpha_1)=c(1,\ \alpha_1)+(1,\ c)-(\alpha_1,\ c\alpha_1)-(1,\ \alpha_1)$ and $\partial((1,\ h\alpha_1 a,\ h\alpha_1)-(1,\ h,\ h\alpha_1)=(h\alpha_1 a,\ h\alpha_1)-(1,\ h\alpha_1)+(1,\ h\alpha_1 a)-(h,\ h\alpha_1)+(1,\ h\alpha_1)-(1,\ h)=(h\alpha_1 a,\ h\alpha_1)+(1,\ h\alpha_1 a)-(h,\ h\alpha_1)-(1,\ h)$, we have

$$\begin{aligned}g_1=&\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}(\partial p(\alpha_1,\ c)\otimes_G((1,\ c,\ c\alpha_1)-(1,\ \alpha_1,\ c\alpha_1))\\&-p(\alpha_1,\ c)\otimes_G(1,\ c)+p(\alpha_1,\ c)\alpha_1\otimes_G(1,\ \alpha_1^{-1}c\alpha_1))\\&+\sum_{\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);a\in H}(\partial u_1(h\alpha_1 a)\otimes_G((1,\ h\alpha_1 a,\ h\alpha_1)+(1,\ h,\ h_1))\\&-u_1(h\alpha_1 a)h\alpha_1\otimes_G(a,\ 1)+u_1(h\alpha_1 a)\otimes_G(1,\ h))\end{aligned}$$

So $g\sim g_1\sim \iota f$ in $M\otimes_G X(G)$ with

$$(16)\quad \begin{aligned}f=&\sum_{\alpha_1\in\mathfrak{G}_1;c\in H\cap\alpha_1H\alpha_1^{-1}}(-p(\alpha_1,\ c)\otimes_H(1,\ c)+(p(\alpha_1,\ c)\alpha_1\otimes_H(1,\ \alpha_1^{-1}c\alpha_1))\\&+\sum_{\alpha_1\in\mathfrak{G}_1;h\in\mathfrak{H}(\alpha_1);a\in H}(-u(h\alpha_1 a)h\alpha_1\otimes_H(a,\ 1)+u_1(h\alpha_1 a)\otimes_H(1,\ h)).\end{aligned}$$

Here f is a cycle, since $0=\partial g=\partial g_1=\partial\iota f=\iota\partial f$ and $\iota: M\otimes_H X(H)\to M\otimes_G X(G)$ is monomorphic. The sequence (13) is thus exact at $H_1(G,\ M)$.

Let h be a 2-cycle in $M\otimes_G X(G,\ H)$. Its homology class is mapped by τ to the element of $H_1(H,\ M)_I$ represented by the homology class of a 1-cycle f in $M\otimes_H X(H)$ such that there is an element g of $M\otimes_G X_2(G)$ satisfying (10). The homology class of the cycle f is mapped by ι to 0, because of the latter relation of (10). Consider next conversely any 1-cycle f in $M\otimes_H X(H)$ such that f is a boundary in $M\otimes_G X(G)$, say $-\partial g(g\in M\otimes_G X_2(G))$. We have $\partial\varphi g=\varphi\partial g=-\varphi\iota f$. As $(H,\ H)=\partial(H,\ H,\ H)$, $\varphi\iota f$ has a form $\partial\varphi\iota k$ with $k\in M\otimes_H X_2(H)$. Set $h=\varphi(g+\iota k)$. Then h is a 2-cycle in $M\otimes_G X(G,\ H)$ and $\iota(f-\partial k)=-\partial(g+\iota k)$. Hence the class in $H_1(H,\ M)_I$ of the homology class of f (i.e. that of $f-\partial k$) is the image of the homology class of the cycle h by τ. So (13) is exact at $H_1(H,\ M)_I$.

Let next g be any 2-cycle in $M\otimes_G X(G)$. Then $h=\varphi g$ is mapped by τ to 0 since $0=-\partial g$. Consider conversely any 2-cycle h in $M\otimes_G X(G,\ H)$ such that its homology class is mapped by τ to 0 in $H_1(H,\ M)$. This means the existence of elements $k,\ g$ in $M\otimes_H X_2(H)$, $M\otimes_G X_2(G)$ respectively such that $h=\varphi g$, $-\partial g=\iota\partial k$. Here $\varphi\iota k=0$ since $\varphi\iota k$ has a form $u\otimes_G(H,\ H,\ H)$ and $\partial\varphi\iota k$

$=\varphi_{\iota}\partial k=-\varphi\partial g=-\partial\varphi g=-\partial h=0$ implies $u=0$. Hence $h=\varphi(g-\iota k)$ and here $g-\iota k$ is a cycle. This proves the exactness of (13) at $H_2(G, H, M)$, and completes the proof of our proposition.

4. Supplement to the principal lemma

Let M be, as hitherto, a G-module. Let K be a subgroup of G, and ξ an element of G. For an element

$$(17)\quad k=\sum_{\kappa_0,\ldots,\kappa_n\in K} u(\kappa_0, \kappa_1, \ldots, \kappa_n)\otimes_K(\kappa_0, \kappa_1, \ldots, \kappa_n)(u(\kappa_0, \ldots, \kappa_n)\in M)$$

of $M\otimes_K X_n(K)$ we set

$$(18)\quad T_\xi k=\sum_{\kappa_0,\ldots,\kappa_n\in K} u(\kappa_0, \kappa_1, \ldots, \kappa_n)\otimes_{\xi^{-1}K\xi}(\xi^{-1}\kappa_0\xi, \xi^{-1}\kappa_1\xi, \ldots, \xi^{-1}\kappa_n\xi).$$

This is an element of $M\otimes_{\xi^{-1}K\xi}X_n(\xi^{-1}K\xi)$ and is determined uniquely by k, independently of the special choice of the form (17) expressing k. If k is a cycle, so is $T_\xi k$.

Now, let H be a subgroup and k an element of $M\otimes_K X_n(K)$ with $K=H\cap\xi H\xi^{-1}$, where ξ is, as above, an element of G. Then $T_\xi f$ is, thus, an element of $M\otimes_{\xi^{-1}K\xi}X_n(\xi^{-1}K\xi)$ and here $\xi^{-1}K\xi=H\cap\xi^{-1}H\xi$. If $\iota_{H\cap\xi H\xi^{-1},H}$, $\iota_{H\cap\xi^{-1}H\xi,H}$ denote the injection maps of $M\otimes_{H\cap\xi H\xi^{-1}}X_n(H\cap\xi H\xi^{-1})$, $M\otimes_{H\cap\xi^{-1}H\xi}X_n(H\cap\xi^{-1}H\xi)$ into $M\otimes_H X_n(H)$,

$$(19)\qquad \iota_{H\cap\xi H\xi^{-1},H}\,k-\iota_{H\cap\xi^{-1}H\xi,H}\,T_\xi k$$

is an element of $M\otimes_H X_n(H)$. In case f is a cycle our element is a cycle too. So, we consider, taking $n=1$, the subgroup of $M\otimes_H X_1(H)$ generated by boundaries and cycles of form (19) with cycles k in $M\otimes_{H\cap\xi H\xi^{-1}}X_1(H\cap\xi H\xi^{-1})$, ξ varying in G, and we supplement our principal Lemma 1 with the following

LEMMA 2. *Let h be a 2-cycle in $M\otimes_G X(G, H)$. If we take, as in Lemma 1, a 2-chain (6) in $M\otimes_G X(G)$ having h as its image by φ and consider the expression f in (12), f is determined by h uniquely modulo the above subgroup of $M\otimes_H X_1(H)$,* i.e. *modulo boundary cycles and cycles of form (19) with 1-cycles k in $M\otimes_{H\cap\xi H\xi^{-1}}X(H\cap\xi H\xi^{-1})$.*

Proof. First fix a choice of (6). Then we have merely to consider different choices of q in (9). If q, q' are two different choices and if we set $s(\alpha_1, c)=q(\alpha_1, c)-q'(\alpha_1, c)(\alpha_1\in\mathfrak{G}_1, c\in H\cap\alpha_1H\alpha_1^{-1})$, then we have $\sum_{c\in H\cap\alpha_1H\alpha_1^{-1}}(s(\alpha_1, c)c$

$-s(\alpha_1, c))=0$ for every $\alpha_1 \in \mathfrak{G}_1$. This shows that, for each $\alpha_1 \in \mathfrak{G}_1$,

$$\sum_{c \in H \cap \alpha_1 H \alpha_1^{-1}} s(\alpha_1, c) \tag{20}$$

is a 1-cycle in $M \otimes_{H \cap \alpha_1 H \alpha_1^{-1}} X(H \cap \alpha_1 H \alpha_1^{-1})$. The difference of f by our two choices q, q' of q is, on the other hand,

$$\begin{aligned} &\sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} (s(\alpha_1, c) \otimes_H (1, c) - s(\alpha_1, c)\alpha_1 \otimes_H (1, \alpha_1^{-1} c \alpha_1)) \\ &\qquad - \sum_{\alpha_1 \in \mathfrak{G}_1; c \in H \cap \alpha_1 H \alpha_1^{-1}} (s(\alpha_1, c) - s(\alpha_1, c)\alpha_1) \otimes_H (1, 1). \end{aligned} \tag{21}$$

Here the latter sum is a boundary (since $(1, 1) = \partial(1, 1, 1)$) while the former sum is a sum of cycles of form (19) with cycles k (since (20) is, for each $\alpha_1 \in \mathfrak{G}_1$, a cycle).

Next we turn to examining the change of f by different choices of a 2-chain (6) in $M \otimes_G X(G)$ having h as its image by φ. If $\sum_{\xi, \eta \in G} v'(\xi, \eta) \otimes_G (1, \xi, \eta)$ is a second choice, and if we set $w = (\xi, \eta) = v(\xi, \eta) - v'(\xi, \eta)$, then we have $\sum_{\xi, \eta \in G} w(\xi, \eta) \otimes_G (H, \xi H, \eta H) = 0$, or, what is equivalent to that for every pair (α_2, β_2) in $\mathfrak{G}_2$ the class of

$$\sum_{a, b \in H} \sum_{h \in \mathfrak{H}(\alpha_2, \beta_2)} w(h\alpha_2 a, h\beta_2 b)h$$

in the residue module $M_{H \cap \alpha_2 H \alpha_2^{-1} \cap \beta_2 H \beta_2^{-1}}$ is 0. Every such system $w(\xi, \eta)$ is a sum of systems of following forms:

i) $w(\alpha_2, \beta_2) = uk - u$ with $u \in M$, $k \in H \cap \alpha_2 H \alpha_2^{-1} \cap \beta_2 H \beta_2^{-1}$ for one pair (α_2, β_2) in $\mathfrak{G}_2$, and $w(\xi, \eta) = 0$ for $(\xi, \eta) \neq (\alpha_2, \beta_2)$;

ii) $w(\alpha_2, \beta_2) = -w(h_0\alpha_2 a_0, h_0\beta_2 b_0)h_0 = u \in M$ for one system $(\alpha_2, \beta_2) \in \mathfrak{G}_2$, $h_0 \in \mathfrak{H}(\alpha_2, \beta_2)$ and $a_0, b_0 \in H$, with $(h_0\alpha_2 a_0, h_0\beta_2 b_0) \neq (\alpha_2, \beta_2)$, and $w(\xi, \eta) = 0$ for all (ξ, η) different from (α_2, β_2) and $(h_0\alpha_2 a_0, h_0\beta_2 b_0)$.

We first consider $w(\xi, \eta)$ as in i). For them we have for each $\alpha_1 \in \mathfrak{G}_1$

$$\begin{aligned} &\sum_{a \in H; h \in \mathfrak{H}(\alpha_1); \eta \in G} (w(\eta, \eta h\alpha_1 a)\eta - w(\eta, h\alpha_1 a) + w(h\alpha_1 a, \eta))h \\ &= \delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H} w(\alpha_2, \alpha_2 h'\alpha_1 a')\alpha_2 h' - \delta_{H\alpha_1 H, H\beta_2 H} w(\alpha_2, h''\alpha_1 a'')h'' \\ &\quad + \delta_{H\alpha_1 H, H\alpha_2 H} w(h'''\alpha_1 a''', \beta_2)h''' \\ &= \delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H}(u\kappa - u)\alpha_2 h' - \delta_{H\alpha_1 H, H\beta_2 H}(u\kappa - u)h'' \\ &\quad + \delta_{H\alpha_1 H, H\alpha_2 H}(u\kappa - u)h''' \end{aligned}$$

where δ's are Kronecker δ's and $h', h'', h''' \in \mathfrak{H}(\alpha_1)$ and $a', a'', a''' \in H$ are, when exist, such that $h'\alpha_1 a' = \alpha_2^{-1}\beta_2$, $h''\alpha_1 a'' = \beta_2$, $h'''\alpha_1 a''' = \alpha_2$; each of the equation is solvable (indeed uniquely) if and only if the corresponding δ is 1. This is expressed also as $\sum_{c\in H\cap\alpha_1 H\alpha_1^{-1}}(r(\alpha_1, c)c - r(\alpha_1, c))$ with $r(\alpha_1, c) = \delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H} r'(\alpha_1, c) + \delta_{H\alpha_1 H, H\beta_2 H} r''(\alpha_1, c) + \delta_{H\alpha_1 H, H\alpha_2 H} r'''(\alpha_1, c)$, where $r'(\alpha_1, c) = u\alpha_2 h'$ or 0 according as $c = h'^{-1}\alpha_2^{-1}k\alpha_2 h'$ or not, $r''(\alpha_1, c) = -uh''$ or 0 according as $c = h''^{-1}kh''$ or not, and $r'''(\alpha_1, c) = uh'''$ or 0 according as $c = h'''kh'''$ or not; observe that we have, by $k\in H\cap\alpha_2 H\alpha_2^{-1}\cap\beta_2 H\beta_2^{-1}$ and the defining properties of h', h'', h''', a', a'', a''',

$$h'^{-1}\alpha_2^{-1}k\alpha_2 h' \in h'^{-1}Hh'\cap\alpha_1 a' H a'^{-1}\alpha_1^{-1} = H\cap\alpha_1 H\alpha_1^{-1},$$
$$h''^{-1}kh'' \in h''^{-1}Hh''Hh''\cap\alpha_1 a'' H a''^{-1}\alpha_1^{-1} = H\cap\alpha_1 H\alpha_1^{-1},$$
$$h'''^{-1}kh''' \in h'''^{-1}Hh'''\cap\alpha_1 a''' H a'''^{-1}\alpha_1^{-1} = H\cap\alpha_1 H\alpha_1^{-1}.$$

Thus $q'(\alpha_1, c) = q(\alpha_1, c) - r(\alpha_1, c)$ satisfy (9) with $v(\xi, \eta)$ replaced by $v'(\xi, \eta) = v(\xi, \eta) - w(\xi, \eta)$. So the expression (12) for $v'(\xi, \eta)$ in place of $v(\xi, \eta)$ differs, with our choice of $q'(\alpha_1, c)$, from the original one, f, by

$$\begin{aligned}
(22)\qquad & w(\alpha_2, \alpha_2 h'\alpha_1' a')\alpha_2 h'\alpha_1'\otimes_H(1, a') - w(\alpha_2, h''\alpha_1'' a'')h''\alpha_1''\otimes_H(1, a'') \\
&\quad + w(h'''\alpha_1''' a''', \beta_2)h'''\alpha_1'''\otimes_H(1, a''') \\
&\quad + w(\alpha_2, \alpha_2 h'\alpha_1' a')\alpha_2\otimes_H(1, h') - w(\alpha_2, h''\alpha_1'' a'')\otimes_H(1, h'') \\
&\quad + w(h'''\alpha_1''' a''', \beta_2)\otimes_H(1, h''') \\
&\quad - \sum_{\alpha_1\in\mathfrak{G}_1;\, c\in H\cap\alpha_1 H\alpha_1^{-1}}(r(\alpha_1, c)\otimes_H(1, c) - r(\alpha_1, c)\alpha_1\otimes_H(1, \alpha_1^{-1}c\alpha_1)) \\
&\quad + \sum_{\alpha_1\in\mathfrak{G}_1;\, c\in H\cap\alpha_1 H\alpha_1^{-1}}(r(\alpha_1, c) - r(\alpha_1, c)\alpha_1)\otimes_H(1, 1) \\
= {} & w(\alpha_2, \beta_2)\alpha_2 h'\alpha_1'\otimes_H(1, a') - w(\alpha_2, \beta_2)h''\alpha_0''\otimes_H(1, a'') \\
&\quad + w(\alpha_2, \beta_2)h'''\alpha_0'''\otimes_H(1, a''') \\
&\quad + w(\alpha_2, \beta_2)\alpha_2\otimes_H(1, h') - w(\alpha_2, \beta_2)\otimes_H(1, h'') + w(\alpha_2, \beta_2)\otimes_H(1, h''') \\
&\quad - r'(\alpha_1', h'^{-1}\alpha_2^{-1}k\alpha_2 h')\otimes_H(1, h'^{-1}\alpha_2^{-1}k\alpha_2 h') \\
&\quad - r''(\alpha_1'', h''^{-1}kh'')\otimes_H(1, h''^{-1}kh'') \\
&\quad - r'''(\alpha_1''', h'''^{-1}kh''')\otimes_H(1, h'''^{-1}kh''') \\
&\quad + r'(\alpha_1', h'^{-1}\alpha_2^{-1}k\alpha_2 h')\alpha_1'\otimes_H(1, \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1') \\
&\quad + r''(\alpha_1'', h''^{-1}kh'')\alpha_1''\otimes_H(1, \alpha_1''^{-1}h''^{-1}kh''\alpha_1'') \\
&\quad + r'''(\alpha_1''', h'''^{-1}kh''')\alpha_1'''\otimes_H(1, \alpha_1'''^{-1}h'''^{-1}kh'''\alpha_1''') \\
&\quad + (r'(\alpha_1', h'^{-1}\alpha_2^{-1}k\alpha_2 h') - r'(\alpha_1', h'^{-1}\alpha_2^{-1}k\alpha_2 h')\alpha_1')\otimes_H(1, 1) \\
&\quad + (r''(\alpha_1'', h''^{-1}kh'') - r''(\alpha_1'', h''^{-1}kh'')\alpha_1'')\otimes_H(1, 1) \\
&\quad + (r'''(\alpha_1''', h'''^{-1}kh''') - r'''(\alpha_1''', h'''^{-1}kh''')\alpha_1''')\otimes_H(1, 1)
\end{aligned}$$

$$\begin{aligned}
&= (uk-u)\alpha_2 h'\alpha_1' \otimes_H (1, a') - (uk-u)h''\alpha_1'' \otimes_H (1, a'') \\
&\quad + (uk-u)h'''\alpha_1''' \otimes_H (1, a''') \\
&\quad + (uk-u)\alpha_2 \otimes_H (1, h') - (uk-u) \otimes_H (1, h'') + (uk-u) \otimes_H (1, h''') \\
&\quad - u\alpha_2 h' \otimes_H (1, h'^{-1}\alpha_2^{-1}k\alpha_2 h') + uh'' \otimes_H (1, h''^{-1}kh'') \\
&\quad\quad - uh''' \otimes_H (1, h'''kh''') \\
&\quad + u\alpha_2 h'\alpha_1' \otimes_H (1, \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1') \\
&\quad\quad - uh''\alpha_1'' \otimes_H (1, \alpha_1''^{-1}h''^{-1}kh''\alpha_1'') \\
&\quad\quad + uh'''\alpha_1''' \otimes_H (1, \alpha_1'''^{-1}h'''^{-1}kh'''\alpha_1''') \\
&\quad + (u\alpha_2 h' - u\alpha_2 h'\alpha_1') \otimes_H (1, 1) - (uh'' - uh''\alpha_1'') \otimes_H (1, 1) \\
&\quad\quad + (uh''' - uh'''\alpha_1''') \otimes_H (1, 1)
\end{aligned}$$

where $\alpha_1', \alpha_1'', \alpha_1''' \in \mathfrak{G}_1$ are such that

$$H\alpha_1'H = H\alpha_2^{-1}\beta_2 H, \quad H\alpha_1''H = H\beta_2 H, \quad H\alpha_1'''H = H\alpha_2 H \tag{23}$$

and $h' \in \mathfrak{H}(\alpha_1')$, $h'' \in \mathfrak{H}(\alpha_1'')$, $h''' \in \mathfrak{H}(\alpha_1''')$ and $a', a'', a''' \in H$ are such that

$$h'\alpha_1' a' = \alpha_2^{-1}\beta_2, \quad h''\alpha_1'' a'' = \beta_2, \quad h'''\alpha_1''' a''' = \alpha_2; \tag{24}$$

thus $h', h'', h''', a', a'', a'''$ are the same as what were denoted, when they existed, by the same letters a few lines above.

The last three terms in the last sum are evidently boundaries. The sum of the 1., 4., 7. and 10. terms, concerned with α_1', h', a', is

$$\begin{aligned}
&uk\alpha_2 h'\alpha_1' \otimes_H (1, a') - u\alpha_2 h'\alpha_1' \otimes_H (1, a') + uk\alpha_2 \otimes_H (1, h') - u\alpha_2 \otimes_H (1, h') \\
&- u\alpha_2 h' \otimes_H (1, h'^{-1}\alpha_2^{-1}k\alpha_2 h') + u\alpha_2 h'\alpha_1' \otimes_H (1, \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1') \\
&= uk\alpha_2 h'\alpha_1' \otimes_H (1, a') - \partial(u\alpha_2 h'\alpha_1' \otimes_H (1, a', \alpha_1'^{-1}h'\alpha_2^{-1}k\alpha_2 h'\alpha_1')) \\
&+ u\alpha_2 h'\alpha_1' \otimes_H (a', \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1') + uk\alpha_2 \otimes_H (1, h') - u\alpha_2 \otimes_H (1, h') \\
&- u\alpha_2 h' \otimes_H (1, h'^{-1}\alpha_2^{-1}k\alpha_2 h') = \partial(uk\alpha_2 h'\alpha_1' \otimes_H ((1, a', 1) + (1, 1, 1))) \\
&- uk\alpha_2 h'\alpha_1' \otimes_H (a', 1) - \partial(u\alpha_2 h'\alpha_1' \otimes_H (1, a', \alpha_1^{-1}h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1')) \\
&+ uk\alpha_2 h'\alpha_1' \otimes_H (\alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1' a', 1) + uk\alpha_2 \otimes_H (1, h') \\
&- uk\alpha_2 \otimes_H (\alpha_2^{-1}k\alpha_2, \alpha_2^{-1}k\alpha_2 h') - uk\alpha_2 \otimes_H (\alpha_2^{-1}k^{-1}\alpha_2 h', h') \\
&= \partial(uk\alpha_2 h'\alpha_1' \otimes_H ((1, a', 1) + (1, 1, 1))) \\
&+ \partial(uk\alpha_2 h'\alpha_1' \otimes_H (a', \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1' a', 1)) \\
&- uk\alpha_2 h'\alpha_1' \otimes_H (a', \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1' a') \\
&- \partial(u\alpha_2 h'\alpha_1' \otimes_H (1, a', \alpha_1 h'^{-1}\alpha_2^{-1}k\alpha_2 h'\alpha_1')) + \partial(uk\alpha_2 \otimes_H ((\alpha_2^{-1}k^{-1}\alpha_2 h', 1, h') \\
&- (\alpha_2^{-1}k\alpha_2, \alpha_2^{-1}k\alpha_2 h', 1))) - uk\alpha_2 \otimes_H (\alpha_2^{-1}k\alpha_2, 1) = \partial(uk\alpha_2 h'\alpha_1' \otimes_H ((1, a', 1) \\
&+ (1, 1, 1) + (a', \alpha_1'^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1' a', 1)
\end{aligned}$$

$-(\alpha_1^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1', \alpha_1^{-1}h'^{-1}\alpha_2^{-1}k^{-1}\alpha_2 h'\alpha_1' a', 1))$
$+uk\alpha_2\otimes_H((\alpha_2^{-1}k^{-1}\alpha_2 h', 1, h')-(\alpha_2^{-1}k\alpha_2, \alpha_2^{-1}k\alpha_2 h', 1)))$
$-uk\beta_2\otimes_H(1, \beta_2^{-1}k\beta_2)-uk\alpha_2\otimes_H(\alpha_2^{-1}k\alpha_2, 1).$

The sum of the 2., 5., 8. and 11. terms (concerned with α_1'', h'', a'') of the above sum is, on the other hand,

$-ukh''\alpha_1''\otimes_H(1, a'')+uh''\alpha_1''\otimes_H(1, a'')-uk\otimes_H(1, h'')+u\otimes_H(1, h'')$
$+uh''\otimes_H(1, h''^{-1}kh'')-uh''\alpha_1''\otimes_H(1, \alpha_1''^{-1}h''^{-1}kh''\alpha_1'')=-ukh''\alpha_1''\otimes_H(1, a'')$
$+\partial(uh''\alpha_1''\otimes_H(1, a'', \alpha_1''^{-1}h''^{-1}kh''\alpha_1''))-uh''\alpha_1''\otimes_H(a'', \alpha_1''^{-1}h''^{-1}kh''\alpha_1'')$
$-uk\otimes_H(1, h'')+\partial(u\otimes_H(1, h'', kh''))+u\otimes_H(1, kh'')$
$=-\partial(ukh''\alpha_1''\otimes_H((1, a'', 1)-(1, 1, 1)))+ukh'''\alpha_1''\otimes_H(a'', 1)$
$+\partial(uh''\alpha_1''\otimes_H(1, a'', \alpha_1''^{-1}h''^{-1}kh''\alpha_1''))-ukh''\alpha_1''\otimes_H(\alpha_1''^{-1}h''^{-1}k''^{-1}h''\alpha_1''a'', 1)$
$-uk\otimes_H(1, h'')+\partial(u\otimes_H(1, h'', kh''))+uk\otimes_H(k^{-1}, h'')$
$=-\partial(ukh''\alpha_1''\otimes_H((1, a'', 1)-(1, 1, 1)))$
$+\partial(ukh''\alpha_1''\otimes_H(\alpha_1''^{-1}h''^{-1}k^{-1}h''\alpha_1''a'', a'', 1))$
$-ukh''\alpha_1''\otimes_H(\alpha_1''h''^{-1}k^{-1}h''\alpha_1''a'', a'')+\partial(uh''\alpha_1''\otimes_H(1, a'', \alpha_1''^{-1}h''^{-1}kh''\alpha_1''))$
$+\partial(uk\otimes_H(1, k^{-1}, h'')-uk\otimes_H(1, k^{-1})+\partial(u\otimes_H(1, h'', kh''))$
$=\partial(ukh''\alpha_1''\otimes_H(-(1, a'', 1)+(1, 1, 1)+(\alpha_1''^{-1}h''^{-1}k^{-1}h''\alpha_1''a'', a'', 1)$
$+(\alpha_1''^{-1}h''^{-1}h''\alpha_1'', \alpha_1''^{-1}h''^{-1}k^{-1}h''\alpha_1''a'', 1))+uk\otimes_H(1, k^{-1}, h'')$
$+u\otimes_H(1, h'', kh''))-uk\beta_2\otimes_H(\beta_2^{-1}k^{-1}\beta_2, 1)-uk\otimes_H(1, k^{-1}).$

Further, the sum of the 3., 6., 9. and 12. terms (concerned with α_1''', h''', a''') of the above sum is, similarly,

$ukh'''\alpha_1'''\otimes_H(1, a''')-uh'''\alpha_1'''\otimes_H(1, a''')+uk\otimes_H(1, h''')-u\otimes_H(1, h''')$
$-uh'''\otimes_H(1, h'''^{-1}kh''')+uh'''\alpha_1'''\otimes_H(1, \alpha_1'''^{-1}h'''^{-1}kh'''\alpha_1''')$
$=\partial(ukh'''\alpha_1'''\otimes_H((1, a''', 1)-(1, 1, 1)-(\alpha_1'''^{-1}h'''^{-1}k^{-1}h'''\alpha_1'''a''', a''', 1)$
$+(\alpha_1'''^{-1}h'''^{-1}k^{-1}h'''\alpha_1''', \alpha_1'''h'''^{-1}k^{-1}h'''\alpha_1'''a''', 1))-uk\otimes_H(1, k^{-1}, h''')$
$-u\otimes_H(1, h''', kh'''))+uk\alpha_2\otimes_H(\alpha_2^{-1}k\alpha_2, 1)+uk\otimes_H(1, k^{-1}).$

So, in altogether, the above sum (which is the change of the expression (12) corresponding to the change of v (and q) to v' (and q')) is $\sim -uk\beta_2\otimes_H(1, \beta_2^{-1}k^{-1}\beta_2)-uk\beta_2\otimes_H(\beta_2^{-1}k\beta_2, 1)=-\partial(uk\beta_2\otimes_H((1, \beta_2^{-1}k^{-1}\beta_2, 1)+(1, 1, 1))\sim 0.$

Next we consider $w(\xi, \eta)$ as in ii). For them we have for each $\alpha_1\in\mathfrak{G}_1$

$$\sum_{a\in H;h\in\mathfrak{H}(\alpha_1);\eta\in G}(w(\eta, \eta h\alpha_1 a)\eta-w(\eta, h\alpha_1 a)+w(h\alpha_1 a, \eta))h$$
$$=\delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H}(w(\alpha_2, \beta_2)\alpha_2 h'+w(h_0\alpha_2 a, h_0\beta_2 b_0)h_0\alpha_2 a_0 h^{-1})$$

$$\begin{aligned}
&- \delta_{H\alpha_1 H, H\beta_2 H}(w(\alpha_2, \beta_2)h'' + w(h_0\alpha_2 a, h_0\beta_2 b)\bar{h}'') \\
&+ \delta_{H\alpha_1 H, H\alpha_2 H}(w(\alpha_2, \beta_2)h''' + w(h_0\alpha_2 a, h_0\beta_2 b)\bar{h}''') \\
= {} & \delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H}(u\alpha_2 h' - u\alpha_2 a_0 h^{-1}) - \delta_{H\alpha_1 H, H\beta_2 H}(uh'' - uh_0^{-1}\bar{h}'') \\
&+ \delta_{H\alpha_1 H, H\alpha_2 H}(uh''' - uh_0^{-1}\bar{h}''')
\end{aligned}$$

where h', $\bar{h}'$, h'', $\bar{h}''$, h''', $\bar{h}''' \in \mathfrak{H}(\alpha_1)$ are, when exist, such that $\alpha_2 h'\alpha_1 H = \beta_2 H$ (i.e. $h'\alpha_1 H = \alpha_2^{-1}\beta_2 H$), $h_0\alpha_2 a\bar{h}'\alpha_1 H = h_0\beta_2 H$ (i.e. $\bar{h}'\alpha_1 H = a_0^{-1}\alpha_2^{-1}\beta_2 H$), $h''\alpha_1 H = \beta_2 H$, $\bar{h}''\alpha_1 H = h_0\beta_2 H$, $h'''\alpha_1 H = \alpha_2 H$, $\bar{h}'''\alpha_1 H = h_0\alpha_2 H$. This is expressed also as $\sum_{c\in H\cap\alpha_1 H\alpha^{-1}}(r(\alpha_1, c)c - r(\alpha_1, c))$ with $r(\alpha_1, c) = -\delta_{H\alpha_1 H, H\alpha_2^{-1}\beta_2 H}\delta_{c, h'^{-1}a_0\bar{h}'}u\alpha_2 h' + \delta_{H\alpha_1 H, H\beta_2 H}\delta_{c, h''^{-1}h_0^{-1}\bar{h}''}uh'' - \delta_{H\alpha_1 H, H\alpha_2 H}\delta_{c, h'''^{-1}h_0^{-1}\bar{h}'''}uh'''$. $q'(\alpha_1, c) = q(\alpha_1, c) - r(\alpha_1, c)$ sitisfy (9) with $v(\xi, \eta)$ replaced by $v'(\xi, \eta) = v(\xi, \eta) - w(\xi, \eta)$.

The expression (12) for $v'(\xi, \eta)$, with this choice of $q'(\alpha_1, c)$, differs from the expression (12) for $v(\xi, \eta)$ by

$$\begin{aligned}
(25)\qquad & w(\alpha_2, \beta_2)\alpha_2 h'\alpha_1'\otimes_H(1, a') + w(h_0\alpha_2 a_0, h\beta_2 b_0)h_0\alpha_2 a\bar{h}'\alpha_1'\otimes_H(1, \bar{a}') \\
&- w(\alpha_2, \beta_2)h''\alpha_1''\otimes_H(1, a'') - w(h_0\alpha_2 a, h_0\beta_2 b_0)\bar{h}''\alpha_1''\otimes_H(1, \bar{a}'') \\
&+ w(\alpha_2, \beta_2)h''\alpha_1'''\otimes_H(1, a''') + w(h_0\alpha_2 a, h_0\beta_2 b_0)\bar{h}'''\alpha_1'''\otimes_H(1, \bar{a}''') \\
&+ w(\alpha_2, \beta_2)\alpha_2\otimes_H(1, h') + w(h_0\alpha_2 a_0, h_0\beta_2 b_0)h_0\alpha_2 a_0\otimes_H(1, \bar{h}') \\
&- w(\alpha_2, \beta_2)\otimes_H(1, h'') - w(h_0\alpha_2 a_0, h_0\beta_2 b_0)\otimes_H(1, \bar{h}'') \\
&+ w(\alpha_2, \beta_2)\otimes_H(1, h''') + w(h_0\alpha_2 a_0, h_0\beta_2 b_0)\otimes_H(1, h''') \\
&- \sum_{\alpha_1\in\mathfrak{G}_1; c\in H\cap\alpha_1 H\alpha_1^{-1}}(r(\alpha_1, c)\otimes_H(1, c) - r(\alpha_1, c)\alpha_1\otimes_H(1, \alpha_1^{-1}c\alpha_1)) \\
&+ \sum_{\alpha_1\in\mathfrak{G}_1; c\in H\cap\alpha_1 H\alpha_1^{-1}}(r(\alpha_1, c) - r(\alpha_1, c)\alpha_1)\otimes_H(1, 1) \\
= {} & u\alpha_2 h'\alpha_1'\otimes_H(1, a') - u\alpha_2 a_0\bar{h}'\alpha_1'\otimes_H(1, \bar{a}') - uh''\alpha_1''\otimes_H(1, a'') \\
&+ uh_0^{-1}\bar{h}''\alpha_1''\otimes_H(1, \bar{a}'') + uh'''\alpha_1'''\otimes_H(1, a''') \\
&- uh_0^{-1}\bar{h}'''\alpha_1'''\otimes_H(1, \bar{a}''') \\
&+ u\alpha_2\otimes_H(1, h') - u\alpha_2 a_0\otimes_H(1, \bar{h}') - u\otimes_H(1, h'') \\
&+ uh_0^{-1}\otimes_H(1, \bar{h}'') + u\otimes_H(1, h''') - uh_0^{-1}\otimes_H(1, \bar{h}''') \\
&+ u\alpha_2 h'\otimes_H(1, h'^{-1}a_0\bar{h}') - uh''\otimes_H(1, h''^{-1}h_0\bar{h}'') \\
&+ uh'''\otimes_H(1, h'''^{-1}h_0^{-1}\bar{h}''') \\
&- u\alpha_2 h'\alpha_1'\otimes_H(1, \alpha_1'^{-1}h'^{-1}a_0\bar{h}'\alpha_1) + uh''\alpha_1''\otimes_H(1, \alpha_1''^{-1}h''^{-1}h_0^{-1}\bar{h}''\alpha_1'') \\
&- uh'''\alpha_1'''\otimes_H(1, \alpha_1'''^{-1}h'''^{-1}h_0\bar{h}'''\alpha_1''') + (-u\alpha_2 h' + uh'' - uh''' \\
&+ u\alpha_2 h'\alpha_1' - uh''\alpha_1'' + uh'''\alpha_1''')\otimes_H(1, 1)
\end{aligned}$$

where α_1', α_1'', $\alpha_1''' \in \mathfrak{G}_1$ are the same as in (23) and $h' \in \mathfrak{H}(\alpha_1')$, $h'' \in \mathfrak{H}(\alpha_1'')$, $h''' \in \mathfrak{H}(\alpha_1''')$ and a', a'', $a''' \in H$ are the same as in (24) while $\bar{h}' \in \mathfrak{H}(\alpha_1')$,

$\bar{h}'' \in \mathfrak{H}(\alpha_1'')$, $\bar{h}''' \in \mathfrak{H}(\alpha_1''')$ and $a', a'', a''' \in H$ are such that

$$h_0\alpha_2 a_0 \bar{h}'\alpha_1'\bar{a}' = h_0\beta_2 b \text{ (i.e. } \bar{h}'\alpha_1'\bar{a}' = a_0^{-1}\alpha_2^{-1}\beta_2 b_0), \tag{26}$$
$$\bar{h}''\alpha_1''\bar{a}'' = h_0\beta_2 b_0, \quad \bar{h}'''\alpha_1'''\bar{a}''' = h_0\alpha_2 a_0.$$

The last term of the above sum is evidently a boundary. The sum of the 1., 2., 7., 8., 13., 14. terms is

$$\begin{aligned} &u\beta_2 a'^{-1}\otimes_H(1, a') - u\beta_2 b_0\bar{a}'^{-1}\otimes_H(1, a') + u\alpha_2\otimes_H(1, h') - u\alpha_2\otimes_H(a_0, a_0\bar{h}') \\ &\quad + u\alpha_2\otimes_H(h', a_0\bar{h}') - u\beta_2 a'^{-1}\otimes_H(1, a'b_0\bar{a}'^{-1}) \\ &= u\beta_2\otimes_H((a'^{-1}, 1) - (b_0\bar{a}'^{-1}, b_0) - (a'^{-1}, b_0\bar{a}'^{-1})) \\ &\quad + u\alpha_2\otimes_H((1, h') - (a_0, a_0\bar{h}') + (h', a_0\bar{h}')) \\ &= \partial(-u\beta_2\otimes_H((a', b_0\bar{a}'^{-1}, 1) + (b_0\bar{a}'^{-1}, b_0, 1)) \\ &\quad + u\alpha_2\otimes_H((a_0, h', a_0\bar{h}') + (a_0, 1, h')) + u\beta_2\otimes_H(b_0, 1) - u\alpha_2\otimes_H(a_0, 1). \end{aligned}$$

The sum of the 3., 4., 9., 10., 15., 16. terms is

$$\begin{aligned} &-u\beta_2 a''^{-1}\otimes_H(1, a'') + u\beta_2 b\bar{a}''^{-1}\otimes_H(1, a'') - u\otimes_H(1, h'') + uh_0^{-1}\otimes_H(1, \bar{h}'') \\ &-uh''\otimes_H(1, h''^{-1}h_0^{-1}\bar{h}'') + u\beta_2 a''^{-1}\otimes_H(1, a''b_0\bar{a}''^{-1}) \\ &= -u\beta_2\otimes_H((a'', 1) - (b_0\bar{a}''^{-1}, b_0) - (a''^{-1}, b_0\bar{a}''^{-1})) \\ &-u\otimes_H((1, h'') - (h_0^{-1}, h_0^{-1}\bar{h}'') + (h''^{-1}, h_0^{-1}\bar{h}'')) \end{aligned}$$

and is, by exactly the same computation as in the last stage of the above calculation,

$$\begin{aligned} &= \partial(u\beta_2\otimes_H((a''^{-1}, b_0\bar{a}''^{-1}, 1) + (b_0\bar{a}''^{-1}, b_0, 1)) - u\otimes_H((h_0^{-1}, h_0', h_0^{-1}\bar{h}') \\ &\quad + (h_0^{-1}, 1, h')) - u\beta_2\otimes_H(b_0, 1) + u\otimes_H(h_0^{-1}, 1). \end{aligned}$$

Similarly the sum of the 5., 6., 11., 12., 17., 18. terms is

$$\begin{aligned} &\partial(-u\alpha_2\otimes_H((a'''^{-1}, a_0\bar{a}'''^{-1}, 1) + (a_0\bar{a}'''^{-1}, a_0, 1)) + u\otimes_H((h_0^{-1}, h', h_0^{-1}\bar{h}') \\ &\quad + (h_0^{-1}, 1, h')) + u\alpha_2\otimes_H(a_0, 1) - u\otimes_H(h_0^{-1}, 1). \end{aligned}$$

Hence, in altogether, the above sum is ~ 0.

Having thus examined the change of the expression f in (12), first by a fixed choice of $v(\xi, \eta)$, in (6), and then by different choices of $v(\xi, \eta)$, we have proved Lemma 2.

5. Characterization of $H_1(H, M)_I$

Now we prove

PROPOSITION 2. *The kernel of the canonical map* $H_1(H, M) \to H_1(H, M)_I$, i.e. *the subgroup of* $H_1(H, M)$ *formed by the homology classes represented by cycles* f_0 *which satisfy* $\iota f_0 = -\partial g_0$ *with some* $g_0 \in M \otimes_G X_2(G)$ *such that* $\varphi g_0 = 0$, *coincides with the subgroup of* $H_1(H, M)$ *generated by the homology classes of* (19) *with cycles* k *in* $M \otimes_{H \cap \xi H \xi^{-1}} X_1(H \cap \xi H \xi^{-1})$ (ξ *varying in* G).

Proof. Let k be a 1-cycle in $M \otimes_{H \cap \xi H \xi^{-1}} X(H \cap \xi H \xi^{-1})$ where ξ is an element of G. We may assume without loss in generality that $\mathfrak{G}_1$ is taken so as to contain ξ, and thus ξ is some α_1 in $\mathfrak{G}_1$. When $k = \sum_{c \in H \cap \xi H \xi^{-1}} q(c) \otimes_{H \cap \xi H \xi^{-1}} (1, c)$, we set $q(\alpha_1, c) = 0$ for $\alpha_1 \in \mathfrak{G}_1$, $\alpha_1 \neq \xi$, $c \in H \cap \alpha_1 H \alpha_1^{-1}$, and $q(\xi, c) = q(c)$ *for* $c \in H \cap \xi H \xi^{-1}$. Then (9) holds for every $\alpha_1 \in \mathfrak{G}_1$ with all $v(\xi, \eta) = 0$. The cycle f in (12) for the present $v(\xi, \eta)$, $q(\alpha_1, c)$ coincides up to the last sum in it, which is evidently a boundary cycle, with (19). Hence the image by $\iota : M \otimes_H X_1(H) \to M \otimes_G X_1(G)$ of (19) plus some boundary cycle is $-\partial g$ with g in (11) for our $v(\xi, \eta)$, $q(\alpha_1, c)$. Since here all $v(\xi, \eta) = 0$, we have $\varphi g = 0$. Thus (19) is a cycle f_0 as in our lemma modulo boundary cycles, and one half of the proposition is proved. Our proof, which depends on a part of the proof of Lemma 1, may simply be summarized in the equation

$$\begin{aligned} &\iota\Big(\sum_{c \in H \cap \xi H \xi^{-1}} (q(c) \otimes_H (1, c) - q(c)\xi \otimes_H (1, \xi^{-1} c \xi))\Big) \\ &\quad - \partial \sum_{c \in H \cap \xi H \xi^{-1}} (q(c) - q(c)\xi) \otimes_H (1, 1, 1)) \\ &= -\partial \sum_{c \in H \cap \xi H \xi^{-1}} q(c) \otimes_G (((1, \xi, c\xi) - (1, \xi, \xi)) + ((1, 1, \xi) - (1, c, c\xi))) \end{aligned}$$

which can naturally be verified directly.

To prove the other half of the proposition, we assume, as we may, that $\mathfrak{G}_1$ is chosen so as to contain 1 and $\mathfrak{H}(1) = \{1\}$. Let f_0, g_0 be as in the proposition. If we express g_0 in the form (6), the relation $-\partial g_0 = \iota f_0$ implies that $\sum_{\eta \in G} (v(\eta, \eta h \alpha_1 a)\eta - v(\eta, h\alpha_1 a) + v(h\alpha_1 a, \eta)) = 0$ for all $a \in H$ except when $(\alpha_1, h) = (1, 1)$ (cf. (7)). It follows that the image by ι of the first sum in f of (12), for the present $v(\xi, \eta)$, is $-\partial g_0$ (which is equal to ιf_0). Hence $f - f_0$ is equal to the sum of the 2., 3., and 4. sums in (12), where $q(\alpha_1, c)$ are chosen so as to satisfy (9) for the present $v(\xi, \eta)$. By virtue of the above relation we may simply set $q(\alpha_1, c) = 0$ except in case $\alpha_1 = 1$. Hence, with this choice of $q(\alpha_1, c)$, the 3. and 4. sums, in question, are 0. As to the 2. sum, it is a boundary cycle, since, again by the above remark, only $h = 1$ matters there. Thus $f - f_0$ is a boundary cycle.

On the other hand, the present f is an expression (12) for $h=\varphi g_0=0$, and is hence, by virtue of Lemma 2, modulo boundary cycles a sum of cycles of form (19) with 1-cycles k in $M\otimes_{H\cap\xi H\xi^{-1}}X(H\cap\xi H\xi^{-1})$. The same holds for $f_0(=f-(f-f_0))$ by the above observation. Our proposition is thus proved.

6. Transgression map in higher dimensions

A G-module A is called (relatively) (G, H)-regular when there exists an H-module B such that A is G-isomorphic to $B\otimes_H Z[G]$, the G-module structure of the last module being defined by $(b\otimes_H\xi)\sigma=b\otimes_H\xi\sigma$. For every $n\geqq 1$, the homology group $\mathrm{H}_n(G, H, A)$ in such a module A vanishes. To prove this (well-known) fact, we recall that we have $x=\partial\psi_n x+\psi_{n-1}\partial x$ for every $x\in X_n(G, H)$ with the H-homomorphisms $\psi_m: X_m(G, H)\to X_{m+1}(G, H)$ $(m=n-1, n)$ defined by $\psi_m(\sigma_0 H, \ldots, \sigma_m H)=(H, \sigma_0 H, \ldots, \sigma_m H)$. If $\sum_{\xi\in G;x\in X_n(G,H)} b(\xi, x)\otimes_H\xi\otimes_G x$ is an n-chain in $B\otimes_H Z[G]\otimes_G X(G, H)$, we have

$$\sum_{\xi,x} b(\xi, x)\otimes_H\xi\otimes_G x=\sum_{\xi,x} b(\xi, x)\otimes_H 1\otimes_G\xi x$$
$$=\sum_{\xi,x} b(\xi, x)\otimes_H 1\otimes_G(\partial\psi_n\xi x+\psi_{n-1}\partial\xi x)=\partial\Big(\sum_{\xi,x} b(\xi, x)\otimes_H 1\otimes_G\psi_n\xi x\Big)$$
$$+\sum_{\xi,x} b(\xi, x)\otimes_H 1\otimes_G\psi_{n-1}\xi\partial x.$$

Now, for each element $\sum_{\xi,x} b(\xi, x)\otimes_H\xi\otimes_G x$ in $B\otimes_H Z[G]\otimes_G X_n(G, H)$ the element $\sum_{\xi,x} b(\xi, x)\otimes_H 1\otimes_G\psi_{n-1}\xi x$ is uniquely determined (independent of its special expression) as we readily see from the H-allowability of ψ_{n-1}. The last sum in the right-hand side of the above equation is the image of $\sum_{\xi,x} b(\xi, x)\otimes_H\xi^{-1}\otimes\partial x$ by the endomorphism of $B\otimes_H Z[G]\otimes_G X_n(G, H)$ thus obtained. It is, hence, 0 in case the chain $\sum_{\xi,x} b(\xi, x)\otimes_H\xi\otimes_G x$ is a cycle. The above equation thus shows that every n-cycle in $B\otimes_H Z[G]\otimes_G X(G, H)$ is a boundary, as was asserted. (A (G, H)-regular module A is a (G, H)-projective module in the sense of relative homological algebra (Hochschild [5]), and that $H_n(G, H, A)$ are 0 follows also from the relation $H_n(G, H, A)=\mathrm{Ext}_n^{G,H}(A, Z)$).

A $(G, 1)$-regular module is called G-regular, and is (G, H)-regular for every subgroup H. Now, with any G-module M a G-regular module $\tilde{M}$ is constructed as follows: $\tilde{M}=M\otimes Z[G]$ and its G-module structure is defined by $(u\otimes\xi)\sigma=u\sigma\otimes\xi\sigma$. Indeed, the totality of elements of form $u\otimes 1$ in $\tilde{M}$ forms a (not G-, but mere) module B and we have $\tilde{M}\approx B\otimes Z[G]$ in the sense of the construction as in the opening of this number (by the correspondence $u\otimes\xi$

$\to (u\xi^{-1} \otimes 1) \otimes \xi)$. Further the map $u \otimes \xi \to u (u \in M, \ \xi \in G)$ defines a G-epimomorphism of $\tilde{M}$ to M. Denoting the kernel of the epimorphism by N we have an exact sequence

$$(27) \qquad 0 \to N \to \tilde{M} \to M \to 0.$$

Since $\tilde{M}$ is G-regular, we have $H_r(U, \tilde{M}) = 0 \ (r \geqq 1)$ for any subgroup U of G. Hence we have

$$(28) \qquad H_r(U, M) \approx H_{r-1}(U, N) \qquad (r \geqq 2)$$

for any subgroup U of G.

We next prove

LEMMA 3. *Let*

$$(29) \qquad 0 \to N \to R \to M \to 0$$

be an exact sequence of G-modules, and assume

$$(30) \qquad H_1(U, M) = 0$$

for every subgroup U of G which is an intersection of H and its $r-1$ conjugates in G. Then we obtain an exact sequence of relative homology groups

$$(31) \qquad H_r(G, H, N) \to H_r(G, H, R) \to H_r(G, H, M) \to H_{r-1}(G, H, N)$$
$$\to H_{r-1}(G, H, R) \to \cdots \to H_1(G, H, M)$$

where the maps are those induced by maps in (29) *and the connecting homomorphisms of an exact sequence of relative chains corresponding to* (29) (*which exists because of our assumption*).

Proof. From $H_1(U, M) = 0$ we get an exact sequence

$$(32) \qquad (H_1(U, M) =)0 \to N_U \to R_U (\to M_U \to 0)$$

with maps induced by those in (29). It follows that the sequence

$$(33) \qquad 0 \to N \otimes_G X_t(G, H) \to R \otimes_G X_t(G, H) \to M \otimes_G X_t(G, H) \to 0$$

is exact, for every $t \leqq r-1$, where the maps are again induced by those in (29). For, it suffices to show that the second arrow is monomorphic, and this last follows, by our consideration in the number 1 applied to N, R in place of M, from that the map $N_U \to R_U$ is monomorphic for every subgroup U in G of form $H \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_t H \sigma_t^{-1}$. The exactness of the sequence (31) of homology groups follows now in the usual manner (from this and the trivial exactness of $R\otimes_G X_r(G, H) \to M\otimes_G X_r(G, H) \to 0$).

LEMMA 4. *Let M be a G-module such that for every $i=1, \ldots, n-1$ we have $H_i(U, M)=0$ for every subgroup U of G of form $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-i+1}H\sigma_{n-i+1}^{-1}$. Then, for every cycle h in $M\otimes_G X_{n+1}(G, H)$ there exist an element g in $M\otimes_G X_{n+1}(G)$ and a cycle f in $M\otimes_H X(H)$ satisfying*

$$h=\varphi g, \qquad \iota f=(-1)^n\partial g. \tag{34}$$

Proof. $\tilde{M}=M\otimes Z[G]$ is a regular G-module. Hence we have $H_r(G, H, \tilde{M})=0$ for all $r=1, 2, \ldots$. Also $H_r(U, \tilde{M})=0$ $(r=1, 2, \ldots)$ for every subgroup U of G. Denoting the kernel of the natural epimorphism $\tilde{M}\to M$ by N, we have thus $H_r(U, M)\approx H_{r-1}(U, N)$ for $r=2, 3, \ldots$, as has been observed before too. Hence, because of our assumption on M, we have $H_i(U, N)=0$ for $i=1, \ldots, n-2$ and for every subgroup U of G of form $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-i}H\sigma_{n-i}^{-1}$. So the G-module N satisfies the assumption of our lemma for $n-1$ in place of n (and N in place of M). Now, our lemma for $n=1$ is settled by Lemma 1. So, assume $n>1$, and assume that it is true for $n-1$ in place of n. Then it can be applied to N with $n-1$ in place of n. Further, as $H_1(U, M)=0$ for every subgroup U of G which is an intersection of H and its n conjugates, by assumption, the sequence

$$0\to N\otimes_G X_n(G, H)\to \tilde{M}\otimes_G X_n(G, H)\to M\otimes_G X_n(G, H)(\to 0) \tag{35}$$

is exact, as has been seen in the proof to Lemma 3.

Now, let h be an $n+1$-cycle in $M\otimes_G X(G, H)$. Take any counter-image $\tilde{h}$ of h by the map $\tilde{M}\otimes_G X_{n+1}(G, H)\to M\otimes_G X_{n+1}(G, H)$. Since the kernel of the map $\tilde{M}\otimes_G X_n(G, H)\to M\otimes_G X_n(G, H)$ is $N\otimes_G X_n(G, H)$, $\partial\tilde{h}$ belongs to $N\otimes_G X_n(G, H)$. Denote it by h'.

Since h' is evidently a cycle, there are, by assumption, an element g' of $N\otimes_G X_n(G)$ and a cycle f' in $N\otimes_H X_{n-1}(H)$ such that

$$h'=\varphi g', \qquad f'=-(-1)^n\partial g'. \tag{36}$$

Let f be a cycle in $M\otimes_H X_n(H)$ whose homology class is mapped, by the connecting isomorphism $H_n(H, M)\to H_{n-1}(H, N)$, to the homology class of f'. Thus, if $\tilde{f}$ is a counter-image of f by $\tilde{M}\otimes_H X_n(H)\to M\otimes_H X_n(H)$, then $\partial\tilde{f}=f'+\partial\bar{f}$ with an element $\bar{f}$ of $N\otimes_H X_n(H)$. Replacing $\tilde{f}$ with $\tilde{f}-\bar{f}$, we may assume $\partial\tilde{f}=f'$.

Hence, by (36), $-(-1)^n\partial g'=\iota f'=\iota\partial\tilde{f}=\partial\iota\tilde{f}$, showing that $(-1)^n g'+\iota\tilde{f}$ is a cycle in $\tilde{M}\otimes_G X(G)$. Since $H_n(G, \tilde{M})=0$, there exists an $n+1$-chain $\tilde{g}$ in $M\otimes_G X(G)$ satisfying

$$(-1)^n g'+\iota\tilde{f}=\partial\tilde{g}. \tag{37}$$

Then $\partial\varphi\tilde{g}-\varphi\iota\tilde{f}=(-1)^n\varphi g'=(-1)^n h'=(-1)^n\partial\tilde{h}$ and $\varphi\iota f=\partial(\varphi g-(-1)^n\tilde{h})$ is a cycle. Since $\varphi\iota\tilde{f}$ is of form $\tilde{v}\otimes_G(H, H, \ldots, H)$, this implies, as we see by writing down its boundary, $\varphi\iota\tilde{f}=0$ in case n is even. In case n is odd, on the other hand, $\varphi\iota\tilde{f}$ is a boundary and is in fact equal to $\partial\varphi\iota\tilde{k}=\varphi\partial\iota\tilde{k}$ with $\tilde{k}=\tilde{v}\otimes_H(1, 1, \ldots, 1)\in\tilde{M}\otimes_H X_{n+1}(H)$. Setting $\tilde{k}=0$ in case n is even, we have

$$\varphi\iota\tilde{f}=\varphi\partial\iota\tilde{k} \qquad (\tilde{k}\in M\otimes_H X_{n+1}(H)) \tag{38}$$

in either case. Hence $\varphi\partial\iota\tilde{k}=\partial(\varphi\tilde{g}-(-1)^n\tilde{h})$ and $\varphi\tilde{g}-(-1)^n\tilde{h}-\varphi\iota\tilde{k}$ is a cycle and, since $H_{n+1}(G, H, \tilde{M})=0$, an $n+2$-chain $\tilde{l}$ in $\tilde{M}\otimes_G X(G)$ satisfying

$$\varphi\tilde{g}-(-1)^n\tilde{h}-\varphi\iota\tilde{k}=\partial\varphi\tilde{l}. \tag{39}$$

Let $\tilde{g}$, $\tilde{l}$ be mapped to g, l by $\tilde{M}\otimes_G X(G)\to M\otimes_G X(G)$, and $\tilde{k}$ to k by $\tilde{M}\otimes_G X(H)\to M\otimes_G X(H)$. Then, by (39) and (37)

$$h=(-1)^n\varphi(g-\iota k-\partial l),\quad \partial(g-\iota k-\partial l)=\partial g-\partial\iota k=\iota f-\iota\partial k. \tag{40}$$

Denoting $(-1)^n(g-\iota k-\partial l)\in M\otimes_G X_{n+1}(G)$ and $f-\partial k\in M\otimes_H X_n(H)$ anew with g and f, we have (34). Our lemma is thus proved by induction with respect to n.

Under the assumption of Lemma 4 we obtain thus, as in the case $n=1$, a homomorphism τ of the homology group $H_{n+1}(G, H, M)$ to the residue group $H_n(H, M)_l$ of $H_n(H, M)$ modulo the subgroup consisting of homology classes containing cycles f such that $\iota f=(-1)^n\partial g$, $\varphi g=0$ with some $g\in M\otimes_G X_{n+1}(G)$.

We remark that the kernel of our residue homomorphism $H_n(H, M)\to H_n(H, M)_l$ is contained, evidently, in the kernel of $\iota: H_n(H, M)\to H_n(G, M)$. Thus we obtain a homomorphism $H_n(H, M)_l\to H_n(G, M)$ which we denote also by ι.

LEMMA 5. *Under the assumption of Lemma 4, the kernel of $H_n(H, M)\to H_n(H, M)_l$ corresponds to the kernel of $H_{n-1}(H, N)\to H_{n-1}(H, N)_l$ by the* *connecting isomorphism* $H_n(H, M) \approx H_{n-1}(H, N)$.

Proof. Let f' be an $n-1$-cycle in $N \otimes_H X(H)$ such that there exists an n-chain g' in $N \otimes_G X(G)$ satisfying $\iota f' = -(-1)^n \partial g'$, $\partial g' = 0$. We may apply our proof to Lemma 4 to $h' = 0$, $\tilde{h} = 0$, $h = 0$ and to the present f'. Thus, there is an n-chain $\tilde{f}$ in $\tilde{M} \otimes_H X(H)$ with $\partial \tilde{f} = f'$, and for the image f of $\tilde{f}$ in $M \otimes_H X(H)$ we have $\iota f - \iota \partial k = \partial(g - \iota k - \partial l)$, $(-1)^n \varphi(g - \iota k - \partial l) = h = 0$ with suitable elements g, k, l of $M \otimes_G X_{n+1}(G)$, $M \otimes_H X_{n+1}(H)$, $M \otimes_G X_{n+2}(G)$, respectively. Hence the homology class of f, i.e. that of $f - \partial k$, which corresponds to the homology class of f' by the isomorphism $H_{n-1}(H, N) \approx H_n(H, M)$, is in the kernel of $H_n(H, M) \to H_n(H, M)_I$.

Let conversely f be an n-cycle in $M \otimes_H X(H)$ such that $\iota f = \partial g$ with an $n+1$-chain g in $M \otimes_G X(G)$ satisfying $\varphi g = 0$. Let $\tilde{f} \in \tilde{M} \otimes_H X_n(H)$, $\tilde{g} \in M \otimes_G X_{n+1}(G)$ be mapped to f, g by the maps of respective complexes induced by $\tilde{M} \to M$. Since $\iota f = \partial g$, we have $\iota \tilde{f} - \partial \tilde{g} \in N \in_G X_n(G)$. Further, since $\partial f = 0$, we have $\partial \tilde{f} \in N \otimes_H X_{n-1}(H)$. We denote $\partial \tilde{f}$ by f'. Then we have $\varphi \iota f' = \varphi \iota \partial k'$ with an element k' in $N \otimes_H X_n(H)$; cf. the similar argument we made in our proof to Lemma 4 with respect to $\varphi \iota \tilde{f}$ there. $f' - \partial k'$ is an $n-1$-cycle in $N \otimes_H X(H)$ and satisfies $\iota(f' - \partial k') = \partial g'$ with $g' = (\iota \tilde{f} - \partial \tilde{g}) - \iota k' \in N \otimes_G X_n(G)$. Here $\partial \varphi g' = \varphi \iota f' - \varphi \iota \partial k' = 0$ and $\varphi g'$ is an n-cycle in $N \otimes_G X(G, H)$. To prove $\varphi g' \sim 0$ (in $N \otimes_G X(G, H)$), let $\tilde{h}$ be an $n+1$-chain in $\tilde{M} \otimes_G (G, H)$ satisfying $\varphi g' = \partial \tilde{h}$ and h be its image in $M \otimes_G X(G, H)$. (The following argument is to recast our proof to Lemma 4 with respect to the present h). Since $\partial \varphi \iota(\tilde{f} - k') = \varphi \iota f' - \varphi \iota \partial k' = 0$, there exists, by an argument we have made above already twice (with respect to $\varphi \iota \tilde{f}$, in the old notation, and to $\varphi \iota f'$), an $n+1$-chain $\tilde{k}$ in $M \otimes_H X(H)$ with $-(-1)^n \varphi \iota(\tilde{f} - k') = \varphi \partial \iota \tilde{k}$. Then, as $\partial(-(-1)^n \varphi \tilde{g} - (-1)^n \tilde{h} - \varphi \iota \tilde{k}) = -(-1)^n \varphi \partial \tilde{g} - (-1)^n \varphi g' + (-1)^n \varphi \iota(\tilde{f} - k') = -(-1)^n \varphi \partial \tilde{g} - (-1)^n \varphi(\iota \tilde{f} - \partial \tilde{g} - \iota k') + (-1)^n \varphi \iota(\tilde{f} - k') = 0$, there exists an $n+2$-chain $\tilde{l}$ in $\tilde{M} \otimes_G X(G)$ with $-(-1)^n \varphi \tilde{g} - (-1)^n \tilde{h} - \varphi \iota \tilde{k} = \partial \varphi \tilde{l}$. Let l be the image of $\tilde{l}$ in $M \otimes_G X(G)$, and k the image of $\tilde{k}$ in $M \otimes_H X(H)$. Then

$$h = -\varphi g - (-1)^n \varphi \iota k - (-1)^n \partial \varphi l = -(-1)^n (\varphi \iota k + \partial \varphi l);$$

observe that $\varphi g = 0$. $\varphi \iota k$ is a cycle and ~ 0 by the argument we have used repeatedly. Hence $h \sim 0$. Since the homology class of h corresponds to that of $\varphi g'$ by the isomorphism of $H_{n+1}(G, H, M)$ and $H_n(G, H, N)$, we have $\varphi g'$

~ 0. It follows that the homology class of $f' - \partial k'$, i.e. that of f', belongs to the kernel of $H_{n-1}(H, N) \to H_{n-1}(H, N)_I$.

We have further

LEMMA 6. *Under the assumption of Lemma* 4, *the kernel of the residue class homomorphism* $H_n(H, M) \to H_n(H, M)_I$ *coincides with the subgroup of* $H_n(H, M)$ *generated by the homology classes of cycles of form* (19) *with n-cycles* k *in* $M \otimes_{H \cap \xi H \xi^{-1}} X(H \cap \xi H \xi^{-1})$, ξ *ranging in* G.

Proof. Assume the assertion for $n-1$ in place of n. Then the kernel of the homomorphism $H_{n-1}(H, N) \to H_{n-1}(H, N)_I$ is the subgroup of $H_{n-1}(H, N)$ generated by the cycles of form (19) with cycles k in $N \otimes_{H \cap \xi H \xi^{-1}} X_{n-1}(H \cap \xi H \xi^{-1})$. This subgroup of $H_{n-1}(H, N)$ evidently corresponds by the isomorphism $H_{n-1}(H, N) \approx H_n(H, M)$ to the subgroup of $H_n(H, M)$ in our lemma. On the other hand, the said kernel corresponds, by the same isomorphism, to the kernel in our lemma in virtue of Lemma 6. Hence the kernel and the subgroup in our lemma coincide, and the lemma is proved by induction.

7. Fundamental exact sequence for higher dimensions

Now we have our main

THEOREM. *Let M be a G-module such that for* $i = 1, \ldots, n-1$ *we have* $H_i(U, M) = 0$ *for every subgroup U of G of form* $U = H \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_{n-i+1} H \sigma_{n-i+1}^{-1}$. *Then the subgroup of* $H_n(H, M)$ *generated by homology classes of cycles of form* (19) *with n-cycles k in* $M \otimes_{H \cap \xi H \xi^{-1}} X(H \cap \xi H \xi^{-1})$, ξ *ranging in G, coincides with the totality of homology classes represented by cycles f satisfying* $\iota f = \partial g$ *with some* $g \in M \otimes_G X_{n+1}(G)$ *such that* $\varphi g = 0$, *and we have, when we denote the residue group of* $H_n(H, M)$ *modulo this subgroup by* $H_n(H, M)_I$, *the exact sequence*

$$0 \longleftarrow H_n(G, H, M) \stackrel{\varphi}{\longleftarrow} H_n(G, M) \stackrel{\iota}{\longleftarrow} H_n(H, M)_I \stackrel{\tau}{\longleftarrow} H_{n+1}(G, H, M) \stackrel{\varphi}{\longleftarrow} H_{n+1}(G, M) \tag{41}$$

where τ is the map which maps a homology class in $H_{n+1}(G, H, M)$ *represented by a cycle h to the class in* $H_n(H, M)_I$ *of a homology class represented by a cycle f satisfying* (34) *with some g in* $M \otimes_G X_{n+1}(G)$, *and where ι is the map induced by the ordinary injection map* $\iota : H_n(H, M) \to H_n(G, M)$.

Proof. The case $n=1$ is settled by Proposition 1, and we want to prove our theorem by induction on n. Thus, let $n>1$. Consider again the G-module $\tilde{M}=M\otimes Z[G]$ and let N be the kernel of the natural epimomorphism $\tilde{M}\to M$. As in (28) we have an isomorphism

$$\partial^*: \quad H_r(U, M)\to H_{r-1}(U, N) \tag{42}$$

for every $r\geqq 2$ and every subgroup U of G. In particular, we have isomorphisms

$$\begin{gathered} H_n(G, M)\to H_{n-1}(G, N), \quad H_n(H, M)\to H_{n-1}(H, N), \\ H_{n+1}(G, M)\to H_n(G, N), \end{gathered} \tag{43}$$

which we denote all by ∂^*. Further, the isomorphisms (42) for $r=2, \ldots, n$ and for subgroups U of form $H\cap\sigma_1 H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-r+1}H\sigma_{n-r+1}^{-1}$ shows, in combination with our assumption, that $H_i(U, N)=0$ for $i=1, \ldots, n-2$ and for subgroups U of form $H\cap\sigma_1 H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-i}H\sigma_{n-i}^{-1}$.

Moreover, by Lemma 3 we obtain the exact sequence

$$\begin{aligned} &H_{n+1}(G, H, \tilde{M})\to H_{n+1}(G, H, M)\to H_n(G, H, N)\to H_n(G, H, \tilde{M}) \\ &\quad\to H_n(G, H, M)\to H_{n-1}(G, H, N)\to H_{n-1}(G, H, \tilde{M}). \end{aligned}$$

Here the 1st, 4th and 7th terms are 0, again because $\tilde{M}$ is G-regular. Hence we have isomorphisms

$$\begin{aligned} &H_{n+1}(G, H, M)\to H_n(C, H, N), \\ &H_n(G, H, M)\to H_{n-1}(G, H, N) \end{aligned} \tag{44}$$

which may be denoted both by ∂^* too.

Now, the isomorphism ∂^* between $H_n(H, M)$ and $H_{n-1}(H, N)$ induces a same of the residue groups $H_n(H, M)_I$ and $H_{n-1}(H, N)_I$ as was seen in Lemma 5. It is seen from our proof to Lemma 4, that the diagram

$$\begin{tikzcd} H_{n-1}(H, N)_I & H_n(G, H, N) \arrow[l, "\tau"'] \\ H_n(H, M)_I \arrow[u, "\partial^*"] & H_{n+1}(G, H, M) \arrow[l, "\tau"'] \arrow[u, "\partial^*"'] \end{tikzcd} \tag{45}$$

is commutative, where the left vertical arrow, denoted by ∂^* too, is the isomorphism induced by the isomorphism ∂^* of $H_n(H, M)$ and $H_{n-1}(H, N)$. In fact, we see readily that also other squares in the diagram connecting the sequence

(46)
$$0 \longleftarrow H_{n-1}(G, H, N) \overset{\varphi}{\longleftarrow} H_{n-1}(G, N) \overset{\iota}{\longleftarrow} H_{n-1}(H, N)$$
$$\overset{\tau}{\longleftarrow} H_n(G, H, N) \overset{\varphi}{\longleftarrow} H_n(G, N)$$

with our sequence (41) by the isomorphisms ∂^* are commutative; thus the said diagram is commutative. Now, the sequence (46) is exact, by our induction assumption. It follows that (41) is exact too. Our theorem is thus proved.

Remark. As our proof shows, a partial exact sequence

(47)
$$0 \longleftarrow H_n(G, H, M) \overset{\varphi}{\longleftarrow} H_n(G, M) \overset{\iota}{\longleftarrow} H_n(H, M)$$

holds under a weaker assumption that for $i=1, \ldots, n-1$ we have $H_i(U, M) = 0$ for subgroups U of form $U = H \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_{n-i} H \sigma_{n-i}^{-1}$.

Appendix. Fundamental exact sequences for finite groups

Let now H be a subgroup of finite index in G. Then both the homology and cohomology groups $H_n(G, H, M)$, $H^n(G, H, M)$ are defined for all values $n \gtreqless 0$. In our previous note [9] we gave a certain condition under which we have the relation $H_n(G, H, M) \approx H^{-n-1}(G, H, N)$ ($n \gtreqless 0$) for two G-modules M, N. In re-examining our proof and in observing the dimensions concerned in its steps, we see the followings:

1) Let first $n \geqq 0$. Let for each subgroup K in G of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_n H \sigma_n^{-1}$ there be given a homomorphism $\kappa(K)$: $M \to N$ (written as *right* operator on M contrary to [9]) which naturally induces a homomorphism $M_K \to N^K$ (or, more precisely, such that the kernel of $\kappa(K)$ contains that of $M \to M_K$ and the image of $\kappa(K)$ is contained in N^K) and let $\kappa(\sigma^{-1} K \sigma) = \sigma^{-1} \kappa(K) \sigma$ hold for all $\sigma \in G$ (and such K). Then a natural homomorphism

(48)
$$M \otimes_G X_n \to (N \otimes_Z X_n)^G$$

is defined (by (9) $\to$ (11) in the notation of [9]), where X_n stands for $X_n(G, H)$. If the homomorphism $M_K \to N^K$ induced by $\kappa(K)$ is an epimorphism (resp. a monomorphism) for every such K, then the homomorphism (48) is also an epimorphism (resp. monomorphism). Further, if $n \geqq 1$ and if

(49)
$$\sum_{L \ni \rho \ r.\mathrm{mod}\ K} \kappa(K) \rho^{-1} = \kappa(L)$$

always holds for $K=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_n H\sigma_n^{-1}$ and $L=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_{n-1}H\sigma_{n-1}^{-1}$, then the diagram

$$
\begin{array}{ccc}
M\otimes_G X_n & \longrightarrow & (N\otimes_Z X_n)^G \approx \mathrm{Hom}_G(X_{-n-1},\ N) \\
\partial\big\downarrow & & \big\downarrow\delta \\
M\otimes_G X_{n-1} & \longrightarrow & (N\otimes_Z X_{n-1})^G \approx \mathrm{Hom}_G(X_{-n},\ N)
\end{array}
\tag{50}
$$

is commutative (where the horizontal arrows are the homomorphisms for n and $n-1$ just introduced and the isomorphisms $\approx$ are canonical ones).

2) Let next $n\leqq -1$. Let for each subgroup K in G of form $K=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_{-n-1}M\sigma_{-n-1}^{-1}$ there be given a homomorphism $\kappa(\mathrm{K}):M\to N$ which naturally induces a homomorphism $M_K\to N^K$, and let $\kappa(\sigma^{-1}K\sigma)=\sigma^{-1}\kappa(K)\sigma$ hold for all $\sigma\in G$. Then a natural homomorphism (48) is defined (again by (9) $\to$(11) in the notation of [9]), where X_n stands for the n-component $X_n(G,H)$ of the complete standard complex of $(G,\ H)$ and is thus the G-module $\mathrm{Hom}_Z(X_{-n-1},\ Z)$. Again, if the induced homomorphism $M_K\to N^K$ is an epimorphism (resp. a monomorphism) for every such K, the homomorphism (48) is an epimorphism (resp. a monomorphism). Also, if (49) holds for every pair $K=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_{-n}H\sigma_{-n}^{-1}$, $L=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_{-n-1}H\sigma_{-n-1}^{-1}$, then the diagran (50) is commutative (where the horizontal arrows are the homomorphisms for n and $n-1$).

3) Let there be given a normal subgroup K_0 of finite index in G which is contained in H and a G-homomorphism $\kappa_0:M\to N$ which induces as homomorphism $M_{K_0}\to N^{K_0}$. Set $\kappa(K)=\sum_{K\ni\rho\ \mathrm{mod}\ K_0}\kappa_0\rho^{-1}$ for every subgroup K in G of form $K=\sigma_0 H\sigma_0^{-1}$. Then the diagram (50) with $n=0$ is commutative when the horizontal arrows are the homorphisms defined (with respect present $\kappa(K)$) in 1), 2) for $n=0,\ -1$ respectively.

Now, assume that G itself is finite. For $M=N$ we can then get a system of endomorphisms $\kappa(K)$ of M for subgroups K in 1), 2) or 3) (and $K_0=1$ in 3)) by setting $\kappa(K)=\sum_{\sigma\in K}\sigma$; it is evident that $\kappa(K)$ induces a homomorphism $M_K\to M^K$. The (sufficient) condition for the commutativity of (50), in 3) and in the latter parts of 1), 2), holds also evidently. Let first $n=\geqq 0$. If $H^0(K,M)=0$ (resp. $H^{-1}(K,M)=0$) for every K of form $K=\sigma_0 H\sigma_0^{-1}\cap\cdots\cap\sigma_n H\sigma_n^{-1}$, then our homomorphism $\kappa(K)$ for K induces an epimorphism (resp. a monomorphism) (of M_K to M^K whence) of $M\otimes_G X_n$ to $\mathrm{Hom}_G(X_{-n-1},\ N)$. Hence, if

$n \geqq 1$ and if $H^0(K, M) = 0$ for every K of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_{n+1} H \sigma_{n+1}^{-1}$, and $H^{-1}(K, M) = 0$ for every K of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_n H \sigma_n^{-1}$, then we obtain, as we can see easily, $H_n(G, H, M) \approx H^{-n-1}(G, H, M)$. Let next $n \leqq 1$. If $H^0(K, M) = 0$ (resp. $H^{-1}(K, M) = 0$) for every K of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_{-n-1} H \sigma_{-n-1}^{-1}$, then our homomorphism $\kappa(K)$ induces an epimorphism (resp. a monomorphism) (of M_K to M^K whence) of $M \otimes_G X_n$ to $\mathrm{Hom}_G(X_{-n-1}, N)$. Hence if $n \leqq -2$ and if $H^0(K, M) = 0$ for every K of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_{-n-1} H \sigma_{-n-1}^{-1}$ and $H^{-1}(K, M) = 0$ for every K of form $K = \sigma_0 H \sigma_0^{-1} \cap \cdots \cap \sigma_{-n} H \sigma_{-n}^{-1}$, then we obtain $H_n(G, H, M) \approx H^{-n-1}(G, H, M)$. Further, taking also the case $n = -1$ or 0 respectively into consideration in the case $n = 0$ or -1, we see that the above statement for $n \geqq 1$ rep. $\leqq -2$ remains valid for $n = 0$ or -1 respectively.

Now, we continue to assume that G is a finite group. Then, besides the exact sequences given in our theorem and their cohomological duals, we have the following sequences:

Let M be a G-module such that for $i = 0, \ldots, n-1$ we have $H_{-i}(U, M) = 0$ for every subgroup U of G of form $U = H \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_{n-i+1} H \sigma_{n-i+1}^{-1}$. Then the subgroup of $H_{-n}(H, M)$ generated by homology classes of (transgressive) cycles f satisfying $f = \rho g$, $\lambda h = (-1)^{n-1} \partial g$ with some element g in $M \otimes_G X_{-n}(G)$ and some cycle h in $M \otimes_G X_{-n-1}(G, H)$ coincides with the subgroup of $H_{-n}(H, M)$ consisting of all stable homology classes, i.e. homology classes of cycles f such that $\rho_{H, H \cap \xi^{-1} H \xi} f - \rho_{\xi^{-1} H \xi, H \cap \xi^{-1} H \xi} T_\xi f \sim 0$ for every $\xi \in G$, and if we denote this subgroup by $H_{-n}(H, M)^I$ *then we have the exact sequence*

$$(51) \qquad 0 \longrightarrow H_{-n}(G, H, M) \xrightarrow{\lambda} H_{-n}(G, M) \xrightarrow{\rho} H_{-n}(H, M)^I$$
$$\xrightarrow{\tau} H_{-n-1}(G, H, M) \xrightarrow{\lambda} H_{-n-1}(G, M)$$

where τ is the map mapping the homology class of a transgressive cycle f as above to the homology class of the cycle h.

Dually: Let M be a G-module such that for $i = 0, \ldots, n-1$ we have $H^{-i}(U, M) = 0$ for every subgroup U of G of form $U = H \cap \sigma_1 H \sigma_1^{-1} \cap \cdots \cap \sigma_{n-i+1} H \sigma_{n-i+1}^{-1}$. Then the subgroup of $H^{-n}(H, M)$ generated by cohomology classes of cocycles f satisfying $\iota f = \delta g$ with some g such that $\varphi g = 0$ coincides with the subgroup of $H^{-n}(H, M)$ consisting of all cohomology classes of cocycles of form (19) with cocycles k in $\mathrm{Hom}_{H\cap\xi H\xi^{-1}}(X_{-n}(H\cap\xi H\xi^{-1}), M)$, and *we have the exact sequence*

(52)
$$0\longleftarrow H^{-n}(G, H, M)\overset{\varphi}{\longleftarrow}H^{-n}(G, M)\overset{\iota}{\longleftarrow}H^{-n}(H, M)_I$$
$$\overset{\tau}{\longleftarrow}H^{-n-1}(G, H, M)\overset{\varphi}{\longleftarrow}H^{-n-1}(G, M)$$

where $H^{-n}(H, M)_I$ is the residue group of $H^{-n}(H, M)$ modulo the said subgroup, ι is the map induced by the ordinary injection, and τ is the map mapping the cohomology class of a cocycle h in $\mathrm{Hom}_G(X_{-n-1}(G, H), M)$ to the class in $H^{-n}(H, M)_I$ of the cohomology class of acycle f in $\mathrm{Hom}_H(X_{-n}(H), M)$ such that there exists an element g in $\mathrm{Hom}_G(X_{-n-1}(G), M)$ satisfying $\iota f=(-1)^n\delta g$, $h=\varphi g$.

But, by virtue of the refinement of the result in [9] we made above the case $n\geqq 2$ of (51) can be derived from the sequence (41) (with $n-1$ instead of n) in our theorem and similarly the case $n\geqq 2$ of (52) can be derived from the dual of (41). Indeed, if $n\geqq 2$ the above assumption for (52), for example, includes $H^0(U, M)=0$ for every subgroup U of G of form $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n+1}H\sigma_{n+1}^{-1}$ and $H^{-1}(U, M)=0$ for every U of form $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_nH\sigma_n^{-1}$. In particular, $H^0(U, M)=H^{-1}(U, M)=0$ for every U of form $H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-1}H\sigma_{n-1}^{-1}$. Under this condition, however, the remaining part of our assumption is that for $i=1,\cdots,n-2$ we have $H_i(U, M)=0$ for every subgroup U of G of form $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n-i}H\sigma_{n-i}^{-1}$. Then we have, by our theorem, the exact sequence (41) with n replaced by $n-1$. This sequence is, however, nothing else than (52) because of the relations $H^0(U, M)=0$, $H^{-1}(U, M)=0$ for $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_{n+1}H\sigma_{n+1}^{-1}$, $U=H\cap\sigma_1H\sigma_1^{-1}\cap\cdots\cap\sigma_nH\sigma_n^{-1}$ respectively, as our refinement of the result in [9] we made above shows.

The case $n=0$ can be verified directly (cf. [7] where the normal case of the sequences (51) and (52) was treated), and the case $n=1$ may be derived from it by the argument similar to our transision to higher dimensions in 6, 7. (and the case $n\geqq 2$ can be derived from these again by the same recursive argument).

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