ON ABSOLUTELY SEGREGATED ALGEBRAS

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Cohomology groups of (associative) algebras have been introduced (for higher dimensions) and studied by G. Hochschild in his papers [2], [3] and [4]. 1-, 2-, and 3-dimensional cohomology groups are in closest connection with some classical properties of algebras. In particular, an algebra is absolutely segregated[1] if and only if its 2-dimensional cohomology groups are all trivial. It is thus of use and importance to determine the structure of algebras with universally vanishing 2-cohomology groups, i.e. absolutely segregated algebras; they form a class which is wider than the class of all algebras with universally vanishing 1-cohomology groups, i.e. separable algebras in the sense of the Dickson-Wedderburn theorem.

In the present note we offer a structural characterization of absolutely segregated algebras. As the preliminary we consider some simple lemmas on M_0-modules of an algebra (Definition 1) which have been studied by W. Gaschütz[2] in the case of finite groups and by H. Nagao, T. Nakayama,[3] and the writer[4] in the case of algebras (§1). Combining these lemmas with a criterion for an algebra to have trivial m-dimensional cohomology groups, obtained by G. Hochschild in terms of Hochschild modules (Definition 3), we can refine Hochschild's criterion and show that the m-dimensional cohomology groups of an algebra are all trivial if and only if the same holds for A_K, Where K is an extension of the ground field of A (§2). Next, after showing that A is absolutely segregated if and only if the basic algebra of A is so (§3), we show a direct decomposition of the Hochschild module of the basic algebra of A into two-sided modules (§4). Then, by the direct analysis of Hochschild modules, we have our structural characterization of absolutely segregated algebras (§5).

The writer wishes to express his gratitude to Professor T. Nakayama for his valuable suggestions.

§1. M_0-modules of an algebra

Let A be, throughout this paper, an associative algebra with a finite rank

Received April 17, 1953.

1) An algebra A is called absolutely segregated if any algebra B containing a two-sided ideal C such that $B/C\cong A$ contains a subalgebra A' with $B=C+A'$.

2) Gaschütz [1].

3) Nagao and Nakayama [6].

4) Ikeda [5].

over a field F. Moreover we assume, without mentioning each time, that A has unit element 1. Let

$$A=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}Ae_{\kappa,i}=\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}A$$

be direct decompositions of A into indecomposable left or right ideals respectively. Here $e_{\kappa,i}$ are primitive idempotents such that $\sum_{\kappa=1}^{n}\sum_{i=1}^{f(\kappa)}e_{\kappa,i}=1$ and $Ae_{\kappa,i} \cong Ae_{\lambda,j}$ $(e_{\kappa,i}A\cong e_{\lambda,j}A)$ if and only if $\kappa=\lambda$. For the sake of brevity, we write $e_{\kappa,1}=e_{\kappa}$ for each κ. We use, moreover, matric units $c_{\kappa,i,j}$ with $c_{\kappa,i,j}c_{\lambda,h,k} = \delta_{\kappa,\lambda}\delta_{j,h}c_{\kappa,i,k}$, $c_{\kappa,i,i}=e_{\kappa,i}$ for κ, $\lambda=1,\ldots,n$; i, $j=1\ldots f(\kappa)$ and h, $k=1,\ldots, f(\lambda)$.

DEFINITION 1. Let $\mathfrak{M}$ be an A-module (one-sided or two-sided). $\mathfrak{M}$ is called an *M_0-module* if, for any A-module $\mathfrak{N}$ containing an A-submodule $\mathfrak{N}'$ such that $\mathfrak{N}/\mathfrak{N}'\cong\mathfrak{M}$, there exists an A-submodule $\mathfrak{N}''$ of $\mathfrak{N}$ such that $\mathfrak{N}$ is the direct sum $\mathfrak{N}=\mathfrak{N}'+\mathfrak{N}''$.

Then we can easily verify

LEMMA 1. *Let $\mathfrak{M}$ be an A-left module. If $\mathfrak{M}=\mathfrak{M}_1+\mathfrak{M}_2$ is a direct decomposition of $\mathfrak{M}$ into A-left modules $\mathfrak{M}_1$ and $\mathfrak{M}_2$, then $\mathfrak{M}$ is an M_0-module if and only if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are M_0-modules.*

Recently H. Nagao and T. Nakayama[5] proved

LEMMA 2. *If 1 acts as the identity operator on an A-left module $\mathfrak{M}$, then $\mathfrak{M}$ is an M_0-module if and only if $\mathfrak{M}$ is a restricted direct sum of A-submodules isomorphic to indecomposable left ideals Ae_{κ} of A.*

By Lemma 2 we have

LEMMA 3. *If $\mathfrak{M}$ is an A-left module with finite rank over F on which 1 acts as the identity operator, then $\mathfrak{M}$ is an M_0-module of A if and only if $\mathfrak{M}_K$ is an M_0-module of A_K, where K is an extension of F.*

Proof. The "only if" part is trivial. We prove the "if" part. Assume that $\mathfrak{M}_K$ is an M_0-module of A_K. Then, by Lemma 2, $\mathfrak{M}_K$ is a direct sum of finite number of A_K-submodules isomorphic to indecomposable left ideals of A_K, say $\mathfrak{M}_K\cong\sum_{i=1}^{r}\mathfrak{m}_i$, $\mathfrak{m}_i\cong A_K\tilde{e}_{\kappa_i}$. Now, since $\tilde{e}_{\kappa_i}$ is a primitive idempotent of A_K, we can assume that $A_K\tilde{e}_{\kappa_i}$ appears as a direct component of $(Ae_{\lambda})_K$ for suitable e_{λ}. Since $(Ae_{\lambda})_K$ is a restricted direct sum of A-modules isomorphic to Ae_{λ}, it is an M_0-module of A. Therefore, by Lemma 1, its direct component $A_K\tilde{e}_{\kappa_i}$ is also an M_0-module of A. Then, being the direct sum of submodules isomorphic to

[5] Cf. Nagao and Nakayama [6].

$A_K\tilde{e}_{\kappa_i}$, $\mathfrak{M}_K$ is an M_0-module of A. Since $\mathfrak{M}_K$ is a direct sum of $\mathfrak{M}$ and a suitable A-submodule $\mathfrak{M}'$, $\mathfrak{M}$ is an M_0-module of A.

As for A-two-sided modules,[6] we can consider them as $A\times A'$-left modules where A' is an algebra anti-isomorphic to A, and the above lemmas hold also for them.

§ 2. Hochschild modules and absolutely segregated algebras

Now we turn to lemmas from the cohomology theory.[7]

DEFINITION 2. Let $\mathfrak{M}$, $\mathfrak{N}$ be A-two-sided modules. Then we call an A-two-sided module $\mathfrak{L}$ an extension module of $\mathfrak{N}$ by $\mathfrak{M}$ if $\mathfrak{L}\supset\mathfrak{N}$ and $\mathfrak{L}/\mathfrak{N}\cong\mathfrak{M}$. If a direct decomposition $\mathfrak{L}=\mathfrak{N}+\mathfrak{M}'$ holds with an A-right submodule $\mathfrak{M}'$, which is necessarily (A-right) isomorphic to $\mathfrak{M}$, then we say that $\mathfrak{L}$ is a *right inessential* extension. If a direct decomposition $\mathfrak{L}=\mathfrak{N}+\mathfrak{M}''$ holds with an A-two-sided submodule $\mathfrak{M}''$, which is necessarily isomorphic to $\mathfrak{M}$, then we say the extension splits.

LEMMA 4. (*Hochschild*) *Let $\mathfrak{M}$, $\mathfrak{N}$ be A-two-sided modules. Then every right inessential extension of $\mathfrak{N}$ by $\mathfrak{M}$ splits if and only if $H^{(1)}$ $(A\ ;\ R(\mathfrak{M},\mathfrak{N}))=0$, where $R(\mathfrak{M},\mathfrak{N})$ is an A-two-sided module consisting of right operator homomorphisms of $\mathfrak{M}$ into $\mathfrak{N}$ and the operation of an element a of A on $R(\mathfrak{M},\mathfrak{N})$ is defined by* $(a*\lambda)(m)=a\lambda(m)$, $(\lambda*a)(m)=\lambda(am)$ $(m\in\mathfrak{M},\ \lambda\in R(\mathfrak{M},\mathfrak{N}))$.

DEFINITION 3. Let $P_m=A\otimes\ldots\otimes A$ be the m-fold direct product of the underlying vector space of A. We make P_m into an A-two-sided module as follows: Let $A\in a_0$, $P_m\ni a_1\otimes\ldots\otimes a_m$. Then we define

$$(a_1\otimes\ldots\otimes a_m)*a_0=a_1\otimes\ldots\otimes a_ma_0 \quad\text{and}$$
$$\begin{aligned}a_0*(a_1\otimes\ldots\otimes a_m)=a_0a_1\otimes\ldots\otimes a_m-a_0\otimes a_1a_2\otimes\ldots\otimes a_m+\ldots\\ \ldots+(-1)^ra_0\otimes\ldots\otimes a_ra_{r+1}\otimes\ldots\otimes a_m+\ldots\\ +(-1)^{m-1}a_0\otimes a_1\otimes\ldots\otimes a_{m-1}a_m.\end{aligned}$$

We call P_m thus defined the *m-dimensional Hochschild module* of A.

In distinction from ordinary direct products, we use the notation $\otimes$ for the Hochschild module P_m, while we use the notation $\times$ for ordinary direct products of two-sided modules, that is, $A^{(m)}=A_1\times\ldots\times A_m$ is an A-two-sided module under the operation $a_0(a_1\times\ldots\times a_m)=a_0a_1\times\ldots\times a_m$ and $(a_1\times\ldots\times a_m)a_0=a_1\times\ldots\times a_ma_0$.

LEMMA 5. (*Hochschild*) *The m-dimensional cohomology groups of A are all trivial if and only if every right inessential extension of any A-two-sided*

[6] "A-two-sided module" means "A-double module" (A-Doppelmodul). Namely a module $\mathfrak{M}$ is an A-two-sided module if $\mathfrak{M}$ is an A-right as well as A-left module and satisfies $(am)b=a(mb)$. $(a,\ b\in A,\ m\in\mathfrak{M})$.

[7] Lemmas 4, 5 and 10 are in Hochschild [4].

module by P_m splits.

Since P_m is an M_0-module as an A-right module, every extension of any A-two-sided module by P_m is right inessential. Therefore

LEMMA 6. *The m-dimensional cohomology groups of A are all trivial if and only if the m-dimensional Hochschild module P_m of A is an A-two-sided M_0-module.*

LEMMA 7. *Let $\mathfrak{M}$ be an A-two-sided module. If $\mathfrak{M}$ is an M_0-module as an A-right module and if $1\mathfrak{M}=0$, then $\mathfrak{M}$ is an A-two-sided M_0-module.*

Proof. Since every extension of any two-sided module $\mathfrak{N}$ by $\mathfrak{M}$ is right inessential, it is sufficient to show that $H^{(1)}(A\,;\,R(\mathfrak{M},\,\mathfrak{N}))=0$. From the definition, we have $(\lambda * a)(m)=\lambda(am)=0$ for every $\lambda \in R(\mathfrak{M},\,\mathfrak{N})$ and $m \in \mathfrak{M}$. Therefore $R(\mathfrak{M},\,\mathfrak{N})*A=0$. Let ρ be a 1-cocycle from A into $R(\mathfrak{M},\,\mathfrak{N})$. Then $\delta\rho(a,\,b)=a*\rho(b)-\rho(ab)+\rho(a)*b=0$. Since $R(\mathfrak{M},\,\mathfrak{N})*A=0$, we have $a*\rho(b)=\rho(ab)$. This shows that ρ is an operator homomorphism of A into $R(\mathfrak{M},\,\mathfrak{N})$. Since A has unit element 1, $\rho(a)=a*\rho(1)=a*\rho(1)-\rho(1)*a=(\delta\rho(1))(a)$. Thus any 1-cocycle is a coboundary.

Since $P_m=1*P_m+P_m^{(0)}$ where $P_m^{(0)}$ is the two-sided submodule of P_m consisting of elements annihilated by 1 on the left-hand side, we have, by Lemmas 6 and 7,

LEMMA 8. *The m-dimensional cohomology groups of A are all trivial if and only if $1*P_m$ is an A-two-sided M_0-module, that is, $1*P_m$ is isomorphic to a direct sum of indecomposable left ideals of $A\times A'$.*

On the other hand we have, from Lemmas 3 and 6,

LEMMA 9. *Let K be an extension of F. Then the m-dimensional cohomology groups of A are all trivial if and only if the m-dimensional cohomology groups of A_K are all trivial.*

DEFINITION 4. An algebra A is called *absolutely segregated* if any algebra B containing a two-sided ideal C such that $B/C\cong A$ contains a subalgebra A' with $B=C+A'$.

Then

LEMMA 10. (*Hochschild*) *An algebra A is absolutely segregated if and only if the 2-dimensional cohomology groups of A are all trivial.*

By Lemmas 9 and 10, we have

PROPOSITION 1. *An algebra A is absolutely segregated if and only if A_K is absolutely segregated, where K is an extension of F. If A is an algebra over an algebraic closed field, then A is absolutely segregated if and only if $1*P_2$ is isomorphic to a direct sum of A-two-sided modules isomorphic to the modules of the form $Ae_\kappa\times e_\lambda A$.*

Now we give the next proposition which gives the relation between $1*P_m$ and $A^{(m)}$.

PROPOSITION 2.[8] *By the correspondence* $a_1\times\ldots\times a_m\to a_1*(a_2\otimes\ldots\otimes a_m)$, $A^{(m)}$ *is mapped homomorphically onto* $1*P_{m-1}$ *and the kernel of this homomorphism is isomorphic to* $1*P_m$.

Proof. The above mapping is obviously "onto." Since $(a_0a_1)*(a_2\otimes\ldots\ldots\otimes a_m)=a_0*(a_2\otimes\ldots\otimes a_m))$ and $a_1*(a_2\otimes\ldots\otimes a_ma_{m+1})=a_1*((a_2\otimes\ldots\ldots\otimes a_m)*a_{m+1})=(a_1*(a_2\otimes\ldots\otimes a_m))*a_{m+1}$, this is an A-homomorphism. Since $1*(a_1\otimes\ldots\otimes a_m)=a_1\otimes\ldots\otimes a_m-1\otimes(a_1*(a_2\otimes\ldots\otimes a_m))$, the rest of the proposition is clear.

Remark. Since $a_0*(a_1\otimes\ldots\otimes a_m)=a_0a_1\otimes\ldots\otimes a_m-a_0\otimes(a_1*(a_2\otimes\ldots\otimes a_m))$, we see that the left multiplication of an element of A to an element of $1*P_m$ coincides with the ordinary multiplication.

§ 3. The basic algebra of an absolutely segregated algebra

DEFINITION 5. The subalgebra $A_0=EAE$ of A is called the *basic algebra* of A, where $E=\sum_{\kappa=1}^{n}e_\kappa$.

LEMMA 11. (*Hochschild*)[9] *An algebra A is absolutely segregated if and only if any algebra B containing a two-sided ideal C such that $B/C\cong A$ and $C^2=0$, contains a subalgebra A' such that $B=C+A'$.*

PROPOSITION 3. *An algebra A is absolutely segregated if and only if its basic algebra A_0 is absolutely segregated.*

Proof. First we prove the "if" part. Assume that A_0 is absolutely segregated. Let B be an algebra containing a two-sided ideal C such that $B/C\cong A$. Then, by Lemma 11, we can assume $C^2=0$ and consequently we can construct matric units $\{\tilde{c}_{\kappa,i,j}\}$ such that each $\tilde{c}_{\kappa,i,j}$ belongs to the class $c_{\kappa,i,j}$ mod C. Then $(\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1})B(\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1})=B_0$ contains $(\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1})C(\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1})=C_0$ and $B_0/C_0\cong A_0$. Therefore B_0 contains a subalgebra A_0' such that $B_0=C_0+A_0'$. Since $A_0'\cong A_0$, A_0' contains idempotents $\tilde{e}_\kappa'$ corresponding to $e_\kappa=c_{\kappa,1,1}$ and, since $\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1}$ is the unit element of B_0, we have $\sum_{\kappa=1}^{n}\tilde{e}_\kappa'=\sum_{\kappa=1}^{n}\tilde{c}_{\kappa,1,1}$. Then $\tilde{c}_{\kappa,i,i}$ $(i\neq 1)$ and $\tilde{e}_\kappa'$ forms mutually orthogonal primitive idempotents and therefore there exists matric units $\{\tilde{c}_{\kappa,i,j}'\}$ such that $\tilde{c}_{\kappa,i,j}'$ belongs to the class $c_{\kappa,i,j}$ mod. C and $\tilde{c}_{\kappa,i,i}'=\tilde{c}_{\kappa,i,i}$ for $i\neq 1$ and $\tilde{c}_{\kappa,1,1}'=\tilde{e}_\kappa'$. Now we consider $A'=\sum_{\kappa,\lambda,i,j}\tilde{c}_{\kappa,i,1}'A_0'c_{\lambda,1,j}'$. It is clear that

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[8] Cf. Nakayama [7], Lemmas 4,1 and 4,2.

[9] Hochschild [2].

A' is a subalgebra and $B = C \cup A'$. From $C_0 \cap A_0' = 0$, it is clear that $C \cap A' = 0$. Thus A is absolutely segregated.

Next we prove the "only if" part. Assume that A is absolutely segregated and B_0 is an algebra containing a two-sided ideal C_0 such that $B_0/C_0 \cong A_0$. Let $\{\tilde{e}_\kappa\}$ be a system of idempotents in B_0 constructed in such a way that $\tilde{e}_\kappa$ corresponds to e_κ of A_0. Now let $\{\tilde{c}_{\kappa,i,j}\}$ $(\kappa = 1, \ldots, n;\ i, j = 1, \ldots, f(\kappa))$ be a system of symbols. $B_0 = \sum_{\kappa,\lambda} \tilde{e}_\kappa B_0 \tilde{e}_\lambda + \sum_\kappa B_0^{(1)} \tilde{e}_\kappa + \sum_\kappa \tilde{e}_\kappa B_0^{(2)} + B_0^{(3)}$, where $B_0^{(1)}$, $B_0^{(2)}$ and $B_0^{(3)}$ consist of elements annihilated by left, right or two-sided multiplications of $\sum_{\kappa=1}^{n} \tilde{e}_\kappa$, respectively. It is clear that $B_0^{(1)}$, $B_0^{(2)}$ and $B_0^{(3)}$ are contained in C_0. Let B be the direct sum of modules $\tilde{c}_{\kappa,i,1} \tilde{e}_\kappa B_0 \tilde{e}_\lambda \tilde{c}_{\lambda,1,j}$, $B_0^{(1)} \tilde{e}_\kappa \tilde{c}_{\lambda,1,j}$, $\tilde{c}_{\kappa,i,1} \tilde{e}_\kappa B_0^{(2)}$ and $B_0^{(3)}$: $B = \sum_{\kappa,\lambda,i,j} \tilde{c}_{\kappa,i,1} \tilde{e}_\kappa B_0 \tilde{e}_\lambda \tilde{c}_{\lambda,1,j} + \sum_{\kappa,i} B_0^{(1)} \tilde{e}_\kappa \tilde{c}_{\kappa,1,i} + \sum_{\kappa,i} \tilde{c}_{\kappa,i,1} \tilde{e}_\kappa B_0^{(2)} + B_0^{(3)}$. Now we set $\tilde{c}_{\kappa,i,j} \tilde{c}_{\lambda,h,k} = \delta_{\kappa,\lambda} \delta_{j,h} \tilde{c}_{\kappa,i,k}$, $\tilde{c}_{\kappa,1,1} = \tilde{e}_\kappa$, $\tilde{c}_{\kappa,i,j} B_0^{(1)} = 0$, $B_0^{(2)} \tilde{c}_{\kappa,i,j} = 0$ and $\tilde{c}_{\kappa,i,j} B_0^{(3)} = B_0^{(3)} \tilde{c}_{\kappa,i,j} = 0$. Then it is clear that B becomes an algebra. Let $C = \sum_{\kappa,\lambda,i,j} \tilde{c}_{\kappa,i,1} \tilde{e}_\kappa C_0 \tilde{e}_\lambda \tilde{c}_{\lambda,1,j} + \sum_{\kappa,i} B_0^{(1)} \tilde{e}_\kappa \tilde{c}_{\kappa,1,i} + \sum_{\kappa,i} \tilde{c}_{\kappa,i,1} \tilde{e}_\kappa B_0^{(2)} + B_0^{(3)}$, then C is a two-sided ideal of B and it is not hard to verify that $B/C \cong A$. Therefore B contains a subalgebra A' such that $B = C + A'$ and consequently $(\sum_{\kappa=1}^{n} \tilde{e}_\kappa) B (\sum_{\kappa=1}^{n} \tilde{e}_\kappa) = \sum_{\kappa,\lambda} \tilde{e}_\kappa B_0 \tilde{e}_\lambda$ contains $(\sum_{\kappa=1}^{n} \tilde{e}_\kappa) A' (\sum_{\kappa=1}^{n} \tilde{e}_\kappa) = A_0'$ and $\sum_{\kappa,\lambda} \tilde{e}_\kappa B_0 \tilde{e}_\lambda = A_0' + (\sum_{\kappa=1}^{n} \tilde{e}_\kappa) C (\sum_{\kappa=1}^{n} \tilde{e}_\kappa)$. Since $B_0 = \sum_{\kappa,\lambda} \tilde{e}_\kappa B_0 \tilde{e}_\lambda \cup C_0$ and $A_0' \cap C_0 = 0$, we have $B_0 = A_0' + C_0$. This shows that A_0 is absolutely segregated.

§ 4. A direct decomposition of $1*P_2$ into two-sided submodules

In this section we assume that A is an algebra with rank m over an algebraically closed field Ω and coincides with its basic algebra, i.e. satisfies the condition (B): if $A = \sum_{\kappa=1}^{n} A e_\kappa = \sum_{\kappa=1}^{n} e_\kappa A$ are direct decompositions into indecomposable left and right ideals of A, respectively, then $A e_\kappa \not\cong A e_\lambda$ $(e_\kappa A \not\cong e_\kappa A)$ for $\kappa \neq \lambda$.

LEMMA 12. $(1*P_2 : \Omega) = m^2 - m$.

Proof. By Proposition 2, $A^{(2)}/\mathfrak{M} \cong 1*P_1 \cong A$ and $\mathfrak{M} \cong 1*P_2$. Therefore $(1*P_2 : \Omega) = (A^{(2)} : \Omega) - (A : \Omega) = m^2 - m$.

LEMMA 13. *Let $\{u_i(\kappa, \lambda)\}$, $\kappa \neq \lambda$, be an Ω-basis of $e_\kappa A e_\lambda$, and let $\{u_i(\kappa, \kappa)\}$ be an Ω-basis of $e_\kappa N e_\kappa$. Then, if we put $v_i(\kappa, \lambda) = e_\kappa \otimes u_i(\kappa, \lambda) - u_i(\kappa, \lambda) \otimes e_\lambda$, $A*v_i(\kappa, \lambda)$ and $v_i(\kappa, \lambda)*A$ are contained in $1*P_2$, $A*v_i(\kappa, \lambda)$ is A-left-isomorphic to $A e_\kappa$ and $v_i(\kappa, \lambda)*A$ is A-right-isomorphic to $e_\lambda A$. Moreover the sums $\bigcup_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)$ and $\bigcup_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$ are direct.*

Proof. Since $a*v_i(\kappa, \lambda) = a e_\kappa \otimes u_i(\kappa, \lambda) - a \otimes u_i(\kappa, \lambda) - a u_i(\kappa, \lambda) \otimes e_\lambda + a \otimes u_i(\kappa, \lambda) = a e_\kappa \otimes u_i(\kappa, \lambda) - a u_i(\kappa, \lambda) \otimes e_\lambda$, $1*v_i(\kappa, \lambda) = v_i(\kappa, \lambda) \in 1*P_2$. Therefore $A*v_i(\kappa,$

$\lambda)$ and $v_i(\kappa, \lambda)*A$ are contained in $1*P_2$. If $\sum_{\kappa,\lambda,i} a(\kappa, \lambda, i)*v_i(\kappa, \lambda)=0$ for some $a(\kappa, \lambda, i)\in A$, then $(\sum_{\kappa,\lambda,i} a(\kappa, \lambda, i)*v_i(\kappa, \lambda)=)0=\sum_{\kappa,\lambda,i}(a(\kappa, \lambda, i)e_\kappa \otimes u_i(\kappa, \lambda)-a(\kappa, \lambda, i)u_i(\kappa, \lambda)\otimes e_\lambda)=\sum_{\kappa,\lambda,i} a(\kappa, \lambda, i)e_\kappa \otimes u_i(\kappa, \lambda)-\sum_{\lambda}(\sum_{\kappa,i} a(\kappa, \lambda, i)u_i(\kappa, \lambda))\otimes e_\lambda$. Since $u_i(\kappa, \lambda)$ and e_κ form an Ω-basis of A, we have $a(\kappa, \lambda, i)e_\kappa=0$ and consequently $a(\kappa, \lambda, i)*v_i(\kappa, \lambda)=0$. This shows that the sum $\bigcup_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)$ is direct. At the same time, this shows that $A*v_i(\kappa, \lambda)=Ae_\kappa*v_i(\kappa, \lambda)\cong Ae_\kappa$. By the same way we have that the sum $\bigcup_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$ is direct and $v_i(\kappa, \lambda)*A\cong e_\lambda A$.

For the sake of brevity, we put $(Ae_\kappa : \Omega)=s_\kappa$, $(e_\kappa A : \Omega)=r_\kappa$ and $(e_\kappa A e_\lambda : \Omega)=c_{\kappa,\lambda}$.

LEMMA 14. $1*P_2=\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)=\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$.

Proof. By direct computation, we see $Ae_\kappa \otimes e_\lambda A\subset 1*P_2$ if $\kappa\neq\lambda$. Since $P_2=\sum_{\kappa,\lambda} Ae_\kappa \otimes e_\lambda A$ and since $\sum_{\kappa} Ae_\kappa \otimes e_\kappa A$ contains $\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)$ and $\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$, the sum $(\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A)\cup(\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda))$ and $(\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A)\cup(\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A)$ are direct. We show that these direct sums coincide with $1*P_2$. To prove this, we compute the ranks of $\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} Av_i(\kappa, \lambda)$ and $\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$. By Lemma 13 and the definition of $u_i(\kappa, \lambda)$, $((\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)) : \Omega)=\sum_{\kappa\neq\lambda} s_\kappa r_\lambda+\sum_{\kappa,\lambda} s_\kappa(c_{\kappa,\lambda}-\delta_{\kappa,\lambda})=\sum_{\kappa\neq\lambda} s_\kappa r_\lambda+\sum_{\kappa} s_\kappa(\sum_{\lambda} c_{\kappa,\lambda}-1)=\sum_{\kappa\neq\lambda} s_\kappa r_\lambda+\sum_{\kappa} s_\kappa(r_\kappa-1)=\sum_{\kappa,\lambda} s_\kappa r_\lambda-\sum_{\kappa} s_\kappa=m^2-m=(1*P_2 : \Omega)$. In the same way, we have $((\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A+\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A) : \Omega)=(1*P_2 : \Omega)$.

LEMMA 15. $\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)=\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A=\mathfrak{M}$ *is a two-sided module.*

Proof. Since $\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)\subset\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)*A\subset(\sum_{\kappa} Ae_\kappa \otimes e_\kappa A)\cap 1*P_2$, we have $1*P_2=\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)*A+\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A$ and consequently $\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)*A=\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)$. By the same way, we have $\sum_{\kappa,\lambda,i} A*v_i(\kappa, \lambda)*A=\sum_{\kappa,\lambda,i} v_i(\kappa, \lambda)*A$.

By these two lemmas and the fact that $\sum_{\kappa\neq\lambda} Ae_\kappa \otimes e_\lambda A\cong\sum_{\kappa\neq\lambda} Ae_\kappa \times e_\lambda A$, A is absolutely segregated if and only if $\mathfrak{M}$ is isomorphic to a direct sum of A-two-sided modules $Ae_\kappa \times e_\lambda A$.

LEMMA 16 $\cdot e_\kappa a*v_i(\lambda, \nu)=(e_\kappa \otimes au_i(\lambda, \nu)-e_\kappa au_i(\lambda, \nu)\otimes e_\nu)-(e_\kappa \otimes e_\kappa a e_\lambda-e_\kappa a e_\lambda \otimes e_\lambda)*u_i(\lambda, \nu)$.

§ 5 Structure of absolutely segregated algebras

Consider an absolutely segregated algebra A over an algebraically closed field Ω satisfying (B). As was mentioned above, $\mathfrak{M}$ in Lemma 15 is a direct sum of submodules isomorphic to $Ae_\kappa \times e_\lambda A$, say $\mathfrak{M}\cong\sum_{\kappa,\lambda} t_{\kappa,\lambda}(Ae_\kappa \times e_\lambda A)$ as we want to write.

Now we assume that the indices $1, \ldots, n$ are so arranged as $s_1 \leqq \ldots \leqq s_n$. Then,

LEMMA 17. $s_\lambda = 1 + \sum_\kappa t_{\kappa,\lambda} s_\kappa$ *and* $r_\kappa = 1 + \sum_\lambda t_{\kappa,\lambda} r_\lambda$. $t_{\kappa,\lambda} = 0$ *if* $\kappa \geqq \lambda$.

Proof. Since $\mathfrak{M} = \sum_{\kappa,\lambda,i} A * v_i(\kappa,\lambda) = \sum_{\kappa,\lambda,i} v_i(\kappa,\lambda) * A \cong \sum_{\kappa,\lambda} t_{\kappa,\lambda}(Ae_\kappa \times e_\lambda A)$, we have, comparing indecomposable summands isomorphic to $e_\lambda A$, $\sum_\kappa (c_{\kappa,\lambda} - \delta_{\kappa,\lambda}) = s_\lambda - 1 = \sum_\kappa t_{\kappa,\lambda} s_\kappa$. Since $s_\lambda \leqq s_\kappa$ for $\lambda \leqq \kappa$, $t_{\kappa,\lambda} = 0$ if $\lambda \leqq \kappa$. By the same way, we have $r_\kappa = 1 + \sum_\lambda t_{\kappa,\lambda} r_\lambda$.

COROLLARY. $s_1 = 1$, *that is,* $Ae_1 = \Omega e_1$.

By this corollary, $\mathfrak{M} = \sum_{\lambda,i} A * v_i(1,\lambda) + \sum_{\kappa \neq 1;\lambda,i} A * v_i(\kappa,\lambda) = \sum_{\lambda,i} \Omega v_i(1,\lambda) + \sum_{\kappa \neq 1;\lambda,i} A * v_i(\kappa,\lambda)$. We denote $\sum_{\kappa \neq 1;\lambda,i} A * v_i(\kappa,\lambda)$ by $\mathfrak{M}_1$. On the other hand, $\mathfrak{M} \cong \sum_{\kappa,\lambda} t_{\kappa,\lambda}(Ae_\kappa \times e_\lambda A)$ and consequently $\mathfrak{M} = \sum_{\kappa,\lambda,i} \mathfrak{M}_i(\kappa,\lambda)$, where, for each pair (κ,λ), $\mathfrak{M}_i(\kappa,\lambda)$ are $t_{\kappa,\lambda}$ two-sided submodules of $\mathfrak{M}$ isomorphic to $Ae_\kappa \times e_\lambda A$. Let $m_i(\kappa,\lambda)$ be the element of $\mathfrak{M}_i(\kappa,\lambda)$ corresponding to $e_\kappa \times e_\lambda$ by the above isomorphism, then $\mathfrak{M}_i(\kappa,\lambda)$ is generated by $m_i(\kappa,\lambda)$.

LEMMA 18. $\mathfrak{M}_1 = \sum_{\kappa \neq 1;\lambda,i} \mathfrak{M}_i(\kappa,\lambda)$; *in particular,* $\mathfrak{M}_1$ *is a two-sided module.*

Proof. Since $\mathfrak{M} = \sum_{\lambda,i} \Omega v_i(1,\lambda) + \mathfrak{M}_1$, if $\kappa \neq 1$, $m_i(\kappa,\lambda) * a = e_\kappa * m_i(\kappa,\lambda) * a$ is contained in $e_\kappa \mathfrak{M} = e_\kappa \mathfrak{M}_1 (\subset \mathfrak{M}_1)$ for any $a \in A$. Therefore $m_i(\kappa,\lambda) * A \subset \mathfrak{M}_1$ if $\kappa \neq 1$ and consequently $\mathfrak{M}_1 \supseteqq \sum_{\kappa \neq 1;\lambda,i} A * m_i(\kappa,\lambda) * A = \sum_{\kappa \neq 1,\lambda,i} \mathfrak{M}_i(\kappa,\lambda)$. On the other hand $(\sum_{\lambda,i} \Omega v_i(1,\lambda) : \Omega) = \sum_\lambda (c_{1,\lambda} - \delta_{1,\lambda}) = r_1 - 1 = \sum_\kappa t_{1,\kappa} r_\kappa = (\sum_{\kappa,i} \mathfrak{M}_i(1,\kappa) : \Omega) = (\mathfrak{M} : \Omega) - (\sum_{\kappa \neq 1,\lambda,i} \mathfrak{M}_i(\kappa,\lambda) : \Omega)$. Therefore $\mathfrak{M}_1 = \sum_{\kappa \neq 1;\lambda,i} \mathfrak{M}_i(\kappa,\lambda)$.

By Lemma 18, $\mathfrak{M}/\mathfrak{M}_1 \cong \sum_\kappa t_{1,\kappa}(Ae_1 \times e_\kappa A) = \sum_\kappa t_{1,\kappa}(\Omega e_1 \times e_\kappa A)$. Since $\mathfrak{M} = \sum_{\lambda,i} \Omega v_i(1,\lambda) + \mathfrak{M}_1$, we can, for each κ, take $t_{1,\kappa}$ elements, say $x_h(1,\kappa) = \sum_i \omega_i(\kappa, h) v_i(1,\kappa) (\omega_i(\kappa,h) \in \Omega)$, as the representatives of the $t_{1,\kappa}$ classes corresponding to $t_{1,\kappa}$ $e_1 \times e_\kappa$'s. Then, since $Ae_1 = \Omega e_1$, $\mathfrak{M} = \sum_{\kappa,h} x_h(1,\kappa) * A + \mathfrak{M}_1$. We donote $\sum_i \omega_i(\kappa,h) u_i(1,\kappa)$ by $w_h(1,\kappa)$. Then $w_h(1,\kappa) \in e_1 A e_\kappa$ and $x_h(1,\kappa) = e_1 \otimes w_h(1,\kappa) - w_h(1,\kappa) \otimes e_\kappa$.

LEMMA 19. $e_1 N = \sum_{\kappa,h} w_h(1,\kappa) A$ *and* $w_h(1,\kappa) A \cong e_\kappa A$ *if* $t_{1,\kappa} \neq 0$.

Proof. Assume that $\sum_{\kappa,h} w_h(1,\kappa) e_\kappa a_{\kappa,h} = 0$ for some $a_{\kappa,h} \in Ae_\nu$, where ν is an arbitrarily fixed. Since $t_{1,1} = 0$, $\sum_{\kappa,h} w_h(1,\kappa) e_\kappa a_{\kappa,h} = \sum_{\kappa \neq 1;h} w_h(1,\kappa) e_\kappa a_{\kappa,h} = 0$. Then $\sum_{\kappa \neq 1,h} x_h(1,\kappa) * e_\kappa a_{\kappa,h} = e_1 \otimes (\sum_{\kappa \neq 1,h} w_h(1,\kappa) e_\kappa a_{\kappa,h}) - \sum_{\kappa \neq 1;h} w_h(1,\kappa) \otimes e_\kappa a_{\kappa,h} = -\sum_{\kappa \neq 1;h} w_h(1,\kappa) \otimes e_\kappa a_{\kappa,h}$. We can write $e_\kappa a_{\kappa,h} = e_\kappa a_{\kappa,h} e_\nu = \sum_j \beta(\kappa,h,j) u_j(\kappa,\nu) + \delta_{\kappa,\nu} \beta(h) e_\nu$, where $\beta(\dots)\in\Omega$. Hence $-\sum_{\kappa\neq 1,h} w_h(1,\kappa)\otimes e_\kappa a_{\kappa,h} = -\sum_{\kappa\neq 1:h,j}\beta(\kappa,h,j)(w_h(1,\kappa)\otimes u_j(\kappa,\nu)) - (\sum_h \beta(h)w_h(1,\nu))\otimes e_\nu$. Now let $a(\kappa, j)e_\kappa = -\sum_h \beta(\kappa, h, j)w_h(1,\kappa)$. Then $\mathfrak{M}_1 \ni \sum_{\kappa\neq 1:j} a(\kappa, j)*v_j(\kappa, \nu) = \sum_{\kappa\neq 1;j} a(\kappa, j)e_\kappa \otimes u_j(\kappa, \nu) - (\sum_{\kappa\neq 1:j} a(\kappa, j)u_j(\kappa, \nu))\otimes e_\nu$ $= -\sum_{\kappa\neq 1,h,j}\beta(\kappa, h, j)(w_h(1, \kappa)\otimes u_j(\kappa, \nu)) + (\sum_{\kappa\neq 1;h,j}\beta(\kappa, h, j)w_h(1, \kappa)u_j(\kappa, \nu))\otimes e_\nu$. Since $\sum_{\kappa\neq 1;h} w_h(1, \kappa)e_\kappa a_{\kappa,h} = \sum_{\kappa\neq 1:h,j}\beta(\kappa, h, j)w_h(1, \kappa)u_j(\kappa, \nu) + \sum_h \beta(h)w_h(1, \nu) = 0$, $\sum_{\kappa\neq 1;h,j}\beta(\kappa, h, j)w_h(1, \kappa)u_j(\kappa, \nu) = -\sum_h \beta(h)w_h(1,\nu)$. Therefore $\sum_{\kappa\neq 1;h} x_h(1, \kappa)*e_\kappa a_{\kappa,h}$ $=\sum_{\kappa\neq 1;j} a(\kappa, j)*v_j(\kappa, \nu)\in\mathfrak{M}_1$ and consequently $e_\kappa a_{\kappa,h}=0$.

Thus the sum $\bigcup_{\kappa,h} w_h(1.\kappa)A$ is direct. If $t_{1,\kappa}\neq 0$, then $w_h(1,\kappa)\neq 0$ and, as was shown above, $w_h(1,\kappa)A\cong e_\kappa A$. On the other hand $e_1N\supseteq\sum_{\kappa,h} w_h(1,\kappa)A$ and $(e_1N:\Omega)$ $= r_1 - 1 = \sum_\kappa t_{1,\kappa}r_\kappa = (\sum_{\kappa,h} w_h(1, \kappa)A : \Omega)$. Therefore $e_1N=\sum_{\kappa,h} w_h(1, \kappa)A$.

LEMMA 20. $\mathfrak{M}_1 = \sum_{\kappa\neq 1;\lambda,i} A*v_i(\kappa, \lambda) = \sum_{\kappa\neq 1;\lambda,i} v_i(\kappa, \lambda)*A + \sum_{\kappa\neq 1;\lambda,h,i} w_h(1,\kappa)*v_i(\kappa,\lambda)*A$.

Proof. By Lemma 19, we can take $w_h(1, \kappa)$ and $w_h(1, \kappa)u_i(\kappa, \lambda)$ $(\kappa\neq 1)$ as an Ω-basis of e_1N. By Lemma 16, $w_h(1, \kappa)*v_i(\kappa, \lambda) = (e_1\otimes w_h(1, \kappa)u_i(\kappa, \lambda) - w_h(1, \kappa)u_i(\kappa, \lambda)\otimes e_\lambda) - x_h(1, \kappa)*u_i(\kappa, \lambda)$. Consequently, using the above Ω-basis, we have $\sum_{\lambda,i} v_i(1, \lambda)*A = \sum_{\kappa\neq 1;h} x_h(1, \kappa)*A + \sum_{\kappa\neq 1;h,i} w_h(1, \kappa)*v_i(\kappa, \lambda)*A$. Since $\mathfrak{M} = \sum_{\kappa\neq 1;h} x_h(1, \kappa)*A + \mathfrak{M}_1$ and $(\sum_{\kappa\neq 1;\lambda,h,i} w_h(1, \kappa)*v_i(\kappa, \lambda)*A)\cup(\sum_{\kappa\neq 1;\lambda,i} v_i(\kappa, \lambda)*A)\subseteq\mathfrak{M}_1$, we have $\mathfrak{M}_1 = (\sum_{\kappa\neq 1;\lambda,h,i} w_h(1, \kappa)*v_i(\kappa, \lambda)*A)\cup(\sum_{\kappa\neq 1;\lambda,i} v_i(\kappa, \lambda)*A)$. It is easy to see that $\mathfrak{M}_1 = \sum_{\kappa\neq 1;\lambda,i} v_i(\kappa, \lambda)*A + \sum_{\kappa\neq 1;\lambda,h,i} w_h(1, \kappa)*v_i(\kappa, \lambda)*A$.

LEMMA 21. *The following conditions* (i), (ii) *and* (iii) *hold for every* κ.

(i) $\mathfrak{M}_\kappa = \sum_{\mu>\kappa;\lambda,i} A*v_i(\mu, \lambda) = \sum_{\mu>\kappa;\lambda,i}\mathfrak{M}_i(\mu, \lambda)$.

(ii) *There exist* $t_{\kappa,\lambda}$ *elements* $w_h(\kappa, \lambda)$ *in* $e_\kappa Ae_\lambda$ *such that* $e_\kappa N=\sum_{\lambda,h} w_h(\kappa, \lambda)A$, $Ne_\kappa = \sum_{\lambda<\kappa;h} Aw_h(\lambda, \kappa)$ *and if* $t_{\lambda,\kappa}\neq 0$, $Aw_h(\lambda, \kappa)$ *is A-left -isomorphic to* Ae_λ, *and if* $t_{\kappa,\lambda}\neq 0$, $w_h(\kappa, \lambda)A$ *is A-right-isomorphic to* $e_\lambda A$.

(iii) $\mathfrak{M}_\kappa = \sum_{\mu>\kappa;\lambda,i} v_i(\mu, \lambda)*A + \sum_{\mu\leqq\kappa;\nu>\kappa;\lambda,h,i} w_h(\mu, \nu)*v_i(\nu, \lambda)*A + \dots + \sum_{\substack{\mu_1<\mu_2<\dots<\mu_r\leqq\kappa;\nu>\kappa;\\ \lambda,h_1,\dots,h_r,i}} (w_{h_1}(\mu_1, \mu_2)w_{h_2}(\mu_2, \mu_3)\dots w_{h_r}(\mu_r, \nu))*v_i(\nu, \lambda)*A + \dots + \sum_{\nu>\kappa;\lambda,h_1,\dots,h_\kappa,i}(w_{h_1}(1,2)\dots w_{h_\kappa}(\kappa, \nu))*v_i(\nu, \lambda)*A$.

Proof. We assume that (i), (ii) and (iii) are satisfied for indices $\kappa\leqq p$. (p is a fixed integer.) We want to prove that (i), (ii) and (iii) hold for $\kappa = p+1$. From (ii), we can see, for $\kappa\leqq p$, $s_\kappa = 1+\sum_{\mu<\kappa}^{i} t_{\mu,\kappa} + \sum_{\mu_1<\mu_2<\kappa} t_{\mu_1,\mu_2}t_{\mu_2,\kappa} + \dots + \sum_{\mu_1<\dots<\mu_r<\kappa} t_{\mu_1,\mu_2}t_{\mu_2,\mu_3}\dots t_{\mu_r,\kappa} + \dots + t_{1,2}t_{2,3}\dots t_{\kappa-1,\kappa}$. From (i) and (iii), $A*v_i(p+1, \lambda)\supseteq\Omega v_i(p+1, \lambda) + \sum_{\mu\leqq p;h}\Omega w_h(\mu, p+1)*v_i(p+1, \lambda) + \dots + \sum_{\mu_1<\dots<\mu_r\leqq p;h_1,\dots,h_r}\Omega(w_{h_1}(\mu_1, \mu_2)\dots$

$\ldots w_{h_r}(\mu_r,\ p+1))^*v_i(p+1,\ \lambda)+\ldots+\sum_{h_1,\ldots,h_p}\Omega(w_{h_1}(1,\ 2)\ldots w_{h_p}(p,\ p+1))^*v_i(p+1,\ \lambda)$.

The rank of the right hand side is equal to $1+\sum_{\mu\leqq p}t_{\mu,p+1}+\ldots+\sum_{\mu_1<\mu_2\ldots<\mu_r\leqq p}t_{\mu_2,\mu_1}\ldots t_{\mu r,p+1}+\ldots+t_{1,2}t_{2,3}\ldots t_{p,p+1}$. Since $s_\kappa=1+\sum_{\mu<\kappa}t_{\mu,\kappa}+\ldots+t_{1,2}t_{2,3}\ldots t_{\kappa-1,\kappa}$ for $\kappa\leqq p$, $1+\sum_{\mu\leqq p}t_{\mu,p+1}+\ldots\sum_{\mu_1<\mu_2\ldots<\mu_r\leqq p}t_{\mu_1,\mu_2}\ldots t_{\mu r,p+1}+\ldots+t_{1,2}t_{2,3}\ldots t_{p,p+1}=1+\sum_{\mu=1}^{p}t_{\mu,p+1}s_\mu=s_{p+1}=(A^*v_i(p+1,\ \lambda)\ :\ \Omega)$. This shows that $A^*v_i(p+1,\ \lambda)=\Omega v_i(p+1,\ \lambda)+\sum_{\mu\leqq p;h}\Omega w_h(\mu,p+1)^*v_i(p+1,\lambda)+\ldots+\sum_{\mu_1<\ldots<\mu_r\leqq p;h_1,\ldots,h_r}\Omega(w_{h_1}(\mu_1,\ \mu_2)\ldots w_{hr}(\mu_1,\ p+1))^*v_i(p+1,\ \lambda)+\ldots+\sum_{h_1,\ldots,h_p}\Omega(w_{h_1}(1,2)\ldots w_{h_p}(p,p+1))^*v_i(p+1,\lambda)$. Since $A^*v_i(p+1,\lambda)\cong Ae_{p+1}$, $Ae_{p+1}=\Omega e_{p+1}+\sum_{\mu\leqq p;h}w_h(\mu,\ p+1)+\ldots+\sum_{\mu_1<\ldots<\mu_r\leqq p;h_1,\ldots,h_r}\Omega(w_{h_1}(\mu_1,\ \mu_2)\ldots w_{h_r}(\mu_r,\ p+1))+\ldots+\sum_{h_1,\ldots,h_r}\Omega(w_{h_1}(1,\ 2)\ldots w_{h_p}(p,\ p+1))$. Then it is easy to see that $Ne_{p+1}=\sum_{\kappa<p+1;h}Aw_h(\kappa,\ p+1)$ and $Aw_h(\kappa,\ p+1)=Ae_\kappa w_h(\kappa,\ p+1)\cong Ae_\kappa$. This proves the second part of (ii) for $\kappa=p+1$.

As was shown above, $\sum_{\lambda,i}A^*v_i(p+1,\ \lambda)=\sum_{\lambda,i}\Omega v_i(p+1,\lambda)+\ldots+\sum_{\lambda,i,h_1,\ldots,h_p}\Omega(w_{h_1}(1,\ 2),,,w_{h_p}(p,\ p+1))^*v_i(p+1,\ \lambda)$. Since $\mathfrak{M}_p=\sum_{\lambda,i}A^*v_i(p+1,\ \lambda)+\mathfrak{M}_{p+1}$, we have $\mathfrak{M}_p=(\sum_{\lambda,i}\Omega v_i(p+1,\lambda)+\ldots+\sum_{\lambda,i,h_1,\ldots,h_p}\Omega(w_{h_1}(1,2)\ldots w_{h_p}(p,p+1))^*v_i(p+1,\lambda))+\mathfrak{M}_{p+1}$. Then, by the same way used in Lemma 17, we have $\mathfrak{M}_{p+1}\geqq\sum_{\kappa>p+1;\lambda,i}\mathfrak{M}_i(\kappa,\ \lambda)$. On the other hand, $(\sum_{\lambda,i}A^*v_i(p+1,\ \lambda):\Omega)=s_{p+1}(\sum_\lambda(c_{p+1,\lambda}-\delta_{p+1,\lambda}))=s_{p+1}(r_{p+1}-1)=s_{p+1}(\sum_\lambda t_{p+1,\lambda}r_\lambda)=(\sum_{\lambda,i}\mathfrak{M}_i(p+1,\ \lambda)\ :\ \Omega)=(\mathfrak{M}_p\ :\ \Omega)-(\sum_{\kappa>p+1;\lambda,i}\mathfrak{M}_i(\kappa,\ \lambda))$. Therefore $\mathfrak{M}_{p+1}=\sum_{\kappa>p+1;\lambda,i}\mathfrak{M}_i(\kappa,\ \lambda)$. This proves (i) for $\kappa=p+1$.

Now $\mathfrak{M}_p/\mathfrak{M}_{p+1}\cong\sum_\lambda t_{p+1,\lambda}(Ae_{p+1}\times e_\lambda A)$. Since $\mathfrak{M}_p=\sum_{\lambda,i}\Omega v_i(p+1,\ \lambda)+\ldots+\sum_{\lambda,i,h_1,\ldots,h_p}\Omega(w_{h_1}(1,2)\ldots w_{h_p}(p,p+1))^*v_i(p+1,\lambda)+\mathfrak{M}_{p+1}$, we can take $t_{p+1,\kappa}$ elements, say $x_h(p+1,\ \kappa)=\sum_i\omega_i(\kappa,\ h)v_i(p+1,\ \kappa)(\omega_i(\kappa,\ h)\in\Omega)$ as the representatives of the classes corresponding to $t_{p+1,\kappa}\ e_{p+1}\times e_\kappa$'s. Then $\mathfrak{M}_p=\sum_{\kappa,h}A^*x_h(p+1,\ \kappa)^*A+\mathfrak{M}_{p+1}$. As before, we denote $\sum_i\omega_i(\kappa,\ h)u_i(p+1,\kappa)$ by $w_h(p+1,\ \kappa)(\in e_{p+1}Ae_\kappa)$. If $\sum_{\lambda,h}w_h(p+1,\ \lambda)e_\lambda a_{\lambda,h}=0$ for some $e_\lambda a_{\lambda,h}\in A$, then, since $t_{p+1,\lambda}=0$ for $\lambda\leqq p+1$, $\sum_{\lambda,h}w_h(p+1,\ \lambda)e_\lambda a_{\lambda,h}=\sum_{\lambda>p+1;h}w_h(p+1,\ \lambda)e_\lambda a_{\lambda,h}=0$ and consequently, by the same way used in Lemma 19, we have $\sum_{\lambda>p+1;h}x_h(p+1,\ \lambda)^*e_\lambda a_{\lambda,h}\equiv0\ (\mathfrak{M}_{p+1})$ which implies $e_\lambda a_{\lambda,h}=0$. This shows that $e_{p+1}N\geqq\sum_{\lambda,h}w_h(p+1,\ \lambda)A$ and $w_h(p+1,\ \lambda)A\cong e_\lambda A$ if $t_{p+1,\lambda}\neq0$. Comparing the ranks of $e_{p+1}N$ and $\sum_{\lambda,h}w_h(p+1,\ \lambda)A$, we have $e_{p+1}N=\sum_{\lambda,h}w_h(p+1,\ \lambda)A$. This proves the first part of (ii) for $\kappa=p+1$.

Now we consider (iii). From the facts that $e_{p+1}N=\sum_{\kappa,h}w_h(p+1,\ \kappa)A$ and that $t_{p+1,\kappa}=0$ for $\kappa=1,\ldots,p+1$, we can take $w_h(p+1,\kappa)$ and $w_h(p+1,\kappa)u_i(\kappa,\lambda)$ $(\kappa\neq 1,\ldots,p+1)$ as an Ω-basis of $e_{p+1}N$. Using this Ω-basis, we have $e_{p+1}\otimes w_h(p+1,\kappa)u_i(\kappa,\lambda)-w_h(p+1,\kappa)u_i(\kappa,\lambda)\otimes e_\lambda=w_h(p+1,\kappa)*v_i(\kappa,\lambda)+x_h(p+1,\kappa)*u_i(\kappa,\lambda)$. Consequently $\sum_{\kappa,i}v_i(p+1,\kappa)*A=\sum_{\kappa>p+1:h}x_h(p+1,\kappa)*A+\sum_{\kappa>p+1:\lambda,i,h}w_h(p+1,\kappa)*v_i(\kappa,\lambda)*A$ and $\sum_{\kappa,i}(w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,p+1))*v_i(p+1,\kappa)*A=\sum_{\kappa>p+1;h}(w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,p+1))*x_h(p+1,\kappa)*A+\sum_{\kappa>p+1;\lambda,i,h}w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,p+1)w_h(p+1,\kappa))*v_i(\kappa,\lambda)*A$. Then, by the facts that $\mathfrak{M}_p=\sum_{\kappa,h}A*x_h(p+1,\kappa)*A+\mathfrak{M}_{p+1}=[\sum_{\kappa,i}v_i(p+1,\kappa)*A+\sum_{\mu\leqq p:\kappa,i,h}w_h(\mu,p+1)*v_i(p+1,\kappa)*A+\ldots+\sum_{\substack{\mu_1<\ldots<\mu_r<p;\kappa,i\\ h_1,\ldots,h_r}}(w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,p+1))*v_i(p+1,\kappa)*A+\ldots+\sum_{\kappa,i,h_1,\ldots,h_p}(w_{h_1}(1,2)\ldots w_{h_p}(p,p+1))*v_i(p+1,\kappa)*A]+[\sum_{\kappa>p+1;\lambda,i}v_i(\kappa,\lambda)*A+\ldots+\sum_{\substack{\kappa>p+1;\mu_1<\ldots<\mu_r\leqq p;\lambda,i\\ h_1,\ldots,h_r}}(w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,\kappa))*v_i(\kappa,\lambda)*A+\ldots+\sum_{\kappa>p+1;\lambda,i,h_1,\ldots,h_p}(w_{h_1}(1,2)\ldots w_{h_p}(p,\kappa))*v_i(\kappa,\lambda)*A]$ and that $\mathfrak{M}_{p+1}\supseteqq[\sum_{\kappa>p+1;\lambda,i}v_i(\kappa,\lambda)*A+\ldots+\sum_{\kappa>p+1;\lambda,i,h_1,\ldots,h_p}(w_{h_1}(1,2)\ldots w_{h_p}(p,\kappa))*v_i(\kappa,\lambda)*A]+[\sum_{\kappa>p+1;\lambda,i,h}w_h(p+1,\kappa)*v_i(\kappa,\lambda)*A+\ldots+\sum_{\kappa>p+1;\lambda,i,h_1,\ldots,h_{p+1}}(w_{h_1}(1,2)\ldots w_{h_p}(p,p+1)w_{h_{p+1}}(p+1,\kappa))*v_i(\kappa,\lambda)*A]$, we have that $\mathfrak{M}_{p+1}=\sum_{\kappa>p+1;\lambda,i}v_i(\kappa,\lambda)*A+\ldots+\sum_{\substack{\kappa>p+1;\mu_1<\ldots<\mu_r\leqq p+1;\lambda,i\\ h_1,\ldots,h_r}}(w_{h_1}(\mu_1,\mu_2)\ldots w_{h_r}(\mu_r,\kappa))*v_i(\kappa,\lambda)*A+\ldots+\sum_{\kappa>p+1;\lambda,i,h_1,\ldots,h_{p+1}}(w_{h_1}(1,2)\ldots w_{h_{p+1}}(p+1,\kappa))*v_i(\kappa,\lambda)*A$. This proves (iii) for $\kappa=p+1$. Therefore we have Lemma 21 by induction.

PROPOSITION 4. *Let A be an absolutely segregated algebra over an algebraically closed field satisfying (B), then there exists a system of non-negative integers $\{t_{\kappa,\lambda}\}$ such that $e_\kappa N\cong\sum_\lambda t_{\kappa,\lambda}e_\lambda A$ and $Ne_\kappa\cong\sum_\lambda t_{\lambda,\kappa}Ae_\lambda$ for each κ. Moreover $e_\kappa Ae_\kappa=\Omega e_\kappa$ for each κ.*

Proof. As was shown above, we have that, for each κ, $Ne_\kappa=\sum_{\lambda<\kappa}Aw_h(\lambda,\kappa)$. Since $t_{\lambda,\kappa}=0$ for $\lambda>\kappa$ and $Aw_h(\lambda,\kappa)\cong Ae_\lambda$ if $t_{\lambda,\kappa}\neq 0$, we have $Ne_\kappa\cong\sum_\lambda t_{\lambda,\kappa}Ae_\lambda$. Then it can easily be seen that Ne_κ has only $\bar{A}\bar{e}_\lambda(\lambda<\kappa)$ as its composition residue-modules. This shows that $e_\kappa Ae_\kappa=\Omega e_\kappa$. In the same way, we have $e_\kappa N\cong\sum_\lambda t_{\kappa,\lambda}e_\lambda A$.

Now we consider a general algebra over an algebraically closed field, and prove

PROPOSITION 5. *Let A be an algebra over an algebraically closed field. Then A is absolutely segregated if and only if there exists a system of non-negative integers $\{t_{\kappa,\lambda}\}$ such that $Ne_\kappa\cong\sum_\lambda t_{\lambda,\kappa}Ae_\lambda$, that is, N is an A-left M_0-module.*

Proof. By Proposition 2 and the fact that there exists such a system $\{t_{\kappa,\lambda}\}$ for A if and only if the same holds for the basic algebra A_0 of A, it is sufficient to prove our assertion for an algebra satisfying (B).

As the "only if" part has been settled above, we prove the "if" part. As before we assume that $s_1 \leqq \ldots \leqq s_n$. Then, by the above relation, we have that $s_\kappa - 1 = \sum_\lambda t_{\lambda,\kappa} s_\lambda$ and $t_{\lambda,\kappa} = 0$ if $\lambda \geqq \kappa$. Therefore $Ne_\kappa \cong \sum_{\lambda<\kappa} t_{\lambda,\kappa} A e_\lambda$. Now let $w_h(\lambda, \kappa)$ be $t_{\lambda,\kappa}$ elements corresponding to e_λ by the above isomorphism. Then it is not hard to see that e_κ, $w_{h_1}(\kappa_1, \kappa_2), w_{h_1}(\kappa_1, \kappa_2) w_{h_2}(\kappa_2, \kappa_3), \ldots, w_{h_1}(1, 2) w_{h_2}(2, 3) \ldots w_{h_{n-1}}(n-1, n)$ $(\kappa_i = 1, \ldots, n;\ \kappa_i > \kappa_{i-1})$ form an Ω-basis of A. By this Ω-basis we can decompose $\mathfrak{M}$ (of Lemma 15) into indecomposable left modules. Here, by Lemma 16, $e_{\kappa_1} \otimes w_{h_1}(\kappa_1, \kappa_2) w_{h_2}(\kappa_2, \kappa_3) \ldots w_{h_r}(\kappa_r, \kappa_{r+1}) - w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_r}(\kappa_r, \kappa_{r+1}) \otimes e_{\kappa_{r+1}} = (w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_{r-1}}(\kappa_{1-r}, \kappa_r)) * (e_{\kappa_r} \otimes w_{h_r}(\kappa_r, \kappa_{r+1}) - w_{h_r}(\kappa_r, \kappa_{r+1}) \otimes e_{\kappa_{r+1}}) + (e_{\kappa_1} \otimes w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_{r-1}}(\kappa_{r-1}, \kappa_r) - w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_{r-1}}(\kappa_{r-1}, \kappa_r) \otimes e_{\kappa_r}) * w_{h_r}(\kappa_r, \kappa_{r+1})$. Therefore, by induction, we have $e_{\kappa_1} \otimes w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_r}(\kappa_r, \kappa_{r+1}) - w_{h_1}(\kappa_1, \kappa_2) \ldots w_{h_r}(\kappa_r, \kappa_{r+1}) \otimes e_{\kappa_{r+1}}$ is contained in $\bigcup_{\kappa,\lambda,h} A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A$. This shows that $\mathfrak{M} = \bigcup_{\kappa,\lambda,h} A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A$. On the other hand, $(A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A : \Omega) \leqq s_\kappa r_\lambda$ and consequently $\sum_{\kappa,\lambda,h} (A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A : \Omega) \leqq \sum_{\kappa,\lambda} t_{\kappa,\lambda} s_\kappa r_\lambda = \sum_\lambda r_\lambda (\sum_\kappa t_{\kappa,\lambda} s_\kappa) = \sum_\lambda r_\lambda (s_\lambda - 1) = \sum_{\kappa,\lambda} r_\kappa s_\lambda - (\sum_{\kappa \neq \lambda} A e_\kappa \otimes e_\lambda A : \Omega) - m = (\mathfrak{M} : \Omega)$. Therefore the sum $\bigcup_{\kappa,\lambda,h} A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A$ is direct and $A * (e_\kappa \otimes w_h(\kappa, \lambda) - w_h(\kappa, \lambda) \otimes e_\lambda) * A \cong A e_\kappa \times e_\lambda A$. Thus A is absolutely segregated.

THEOREM. *Let A be an algebra with unit element over a field F. Then A is absolutely segregated if and only if*

(i) *A/N is separable,*

(ii) *the A-left-module N is directly decomposed into submodules isomorphic to some left-ideal direct components Ae_κ of A, i.e. there exists a system of non-negative integers $\{t_{\kappa,\lambda}\}$ such that*

$$Ne_\kappa \cong \sum_\lambda t_{\lambda,\kappa} A e_\lambda.$$

Proof. We prove the "if" part. Assume that A satisfies (i) and (ii). Then from (ii), N is an A-left M_0-module, therefore N_Ω(Ω is an algebraic closer of F), the radical of A_Ω, is also an A_Ω-left M_0-module. Therefore A_Ω is absolutely segregated and consequently A is absolutely segregated.

Next we prove the "only if" part. Assume that A is absolutely segregated and A/N is inseparable. Then $(A/N)_\Omega$ contains a nilpotent element belonging to the centre of $(A/N)_\Omega$. Let c be a representative of that class. Then c belongs to the radical N' of A_Ω and there exists a primitive idempotent of A, say e, such that $ce \notin N_\Omega$. Since the residue class of c mod N_Ω is in the centre of $(A/N)_\Omega$, $ece \neq 0$. Therefore $eA_\Omega e \supset eN'e \neq 0$. This contradicts $eA_\Omega e = \Omega e$. Thus A/N is separable, and N_Ω is an A_Ω-left M_0-module. Hence N is an A-left M_0-module. This completes the proof.

COROLLARY. *Let A be an algebra without unit element, then A is absolutely segregated if and only if $A^* = (1, A)$, the algebra obtained by adjunction of* 1 *to A, has the properties stated in our Theorem.*

Added note. T. Nakayama and H. Nagao have given simpler proofs of our theorem. These will appear in this journal.

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