NOTE ON HYPOELLIPTICITY OF A FIRST ORDER LINEAR PARTIAL DIFFERENTIAL OPERATOR

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§ 1. **Introduction.** Let Ω be a domain in the $(n+1)$-dimensional euclidian space R^{n+1}. A linear partial differential operator P with coefficients in $C^\infty(\Omega)$[1] (resp. in $C^\omega(\Omega)$[1]) will be termed hypoelliptic (resp. analytic-hypoelliptic) in Ω if a distribution u on Ω (i.e. $u \in \mathscr{D}'(\Omega)$) is an infinitely differentiable function (resp. an analytic function) in every open set of Ω where Pu is an infinitely differentiable function (resp. an analytic function).

In the present paper, we consider a linear partial differential operator

$$L = \sum_{j=1}^{n+1} a^j(y) \frac{\partial}{\partial y^j} + a(y), \tag{1}$$

where the coefficients are complex-valued infinitely differentiable functions (or complex-valued analytic functions) in a domain Ω of R^{n+1}.

Now the main result is :

THEOREM. *Suppose that* $n \geqslant 2$. *A linear partial differential operator of the form* (1) *with coefficients in* $C^\infty(\Omega)$ (*resp. in* $C^\omega(\Omega)$) *is hypoelliptic* (*resp. analytic-hypoelliptic*) *in* Ω *if and only if all the functions* $a^j(y)$ $(j = 1, \cdots, n+1)$ *identically vanish in* Ω *and the function* $a(y)$ *vanishes at no point of* Ω.

For $n = 1$, the hypoellipticity and the analytic-hypoellipticity of the operator of the form (1) with coefficients in C^ω are characterized by H. Suzuki [4] under the condition $|a^1(y)| + |a^2(y)| \neq 0$ for every $y \in \Omega$ and $a(y) = 0$ in Ω.

In the next section, we shall first show that if $L(n \geqslant 1)$ has coefficients in $C^\omega(\Omega)$ and satisfies the condition (3) (see § 2), the hypoellipticity of L as well as the analytic-hypoellipticity of L has no respect to the factor $a(y)$ and we shall study relations between the solvability[2] and the hypoellipticity of L. In the last section, we shall prove the theorem.

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1) We denote by $C^\infty(\Omega)$ the totality of complex-valued infinitely differentiable functions in Ω and by $C^\omega(\Omega)$ the totality of complex-valued analytic functions in Ω.

2) A linear partial differential operator defined on Ω is called solvable in a subdomain Ω_0 of Ω if the equation $Pu = f$ has a solution $u \in \mathscr{D}'(\Omega_0)$ for every $f \in C_0^\infty(\Omega_0)$.

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§ 2. Preliminaries.

We denote by L_0 the principal part of L:

$$L_0 = \sum_{j=1}^{n+1} a^j(y) \frac{\partial}{\partial y^j}. \tag{2}$$

In this section, we always assume that

$$\sum_{j=1}^{n+1} |a^j(y)| \neq 0, \qquad \text{for all } y \in \Omega. \tag{3}$$

We first state the following:

LEMMA 1. *Suppose that $n \geqslant 1$. An operator L of the form* (1) *with coefficients in $C^\omega(\Omega)$ and satisfying the condition* (3) *is hypoelliptic* (*resp. analytic-hypoelliptic*) *in Ω, if and only if the operator L_0 is hypoelliptic* (*resp. analytic-hypoelliptic*) *in Ω.*

Proof. Let y_0 be an arbitrary point of Ω. By the Cauchy-Kovalevsky theorem, we can find a solution $h(y)$, analytic in some neighbourhood N of y_0, of the equation

$$L_0 h = a.$$

From this, we can deduce

$$\begin{cases} L_0(e^h u) = e^h L u, \\ L(e^{-h} u) = e^{-h} L_0 u \end{cases} \tag{4}$$

for all $u \in \mathscr{D}'(N)$. We can immediately conclude Lemma 1 from (4), since the notion of hypoellipticity as well as that of analytic-hypoellipticity has a local property.

Q.E.D.

We set

$$\bar{L}_0 = \sum_{j=1}^{n+1} \overline{a^j(y)} \frac{\partial}{\partial y^j}$$

and denote by C the commutator

$$C = [L_0, \bar{L}_0] = L_0 \bar{L}_0 - \bar{L}_0 L_0.$$

We say L satisfy the condition H at a point y_0 of Ω, if C may be represented as a linear combination of L_0 and $\bar{L}_0$ at $y=y_0$. The Hörmander's necessary condition for L to be solvable in a subdomain Ω_0 of Ω is that $L^{\cdot}$ satisfies the condition H at every point of Ω_0 (see Chap. VI of Hörmander [1]).

LEMMA 2. *Suppose that* $n \geqslant 2$. *If* L *with coefficients in* $C^\infty(\Omega)$ (*resp. in* $C^\omega(\Omega)$) *and satisfying the condition* (3) *fulfils the condition* H *at every point of* Ω, *it then follows that* L_0 *is not hypoelliptic* (*resp. not analytic-hypoelliptic*) *in* Ω *and there exists a subdomain of* Ω *where* L *is solvable.*

Proof. The proof was suggested by Nirenberg-Trèves [3]. Let y_0 be a point fixed arbitrarily in Ω. By a suitable coordinate transformation in some neighbourhood of the point y_0, L_0 may be expressed in the form

$$L_0 = g(x,t)\Big(\frac{\partial}{\partial t} + i\sum_{j=1}^{n} b^j(x,t)\frac{\partial}{\partial x^j}\Big),\quad g(x,t)\neq 0,$$

($i=\sqrt{-1}$, $x=(x^1,\cdots,x^n)$) in a neighbourhood N of the origin: $x=0$, $t=0$, so that L is written by the new coordinate as follows:

$$L = g(x,t)\Big(\frac{\partial}{\partial t} + i\sum_{j=1}^{n} b^j(x,t)\frac{\partial}{\partial x^j}\Big) + c(x,t), \tag{5}$$

where $b^j(x,t)$ $(j=1,\cdots,n)$ are real-valued, and the transformation of coordinates and the coefficients of L of the form (5) are both infinitely differentiable (resp. analytic) in N, if the coefficients of L of the form (1) is infinitely differentiable (resp. analytic) in Ω (see [3]).

If L satisfies the condition H in N, it follows that

$$\sum b_t^j(x,t)\xi_j = 0 \quad \text{if } \sum b^j(x,t)\xi_j = 0,\ (x,t)\in N,\ \xi\in R^n, \tag{6}$$

where $b_t^j(x,t) = \dfrac{\partial b^j}{\partial t}(x,t)$.

Let $\boldsymbol{b}(x,t)$ be the real vector $(b^1(x,t),\cdots,b^n(x,t))$ and $|\boldsymbol{b}(x,t)|$ be the length of the vector $\boldsymbol{b}$:

$$|\boldsymbol{b}(x,t)| = (b^1(x,t)^2+\cdots+b^n(x,t)^2)^{\frac{1}{2}}.$$

If $|\boldsymbol{b}(x,t)|$ identically vanishes in N, any function depending only on the variables x is always a solution of the equation $L_0u=0$. Otherwise, we can find a subdomain N_1 of N in which $\boldsymbol{b}(x,t)$ never vanishes. Thus it follows from (6) that there exists a real-valued function $\beta(x,t)$ in $C^\infty(N_1)$ such that

$$(7) \qquad \boldsymbol{b}_t(x,t) = \beta(x,t)\boldsymbol{b}(x,t),$$

where we have put $\boldsymbol{b}_t = (b_t^1, \cdots, b_t^n)$, and from (7) we obtain

$$\frac{d}{dt}(\boldsymbol{b}(x,t)/|\boldsymbol{b}(x,t)|) = 0, \quad \text{in } N_1.$$

Hence the real vector $\boldsymbol{b}(x,t)/|\boldsymbol{b}(x,t)|$ is independent of the variable t. If we put $\boldsymbol{v}(x) = \boldsymbol{b}(x,t)/|\boldsymbol{b}(x,t)|$, L_0 is rewritten in the form

$$L_0 = g(x,t)\Big(\frac{\partial}{\partial t} + |\boldsymbol{b}(x,t)| \sum_{j=1}^{n} v^j(x) \frac{\partial}{\partial x^j}\Big),$$

where $\boldsymbol{v}(x) = (v^1(x), \cdots, v^n(x))$. Any solution of the equation

$$\sum_{j=1}^{n} v^j(x) \frac{\partial u}{\partial x^j} = 0$$

depending only on the variables x is a solution of the equation $L_0 u = 0$. From these fact, we can assert the first half of the lemma and at the same time we can easily see that L has the property (P) (introduced in [3]) in some subdomain N' of N, that is, there is a unit vector $\boldsymbol{v} = \boldsymbol{v}(x)$ depending on the x-variable only such that $\boldsymbol{b}$ is given by $\boldsymbol{b}(x,t) = |\boldsymbol{b}(x,t)|\boldsymbol{v}(x)$ in N'. Thus using Theorem 2. 1 of [3], we obtain the later half of the lemma.

Q.E.D.

LEMMA 3. *Suppose that* $n \geqslant 1$. *If an operator* L *of the form* (1) *with coefficients in* $C^\omega(\Omega)$ *and satisfying the condition* (3) *does not fulfil the condition* H *at some point in* Ω, *it then follows that* L_0 *is not analytic-hypoelliptic in* Ω.

Proof. This lemma is easily deduced from Theorem 4. 1 of Mizohata [2]. But in our case the proof is simpler. We shall give an outline of the proof.

Suppose that L does not fulfil the condition H at a point $y_0 \in \Omega$. Then we can construct a solution w of the equation $L_0 u = 0$ in a neighbourhood N of y_0 such that $w(y_0) = 0$ and the imaginary part of w is positive in N, the point y_0 excepted (see Chap. VI of [1]). If we take a suitable branch, $\sqrt{w(y)}^3$ is continuously differentiable in N and satisfies the equation $L_0 u = 0$. But it is not twice-continuously differentiable at y_0. This gives the proof.

Q.E.D.

Finally, we state the lemma given by Mr. A. Yoshikawa (see [5]).

LEMMA 4. *Let Ω be a domain of $R^{n+1}(n \geqslant 0)$ and P be a general linear partial differential operator with coefficients in $C^\infty(\Omega)$. If P is hypoelliptic in Ω, then the formal adjoint ${}^tP$ of P is solvable in a neighbourhood of each point of Ω. Here the differential operator ${}^tP$ is defined by the identity*

$$\int Pu \cdot v dy = \int u \cdot {}^tPv dy, \qquad u, v \in C_0^\infty(\Omega).$$

Proof. Suppose that P is hypoelliptic in Ω. Let S be the totality of locally square-integrable functions u in Ω such that Pu is in $C^\infty(\Omega)$. We note $S = C^\infty(\Omega)$ and denote by G_P the graph of P on S into $C^\infty(\Omega)$ in the product space $L^2_{loc}(\Omega) \times C^\infty(\Omega)$, that is, $G_P = \{[u, Pu]\,;\, u \in S\}$. Then, by the open mapping theorem of Banach, the projection on G_P onto $C^\infty(\Omega)([u, Pu] \to u)$ is continuous [3]. Thus let y_0 be an arbitrary point of Ω, N_0 be a neighbourhood of y_0 whose closure $\bar{N}_0$ is contained in Ω, and k be an arbitrary integer $\geqslant 0$. There then exists a constant C_0, an integer $s_0 \geqslant 0$ and compact sets K_1, K_2 of Ω depending on k and N_0 such that

$$(8) \qquad |u|_{k, \bar{N}_0} \leq C_0\Big\{\Big(\int_{K_1} |u|^2 dy\Big)^{\frac{1}{2}} + |Pu|_{s_0, K_2}\Big\}, \quad u \in C^\infty(\Omega).$$

If we choose a neighbourhood N of y_0 such that $N \subset N_0$ and

$$C_0\sqrt{\text{Volume of } N} \leq \frac{1}{2},$$

we obtain from (8) that

$$\| \varphi \|_k \leq |P\varphi|_{s_0}, \qquad \varphi \in C_0^\infty(N),$$

where we have put

$$\| \varphi \|_k = \Big(\sum_{|\alpha| \leqslant k} \int |D^\alpha \varphi|^2 dy\Big)^{\frac{1}{2}}$$

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3) By α we denote multi-indices $\alpha = (\alpha_1, \cdots, \alpha_{n+1})$ of non-negative integers. Their sum is denoted by $|\alpha|$. With $D_j = -i\partial/\partial y^j$, we set

$$D^\alpha = D_1^{\alpha_1} \cdots D_{n+1}^{\alpha_{n+1}}.$$

The topology of $C^\infty(\Omega)$ is then defined by the semi-norms $|\cdot|_{m,K}$:

$$|f|_{m,K} = \sum_{|\alpha| \leqslant m} \sup_{y \in K} |D^\alpha f(y)|,$$

where m is any non-negative integer and K is any compact set of Ω. Hence $C^\infty(\Omega)$ is a Fréchet space by this topology.

From this we may deduce the inequality

$$\|\varphi\|_k \leq C\|P\varphi\|_s \quad , \quad \varphi \in C_0^\infty(N), \tag{9}$$

since we have

$$|\varphi|_{s_0} \leq C\|\varphi\|_s \quad , \quad \varphi \in C_0^\infty(N)$$

with some integer $s>0$ and a constant $C>0$.

From (9) we can immediately see that ${}^tP$ is solvable in a neighbourhood of each point of Ω.

Q.E.D.

§ 3. **Proof of Theorem.** Finally we prove the theorem stated in the introduction. We have only to prove the following :

PROPOSITION. *If* $n \geq 2$, *no operator of the form* (1) *with coefficients in* $C^\infty(\Omega)$ (*resp. in* $C^\omega(\Omega)$) *and satisfying the condition* (3) *is hypoelliptic* (*resp. analytic-hypoelliptic*) *in* Ω.

Before proving the proposition, we must state a lemma which is needed in proving the proposition above.

LEMMA 5. *Let* M *be a linear mapping on* $C^\infty(\Omega)$ *onto itself which satisfies the following conditions* :

(i) *The mapping* M *is bijective and bicontinuous* [4].

(ii) *A function* u *belonging to* $C^\infty(\Omega)$ *identically vanishes in a subdomain of* Ω *if and only if* Mu *identically vanishes there.*

Then M *is an operator of multiplication by a non-vanishing function in* $C^\infty(\Omega)$.

Proof of Lemma 5. It is clear that M and its inverse mapping M^{-1} are both linear partial differential operators with coefficients in $C^\infty(\Omega)$:

$$M = P(y, D) = \sum_{|\alpha| \leqslant m_y} a_\alpha(y) D^\alpha,$$
$$M^{-1} = Q(y, D) = \sum_{|\alpha| \leqslant n_y} b_\alpha(y) D^\alpha,$$

where m_y and n_y are exact orders of $P(y, D)$ and $Q(y, D)$ at a point y respectively, and they are bounded when y goes over a compact set of Ω.

First of all, we shall show that m_y and n_y both identically vanish in

[4] The topology of $C^\infty(\Omega)$ is the same one as the topology stated in footnote [3].

Ω. Assume that $n_y \not\equiv 0$ in Ω. There then exists a subdomain Ω_0 of Ω, $\bar{\Omega}_0 \subset \Omega$, where n_y is a positive constant, say n. Put $m = \max_{y \in \bar{\Omega}_0} m_y$. By $P_m(y, \xi)$ and $Q_n(y, \xi)$, we denote the principal parts of the characteristic polynomials $P(y, \xi)$ and $Q(y, \xi)$ $(y \in \Omega_0, \ \xi \in R^{n+1})$ respectively. Clearly we have

$$P_m(y, \xi) Q_n(y, \xi) = 0 \tag{10}$$

for all $y \in \Omega_0$ and all $\xi \in R^{n+1}$. It follows from (10) that $P_m(y, \xi) = 0$ for all $y \in \Omega_0$ and all $\xi \in R^{n+1}$. Hence we have $m = 0$. This is a contradiction, since $P_0(y, \xi) = (M(1))(y)$ in Ω_0. Therefore n_y as well as m_y identically vanishes in Ω. Thus we can assert that M is equal to an operator of multiplication by a nonvanishing factor. This completes the proof of Lemma 5.

Proof of Proposition. Let L be an operator of the form (1) with coefficients in $C^{\omega}(\Omega)$. Assume that the condition (3) is fulfiled. The lemmas 1, 2 and 3 show us that L is not analytic-hypoelliptic in Ω. In the same way, we can deduce from the lemmas 2 and 4 that the principal part L_0 of an operator L of the form (1) with coefficients in $C^{\infty}(\Omega)$ is not hypoelliptic in any subdomain Ω' of Ω under the condition (3), since if L_0 is hypoelliptic in Ω', ${}^tL_0$ is solvable in a neighbourhood of each point of Ω' and L_0 satisfies the condition H at every point of Ω'.

Next, we are going to show that L with coefficients in $C^{\infty}(\Omega)$ is not hypoelliptic in Ω under the condition (3). Assume that L is hypoelliptic in Ω and the condition (3) holds. If there exists a solution v of the equation $Lv = 0$ in a subdomain Ω_1 of Ω such that v does not vanish in Ω_1, we can construct a function $h \in C^{\infty}(\Omega_1)$ satisfying

$$L_0 h = a.$$

In fact we have only to take $h = -\log v$. (Here note that v is in $C^{\infty}(\Omega_1)$ by the assumption on L and that we may, without loss of generality, assume that the range of v is in the upper half-complex plane). By the same method as in the proof of Lemma 1, it follows that L_0 is hypoelliptic in Ω_1. This is a contradiction. Therefore v vanishes in every open set where Lv vanishes. On the other hand, by Lemma 4 and the assumption on L, L satisfies the condition H at each point of Ω. From this and Lemma

2, we can conclude that L is solvable in some subdomain Ω_0 of Ω. Hence the equation

$$Lu = f \tag{11}$$

has a solution $u \in C^\infty(\Omega_0)$ for every $f \in C_0^\infty(\Omega_0)$. Thus we can more generally assert that the equation (11) has a unique solution $u \in C^\infty(\Omega_0)$ for every $f \in C^\infty(\Omega_0)$. Hence L is bijective and continuous mapping on $C^\infty(\Omega_0)$ onto itself. By the open mapping theorem of Banach, the inverse mapping of L is also continuous. Therefore we can apply Lemma 5 to $M = L$. That is, L is equal, in Ω_0, to an operator of multiplication by a function in $C^\infty(\Omega_0)$. Since this contradicts the condition (3), the proof is complete.

Remark. The author was informed that Mr. A. Yoshikawa had proved the following as an application of Lemma 4: If L_0 of the form (2) with coefficients in $C^\omega(\Omega)$ satisfying the condition (3) is hypoelliptic in Ω, then $n \leq 1$ (see [5]).

References

[1] L. Hörmander: Linear partial differential operators, Springer-Ver., Berlin (1963).
[2] S. Mizohata: Solutions nulles et solutions non analytiques, J. Math. Kyoto Univ., **1** (1962), 271–302.
[3] L. Nirenberg and F. Trèves: Solvability of a first order linear partial differential equation, Comm. Pure Appl. Math., **16** (1963), 331–351.
[4] H. Suzuki: Analytic-hypoelliptic differential operators of first order in two independent variables, J. Math. Soc. Japan, **16** (1964), 367–374.
[5] A. Yoshikawa: On the hypoellipticity of differential operators, J. Fac. Sci. Univ. Tokyo, **14** (1967), 81–88.

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