ON FINITE GROUPS WITH GIVEN CONJUGATE TYPES I

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Let G be a finite group. Let $n_1, n_2, \ldots, n_r$, where $n_1 > n_2 > \ldots > n_r = 1$, be all the numbers each of which is the index of the centralizer of some element of G in G. We call the vector $(n_1, n_2, \ldots, n_r)$ the conjugate type vector of G. A group with the conjugate type vector $(n_1, n_2, \ldots, n_r)$ is said to be a group of type $(n_1, n_2, \ldots, n_r)$.

In the present and succeeding papers we want to investigate the structures of groups of simplest types, that is, the structures of groups of type $(n_1, 1)$, of type $(n_1, n_2, 1)$ and of type $(n_1, n_2, n_3, 1)$. Obviously groups of type (1) are abelian and conversely. Therefore we omit these groups from our considerations. In the present paper, we treat the case of groups of type $(n_1, 1)$ and some related problems.

A subgroup which is the centralizer of some element is called, by M. Cipolla, a fundamental subgroup. Now we call, in the present paper, only a fundamental subgroup which is distinct from G a fundamental subgroup. Some intrinsic properties of fundamental subgroups have been obtained by Italian authors, especially by M. Cipolla, G. Scorza and G. Zappa. On the other hand, some results on the structures of groups with given types of fundamental subgroups have been obtained by L. Weisner and S. Čunihin.[1] Some of the latters are generalized in the present paper.

The main results of the present paper are the following: (I) Any group of type $(n_1, 1)$ is nilpotent. Further, n_1 is a power of a prime p: $n_1 = p^a$ and any group of type $(p^a, 1)$ is the direct product of a p-subgroup of type $(p^a, 1)$ and an abelian subgroup. Therefore the structure of groups of type $(n_1, 1)$ is reduced to that of p-groups of type $(p^a, 1)$. Then: (II) For any p-group G of type $(p^a, 1)$, G/A is a group of exponent p, where A is abelian and normal in G. Our considerations are made much complicated by the presence of centres, and generally speaking, the smaller the centre is, the simpler is the structure of the group. For instance, if the centre of a p-group G of type $(p^a, 1)$ is cyclic, then G is of class 2.

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1) L. Weisner (1), S. Čunihin (1).

§ 1. Preliminaries

In this section we give necessary definitions and notations and prove some preliminary theorems on fundamental subgroups and p-groups.

Let F be a fundamental subgroup of a group G. Therefore $F \neq G$ (according to our agreement above). We say that F is minimal if F contains no fundamental subgroup, of G, properly and F is maximal if F is contained in no fundamental subgroup, of G, properly. Further we say that F is free if F is both minimal and maximal. An abelian fundamental subgroup is obviously minimal. Further we say that a group G is of type (F) if every fundamental subgroup of G is free and that G is of type (A) if every fundamental subgroup is abelian. A group of type (A) is obviously a group of type (F).

We denote the normalizer of a subset X of a group by $\mathfrak{N}(X)$ and the centralizer by $\mathfrak{Z}(X)$. Now let F be a fundamental subgroup and let X be an element of F. If $\mathfrak{Z}(X) \geqq F$ we call X a central element of F, and if especially $\mathfrak{Z}(X) = F$ we call X a generating element of F.

PROPOSITION 1.1. Let F be a free fundamental subgroup of a group. Then F is nilpotent. Further if F contains at least two generating elements one of which is of order a power of a prime p and the other of which is of order a power of a prime of q and $q \neq p$, then F is abelian.

Proof. Clearly F contains a generating element X of order a power of a prime p, because of its maximality. Let Y be any element of F of order a power of a prime $q \neq p$. First let us remark the following fact that $\{XY\}$ contains both $\{X\}$ and $\{Y\}$. Then F clearly contains $\mathfrak{Z}(XY)$, whence F must coincide with $\mathfrak{Z}(XY)$, because of its minimality. Then $\mathfrak{Z}(Y)$ contains F, since $\mathfrak{Z}(Y)$ contains $\mathfrak{Z}(XY)$. In other words Y is a central element of F.

We say that a group G has an abelian partition with a kernel subgroup K, if G is a set-theoretical join of some abelian subgroups each pair of which meet only by K. Further we call each of such abelian subgroups a component of the partition. Then we have clearly the following

PROPOSITION 1.2. Every group G of type (A) has an abelian partition with a kernel subgroup Z which is the centre of G and with components which are fundamental subgroups of G.

PROPOSITION 1.3. Let G be a p-group of exponent different from p and let G have an abelian partition with a kernel subgroup E which is the unit group of G. Then a component A which contains at least one element of order greater than p contains all the elements of order greater than p. In particular G/A is of exponent p. Further any other component except A is non-normal.

Proof. Let X be an element of A of order greater than p and let Y be a central element of G of order p. Then since $(XY)^p = X^p \neq E$, A contains XY.

Hence A contains Y. Let X' be any element of G of order greater than p. Then since $(X'Y)^p = X'^p \neq E$, the component which contains X' contains Y. Hence it must coincide with A.

Now we obviously have the following

PROPOSITION 1.4. Let G be a group such that $G = H \times K$, where K is abelian. Then G and H have the same conjugate type vector.

Therefore we shall set a convention that every group G which we consider contains no abelian direct factor.

We denote as usual the upper central series and the lower central series of a nilpotent group G by

$$E = Z_0 \subset Z_1 \subset \dots \subset Z_{c-1} \subset Z_c = G \quad \text{and} \quad H_1 = G \supset H_2 \supset \dots \supset H_c \supset H_{c+1} = E$$

respectively, c being the class of G. We also denote as usual the commutator series of a soluble group G by $G = D_0 \supset D_1 \supset \dots \supset D_{n-1} \supset D_n = E$, n being the rank of G. Finally we denote by S_p a p-Sylow subgroup of a group G.

§ 2. Groups of type $(n_1, 1)$

The purpose of this section is to show that any group of type $(n_1, 1)$ is nilpotent. With the convention of § 1 any nilpotent group of type $(n_1, 1)$ is a p-group for some prime p. Obviously any group of type $(n_1, 1)$ is of type (F). Hence any fundamental subgroup of it is nilpotent by Proposition 1.1.

Now let us assume that there exists a non-nilpotent group G of type $(n_1, 1)$ and we want to show that this assumption leads up to a contradiction. We first obtain the following

PROPOSITION 2.1. Any fundamental subgroup F of G is abelian. In other words, G is of type (A).

Proof. Suppose F be a p-group for some prime p. Since G is not a p-group, G contains a q-Sylow subgroup $S_q(G)$ different from E for some prime q distinct from p. Then the centre $Z_1(G)$ of G contains clearly $S_q(G)$, whence F also contains $S_q(G)$. This is obviously a contradiction. Therefore F is not a p-group for any prime p. Let all the generating elements of prime power order be of order a power of p for some fixed prime p. Then any q-Sylow subgroup $S_q(F) \neq E$ of F, where q is a prime distinct from p, is contained in $Z_1(G)$. Now, by the convention of § 1, G contains a non-central element X of order a power of q. Let us put $F' = \mathfrak{Z}(X)$. Then a q-Sylow subgroup $S_q(F')$ of F' clearly contains $S_q(F)$ properly. This is obviously a contradiction. Thus F contains at least two generating elements one of which is of order a power of p for some prime p and the other of which is of order a power of q for some prime q different from p. Therefore F is abelian by Proposition 1.1.

Since thus G is of type (A), G possesses the abelian partition with the kernel subgroup $Z_1(G)$ by Proposition 1.2. Now

PROPOSITION 2.2. The class of a p-Sylow subgroup $S_p(G)$ of G is equal to 2 for any prime divisor p of order.

Proof. Assume that $S_p(G)$ is abelian. Then $S_p(G)$ is contained in some fundamental subgroup of G, whence it follows that all the fundamental subgroups of G are conjugate one another (under the convention of §1). On the other hand their set-theoretical join contains all the elements of G. This is clearly a contradiction. Since any fundamental subgroup F of G is abelian, F thus contains no $S_p(G)$. Further the centre $Z_1(S_p(G))$ of $S_p(G)$ is clearly contained in $Z_1(G)$.

By a theorem of P. Hall,[2] we have that $[H_2(S_p(G)), Z_2(S_p(G))] = E$, where $H_2(S_p(G))$ and $Z_2(S_p(G))$ are the second terms of the lower and upper central series of $S_p(G)$. Let P be an element of $Z_2(S_p(G))$ not belonging to $Z_1(S_p(G))$. Then P is a non-central element of G. Put $F = \mathfrak{Z}(P)$. Then we have clearly that $F \geqq H_2(S_p(G))$. Since F is abelian, the rank of $S_p(G)$ is equal to 2. Let us assume that the class of $S_p(G)$ is greater than 2 and we show that this assumption leads up to a contradiction. Since $S_p(G) \cdot Z_1(G) \mid Z_1(G) \cong S_p(G) \mid Z_1(S_p(G))$, and since the former is a group with an abelian partition with the kernel subgroup $E = Z_1(G)$, the latter factor group $S_p(G) \mid Z_1(S_p(G))$ is also a group with an abelian partition with the kernel subgroup $E = Z_1(S_p(G))$. Hereby the p-Sylow subgroup $S_p(F)$ of F can be assumed to be one of the component subgroups of this partition. Since $S_p(F) \geqq H_2(S_p(G))$, $S_p(F)$ is normal in $S_p(G)$. Further $S_p(G) \mid Z_1(S_p(G))$ is not abelian, since we assumed that the class of $S_p(G)$ is greater than 2. Now $S_p(G) \mid S_p(F)$ is an abelian group of type $(p, \ldots, p)$ by Proposition 1.3. On the other hand, clearly the normalizer $\mathfrak{N}(F)$ of F contains $S_p(G)$. So let us consider the subgroup $S_p(G)C_p(F)$, where $C_p(F)$ is the p-Sylow complement of F. Then we can consider $S_p(G) \mid S_p(F)$ as a group of automorphisms of $C_p(F) \mid C_p(F) \frown Z_1(G)$. In fact, clearly $C_p(F)$ and $C_p(F) \frown Z_1(G)$ are normal in $S_p(G)C_p(F)$. Further any element of $S_p(G)$ which is commutative with some residue class other than $C_p(F) \frown Z_1(G)$ of $C_p(F) \frown Z_1(G)$ in $C_p(F)$ is contained in $S_p(F)$. To see this let P be an element of $S_p(G)$ which is not contained in $S_p(F)$. Then $\mathfrak{Z}(P) \neq F$. Therefore P is not commutative with any generating element of F which is contained in $C_p(F)$. Suppose that P is commutative with the residue class $Q(C_p(F) \frown Z_1(G)) \neq C_p(F) \frown Z_1(G)$ of $C_p(F) \frown Z_1(G)$ in $C_p(F)$, where Q is an element of $C_p(F)$. Then Q is a generating element of F. The commutator $[P, Q]$ is contained in $C_p(F) \frown Z_1(G) \leqq Z_1(G)$. Therefore we have clearly $[P, Q]^n = [P^n, Q] = [P, Q^n]$ for any integer n, whence $[P, Q] = E$. This is clearly a contradiction. Hence by a theorem

[2] P. Hall (2).

of W. Burnside[3)] $S_p(G)/S_p(F)$ is either cyclic or a generalized quaternion group. Then $S_p(G) | S_p(F)$ is a group of order p. Since for any fundamental subgroup F^* there exists a p-Sylow subgroup $S_p^*(G)$ of G which contains the p-Sylow subgroup $S_p(F^*)$ of F^*, we can assume that all the fundamental subgroups are conjugate in G with one another. In fact, otherwise, a p-Sylow subgroup $S_p(G)$ of G contains the p-Sylow subgroup $S_p(F)$ and $S_p(F^*)$ of two distinct fundamental subgroups F and F^*. Now $S_p(F)$ and $S_p(F^*)$ are clearly different from each other. They are abelian and of index p in $S_p(G)$, whence it follows that the class of $S_p(G)$ is equal to 2. This is clearly a contradiction. Therefore all the fundamental subgroups are conjugate with one another. On the other hand, their set-theoretical join contains all the elements of G. But clearly it is impossible. This completes the proof to Proposition 2.2.

Now we have easily the following

PROPOSITION 2.3. Any fundamental subgroup F of G is normal in G.

Proof. Let $S_p(F)$ be the p-Sylow subgroup of F. Then the normalizers $\mathfrak{N}(F)$ and $\mathfrak{N}(S_p(F))$ of F and $S_p(F)$ must coincide: $\mathfrak{N}(F)=\mathfrak{N}(S_p(F))$. In fact, since F is abelian, we have $\mathfrak{N}(F)\subseteqq\mathfrak{N}(S_p(F))$. Now $S_p(F)$ contains a generating element of F by the covention of § 1. Therefore if $\mathfrak{N}(S_p(F))$ contains $\mathfrak{N}(F)$ properly, then at least one fundamental subgroup F^* which is different from F (one of the conjugates of F) contains a generating element of F, which is clearly a contradiction. On the other hand, since a p-Sylow subgroup of G is of class 2, and $S_p(F)$ contains the centre of a p-Sylow subgroup of G, $\mathfrak{N}(S_p(F))$ clearly contains a p-Sylow subgroup $S_p(G)$ of G. Thus we have that $\mathfrak{N}(F)$ contains a $S_p(G)$ for any prime p, that is, F is normal.

Therefore as a join of its abelian normal subgroup G must be nilpotent. This is finally a contradiction. Thus we have established the required

THEOREM 1. *Any group of type* $(n_1, 1)$ *is nilpotent.*

§ 3. p-groups of type $(p^a, 1)$

Let G be a p-group of type $(p^a, 1)$. We want to show that G/A is of exponent p for a suitable abelian normal subgroup A of G. In fact, we show, more generally, the following:

PROPOSITION 3.1. Let G be a p-group of type (F). Then G/A is of exponent p for a suitable abelian normal subgroup A of G.

Proof. Let Z_2 be the second centre of G. Let $\mathfrak{Z}(Z_2)$ be the centralizer of Z_2. Let X be any element of G not belonging to $\mathfrak{Z}(Z_2)$. Then X^p belongs to Z_1. In fact, since X does not belong to $\mathfrak{Z}(Z_2)$, there exists an element Y of Z_2 such that $[X, Y]\neq E$. Let p^α be the order of $[X, Y]$. Then since $[X, Y]$

3) W. Burnside (1).

is contained in Z_1, we clearly have $[X^{p^m}, Y^{p^n}]=[X, Y]^{p^{m+n}}$. Therefore, as can easily be seen, $\mathfrak{Z}(X^p)$ contains $\mathfrak{Z}(X)$ properly. Since G is of type (F), $\mathfrak{Z}(X^p)$ must coincide with G. In other words X^p belongs to Z_1 as was asserted. If $\mathfrak{Z}(Z_2)$ is abelian, we put $\mathfrak{Z}(Z_2)=A$. Then A satisfies the required condition. Further we want to observe that in this case A is not only normal in G, but also characteristic in G, which is needed in the following. Now let $G'=\mathfrak{Z}(Z_2)$ be non-abelian. Let Z_2 be the second centre of G'. Let $\mathfrak{Z}'(Z_2')$ be the centralizer of Z_2' in G'. Let X be any element of G' not belonging to $\mathfrak{Z}'(Z_2')$. Then X^p belongs to Z_1. In fact, since X does not belong to $\mathfrak{Z}'(Z_2')$ there exists an element Y of Z_2' such that $[X, Y]\neq E$. Then since $[X, Y]$ is contained in the centre Z_1' of G', we have $[X^{p^m}, Y^{p^n}]=[X, Y]^{p^{m+n}}$. Thus it can be seen that $\mathfrak{Z}(X^p)$ contains $\mathfrak{Z}(X)$ properly. Since G is of type (F), $\mathfrak{Z}(X^p)$ must coincide with G. In other words X^p belongs to Z_1. If $\mathfrak{Z}'(Z_2')$ is abelian, we put $\mathfrak{Z}'(Z_2')=A$. Since A is characteristic in G', A satisfies the required condition. Continuing this process, we obtain the assertion. Further we remark that A contains Z_1 and any element of $G-A$ is of order p to Z_1, which is stronger than the stated above.

Remark. In general, A does not coincide with Z_1. For instance, let G be a dihedral group of order $2^n (n \geqq 4)$. Then obviously G is of type (F) and G/Z_1 is not of exponent 2. However, if G is of class 2, then clearly $A=Z_1$. The same is the case if G is regular in P. Hall's sense,[4] since in such a group the following assertion holds: "if $[X, Y]\neq E$, then $[X^p, Y]$ has the order less than that of $[X, Y]$." We have the same also when H_2 is of exponent p. In fact, let X be an element of A and let Y be any element of G. Then $[X^p, Y]=[X, Y]^p$.

Now since clearly any p-group of type $(p^a, 1)$ is of type (F), we obtain as a special case of the Proposition 3.1 the required

THEOREM 2. *Let G be a p-group of type $(p^a, 1)$. Then G/A is of exponent p for a suitable abelian normal subgroup A of G. Especially, if $p=2$, then G is metabelian. Further, if H_2 is of exponent 2, then G is of class 2.*

Remark. The writer has to leave open whether A can be different from Z_1 in this special case. Further, it may be of use to investigate whether there exists a group of type $(p^a, 1)$ and of arbitrarily high class (or rank).

Now let G be a p-group of type $(p^a, 1)$. Let X be an element of G belonging to Z_2 but not to Z_1. Then $\mathfrak{Z}(X)$ contains H_2 by a theorem of P. Hall[5]. The index of $\mathfrak{Z}(X)$ in G is p^a. Now the set of all the commutators $[X, Y]$, where Y runs over all the elements of G, consititutes an elementary abelian

4) P. Hall (2).

5) P. Hall (2).

central subgroup $C(X)$ of order p^a. In fact, $[X^{p^m}, Y^{p^n}]=[X, Y]^{p^{m+n}}$. Therefore X^p is a central element. Let us correspond Y to $[X, Y]$. Then we clearly have the isomorphism $G/\mathfrak{Z}(X)\cong C(X)$. Therefore since $\mathfrak{Z}(X)$ contains the Frattini subgroup of G, we obtain the following

PROPOSITION 3.2. Let G be a p-group of type $(p^a, 1)$. Then the number of elements of any minimal generator system of G is not less than a and the order of the subgroup $W(Z_1)$ of all the elements of order p of Z_1 is not less than p^a.

In particular, if Z_1 is cyclic, then $W(Z_1)$ is of order p. Therefore $a=1$. Then any fundamental subgroup of G is maximal and contains H_2. Therefore Z_1 contains H_2 and G is of class 2. That is,

PROPOSITION 3.3. Let G be a p-group of type $(p^a, 1)$. If Z_1 is cyclic, then $a=1$ and G is of class 2.

It seems rather difficult to determine the exact structure of groups of type $(p^a, 1)$. Here we merely discuss some examples in this connection.

Example 1. Let $p(\geqq 2)$ be a prime. Let G be a group which is defined by the following relation: $a_1^p= \dots =a_n^p=b_{1,2}^p= \dots =b_{n-1,n}^p=1$ and $[a_1, a_2]=b_{1,2}, \dots, [a_{n-1}, a_n]=b_{n-1,n}$. b's are central elements. It can easily be verified by means of O. Schreier's extension theory[6] that G is of order $p^{\frac{n(n-3)}{2}}$. Now G is of type $(p^{n-1}, 1)$. To see this we have only to prove that any non-central element is commutative only with its own powers up to the central elements. Let $a_1^{x_1}\dots a_n^{x_n}$ and $a_1^{x_1'}\dots a_n^{x_n'}$ be commutative with each other. Then we have that $x_i'x_j=x_j'x_i$ $(i, j=1, \dots, n)$, whence follows the assertion.

Example 2. Let $p>2$ be a prime. Let G be a group which is defined by the following relation. $a_1^p=a_2^p=b^p=c_1^p=c_2^p=1, [a_1, a_2]=b, [a_1, b]=c_1, [a_2, b]=c_2$ and c_1, c_2 are central elements. It can easily be verified that G is of order p^5, by means of O. Schreier's extension theory. Now G is of type $(p^2, 1)$. In fact, any fundamental subgroup of G is of order not less then p^3. If there exists a fundamental subgroup F of order p^4, F is maximal and thus contains $H_2=\{b, c_1, c_2\}$. Then there exists an element $a_1^{x_1}a_2^{x_2}\neq 1$ which is commutative with b. But this is absurd.
Our G is of class 3.

Example 3. There exists a group of type $(p, 1)$ and not of type (A). Let G be a group defined by the following relations: $a_1^p=a_2^p=a_3^p=a_4^p=b^p=1, [a_1, a_2]=[a_3, a_4]=b, [a_1, a_3]=[a_1, a_4]=[a_2, a_3]=[a_2, a_4]=1$. Then G satisfies the condition.

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[6] O. Schreier (1).

§ 4. **Groups of type (F) which are non-simple and without centre.** Let G be a non-simple group of type (F). The purpose of this section is to show that if G is without centre then G is a soluble group of rank 2. On the other hand, there exist infinitely many simple groups of type (F), which we shall treat in the subsequent paper. The case "with centre" seems rather complicated and shall be left open.

First we need to prove the following.

PROPOSITION 4.1 (W. Burnside-H. Zassenhaus). Let K be a field and let M be a finite K-module. Let G be a finite group of linear transformations of M, such that any linear transformation $(\neq 1)$ of G fixes no element $(\neq 0)$ of M. Further let every subgroup of G with a non-trivial centre be abelian. Then G is cyclic.

Proof. Let p and q be primes. Then by a theorem of H. Zassenhaus[7] any subgroup of G of order pq is cyclic. Therefore, since a generalized quaternion group (naturally non-abelian one) does not enter as a subgroup of G, any Sylow subgroup of G is cyclic by a theorem of W. Burnside.[8] Therefore G is soluble. By an induction argument, we can assume that any proper subgroup of G is cyclic. By a theorem of P. Hall,[9] we can assume that G is of order $r^a s^b$, where r and s are two distinct primes, and a and b are natural numbers. Further $a=b=1$. Therefore G is cyclic by a theorem of H. Zassenhaus.[10]

Remark. Let us drop the last condition on G. In such a general case W. Burnside[11] thought to have proved that any subgroup of G of order $p^a q^b$ is cyclic. As H. Zassenhaus[12] remarked, that is not true. But, as the writer fails to understand the counter-example of H. Zassenhaus, we wish to give a one: Let ζ be a primitive $3^2\cdot 7$-th root of units. Let G be a matrix group which has the generator system $A=\begin{pmatrix} \zeta^9 & & \\ & \zeta^{18} & \\ & & \zeta^{36} \end{pmatrix}$ and $B=\begin{pmatrix} 0 & 0 & \zeta^{21} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. Then G is of order $3^2\cdot 7$ and it can be verified easily that G constains no matrix $(\neq 1)$ which has a characteristic root 1. Now the following fact can be verified immediately. Let G be a group of order $p^a q^b$, where p and q are two distinct primes $(p>q)$, and $a>1$. Let any proper subgroup of G be cyclic. Then G itself is cyclic. Therefore W. Burnside's proof fails already for a group of order pq^b. Naturally our above example is such a one.

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7) H. Zassenhaus (2).
8) W. Burnside (1).
9) P. Hall (1).
10) H. Zassenhaus (2).
11) W. Burnside (1).
12) H. Zassenhaus (2).

PROPOSITION 4.2. Let G be a group of type (F) and be without centre. Then any fundamental subgroup F of G is abelian. In other words, G is a group of type (A).

Proof. Let assume that F is non-abelian. Then evidently F must be a p-Sylow subgroup $S_p(G)$ of G for some prime p by Proposition 1.1 and the assumptions on G. Since any element except E of $Z_1(S_p(G))$ is a generating element of $S_p(G)$, and since any fundamental subgroup of non-prime power order is abelian, any element of $S_p(G)$ is contained in some conjugate of $Z_1(S_p(G))$. Therefore if $S_p(G)$ is a group of exponent p^r, where $r>1$, then $Z_1(S_p(G))$ is also a group of exponent p^r. Let P be any element of $Z_1(S_p(G))$ with order p^r. Let P^* be any element of $S_p(G)$ of order p^s, where $s<r$. Then $(PP^*)^{p^s} = P^{p^s}$. Since clearly $\mathfrak{Z}(P^{p^s}) = \mathfrak{Z}(P)$ and $\mathfrak{Z}((PP^*)^{p^s}) = \mathfrak{Z}(PP^*)$, it follows that $\mathfrak{Z}(P) = \mathfrak{Z}(PP^*) = S_p(G)$, which shows that P^* is also a generating element of $S_p(G)$, whence $S_p(G)$ is abelian. This is a contradiction. Hence $S_p(G)$ must be of exponent p. If any two distinct conjugates of $S_p(G)$ are disjoint, then clearly $S_p(G)$ must be abelian, which is a contradiction. Let P be a maximal meet of two distinct conjugates of $S_p(G)$. Thus P is different from E. Then a p-Sylow subgroup $S_p(\mathfrak{N}(P))$ of the normalizer $\mathfrak{N}(P)$ of P contains P properly and is non-normal in $\mathfrak{N}(P)$ by a theorem of H. Zassenhaus.[13] Let us consider the subgroup $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))\cdot P$, where $S_q(\mathfrak{N}(P))$ is a q-Sylow subgroup of $\mathfrak{N}(P)$ and $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ is the normalizer in $\mathfrak{N}(P)$ of $S_q(\mathfrak{N}(P))$ and $q \neq p$ is a prime. Assume that the order of $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ is prime to p. Since any element except E of $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ fixes no element except E of P, and since $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ contains no subgroup which is isomorphic to any non-abelian generalized quaternion group, it follows from our assumptions on G that any subgroup of $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ with a non-trivial centre is abelian. Therefore by Proposition 4.1 $\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P)))$ must be cyclic. Then by W. Burnside's splitting theorem,[14] $\mathfrak{N}(P)$ contains the normal q-Sylow complement. If this holds for every prime order divisor of $\mathfrak{N}(P)$ distinct from p, then $S_p(\mathfrak{N}(P))$ is normal in $\mathfrak{N}(P)$. This is a contradiction. Therefore there must exist at least one $S_q(\mathfrak{N}(P))$ for which its p-Sylow subgroup $S_p(\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P))))$ is different from E. Let us consider the subgroup $H = S_p(\mathfrak{N}_{\mathfrak{N}(P)}(S_q(\mathfrak{N}(P))))\cdot S_q(\mathfrak{N}(P))\cdot P$. Then a p-Sylow subgroup $S_p(H)$ of H must coincide with its own normalizer in H, since, otherwise, some element except E of $S_q(\mathfrak{N}(P))$ must be commutative with some element except E of $S_p(H)$, which is a contradiction. In fact, let $S_q(\mathfrak{N}_H(S_p(H)))$ be a q-Sylow subgroup $\neq E$ of the normalizer $\mathfrak{N}_H(S_p(H))$ of $S_p(H)$ in H. Then $S_q(\mathfrak{N}_H(S_p(H))))\cdot P = \mathfrak{N}_H(S_p(H))_\frown S_q(\mathfrak{N}(P))$. P is normal in $\mathfrak{N}_H(S_p(H))$. So $S_q(\mathfrak{N}_H(S_p(H))$ is different from its normalizer in $\mathfrak{N}_H(S_p(H))$, from which the

[13] H. Zassenhaus (1).

[14] W. Burnside (1), H. Zassenhaus (1).

assertion follows immediately. Since $S_p(H)$ is a group of exponent p, it is, *a posteriori*, regular in P. Hall's sense.[15] Therefore H contains the normal p-Sylow complement by a theorem of H. Wielandt,[16] which is clearly a contradiction. This completes the proof.

Remark. In Proposition 4.2, the non-simplicity is not assumed.

PROPOSITION 4.3. Let G be a group of type (F) and be without centre. Further let G be soluble. Then holds the factorization: $G = AZ$, where A is abelian and normal, Z is cyclic and coincides with its own normalizer, and the orders of A and Z are relatively prime. In particular, G is metabelian and of Frobeniusean type.

Proof. Under the assumptions on G, any fundamental subgroup containing a normal subgroup different from E is itself normal. Hence there exists the largest abelian normal subgroup by Proposition 4.2. Let Z be a complemented subgroup of A. Since any element except E of Z fixes no element except E of A, and since G contains no subgroup which is isomorphic to any non-abelian generalized quaternion group, our assumptions on G imply that any subgroup of Z with a non-trivial centre is abelian. Therefore by Proposition 4.1, Z must be cyclic. Clearly again Z coincides with its own normalizer and any two distinct conjugate subgroup of Z are disjoint. Further, the order a of A is evidently larger than the order z of Z and G is of type $(a, z, 1)$.

PROPOSITION 4.4. Let G be a group of type (F) and be without centre. If G is non-simple, then G is soluble.

Proof. Let N be a minimal normal subgroup of G. If N is soluble, then N is abelian and its centralizer $\mathfrak{Z}(N)$ is an abelian normal S-subgroup. Therefore $\mathfrak{Z}(N)$ coincides with some fundamental subgroup by Proposition 4.2. By a theorem of I. Schur,[17] there exists a complemented subgroup K of $\mathfrak{Z}(N)$ in G. Since any element except E of K is commutative with no element except E of $\mathfrak{Z}(N)$, and since G contains no subgroup which is isomorphic to any non-abelian generalized quaternion group, we can easily see, by the assumptions on G, that any subgroup of K with the non-trivial centre is abelian. Therefore by Proposition 4.2, K must be cyclic. Then G is soluble. So let us assume that N is non-soluble. Then N is a direct product of mutually isomorphic simple non-abelian groups and therefore, in our case, N must be a non-abelian simple group. Let $S_p(N) \neq E$ be a p-Sylow subgroup of N. Then $G = N \cdot \mathfrak{N}(S_p(N))$. Now $\mathfrak{N}(S_p(N))$ is soluble by the above consideration. Therefore G/N is soluble. By an induction argument, we can assume that G/N is of prime order, say p.

15) P. Hall (2).

16) H. Wielandt (1).

17) H. Zassenhaus (1).

Suppose p divides the order of N. Then $\mathfrak{N}_N(S_p(N))$ contains $S_p(N)$ properly. In fact, otherwise, N contains the normal p-Sylow complement by W. Burnside's splitting theorem,[18] which is a contradiction. Now $\mathfrak{N}(S_p(N))$ contains a p-Sylow subgroup $S_p(G)$ of G and $S_p(G)$ is normal in $\mathfrak{N}(S_p(N))$ by Proposition 4.3. Therefore also $\mathfrak{N}(S_p(N))$ is abelian by Proposition 4.3, which is a contradiction. Thus p does not divide the order of N. Let q be any prime divisor of the order of N. Then $G = \mathfrak{N}(S_q(N)) \cdot N$. If there exists no $\mathfrak{N}(S_q(N))$ in which $S_p(G)$ is normal, then holds the factorization: $\mathfrak{N}(S_q(N)) = A \cdot Z$, where $Z = S_p(G)$ by Proposition 4.3, whence $\mathfrak{N}_N(S_q(N))$ is abelian, which is a contradiction. Therefore there exists $\mathfrak{N}(S_q(N))$ in which $S_p(G)$ is normal. Then $\mathfrak{N}(S_q(N)) = (\mathfrak{N}(S_q(N)) \frown N) \times S_p(G)$, whence $\mathfrak{N}(S_q(N))$ is abelian, which is a contradiction. This completes the proof.

Thus we have established the following.

THEOREM 3. *Let G be a group of type (F) which is non-simple and without centre. Then G is metabelian and possesses the factorization: $G = AZ$, where A is abelian and normal, Z is cyclic and coincides with its own normalizer, and the orders of A and Z are relatively prime. Further G is a group of type $(a, z, 1)$, where a and z are orders of A and Z respectively.*

Remark. This theorem can be considered as a generalization of a theorem of L. Weisner.[19]

§ 5. A theorem of S. Čunihin

Let G be a finite group. Let p and q be two distinct prime factors of the order of G. Let $(n_1, n_2, \ldots, n_r)$ be the conjugate type vector of G. If every n_i $(i = 1, \ldots, r)$ is prime to either p or q, then we call G, after S. Čunihin, a group of isolated type. On a group of isolated type S. Čunihin[20] formerly obtained the following result: If G is of odd order, then G is not simple. Now we improve this result as follows:

PROPOSITION 5.1. Let G be of isolated type. Let p and q be two distinct prime factors of the order of G having the above property. Then G is either p-nilpotent or q-nilpotent.

Proof. Let P and Q be p- and q-Sylow subgroups of G respectively. Let x be an element the index of which is prime to p. Then $\mathfrak{Z}(x) \supset P^y$, where y is some element of G, whence, by duality, $\mathfrak{Z}(P^y) \ni x$. That is, any element of G the index of which is prime to p is contained in at least one conjugate subgroup of P. The same is the case for q. Then we easily obtain the following inequality

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[18] W. Burnside (1), H. Zassenhaus (2).
[19] L. Weisner (1).
[20] S. Čunichin (1).

$$[G:\ \mathfrak{N}(\mathfrak{Z}(P))]\ [\mathfrak{Z}(P):\ e-1] \\ +[G:\ \mathfrak{N}(\mathfrak{Z}(Q))]\ [\mathfrak{Z}(Q):\ e-1]+1 \geqq G:\ e.$$

Dividing the both sides by $G:\ e$ we obtain

$$\frac{(\mathfrak{Z}(P):\ e)-1}{\mathfrak{N}(\mathfrak{Z}(P)):\ e}+\frac{(\mathfrak{Z}(Q):\ e)-1}{\mathfrak{N}(\mathfrak{Z}(Q)):\ e}+\frac{1}{G:\ e} \geqq 1.$$

From this inequality we obtain either $\mathfrak{Z}(P)=\mathfrak{N}(\mathfrak{Z}(P))$ or $\mathfrak{Z}(Q)=\mathfrak{N}(\mathfrak{Z}(Q))$. By symmetry we may assume without loss of generality that $\mathfrak{Z}(P)=\mathfrak{N}(\mathfrak{Z}(P))$. Now, since $\mathfrak{Z}(P)$ is normal in $\mathfrak{N}(P)$, we have $\mathfrak{N}(\mathfrak{Z}(P)) \geqq \mathfrak{N}(P)$. Hence $\mathfrak{N}(P)=\mathfrak{Z}(P)$. This shows that P is centrally contained in $\mathfrak{N}(P)$. So we see that G is p-nilpotent by W. Burnside's splitting theorem.[21)]

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21) H. Zassenhaus (1).