NOTE ON SCHIFFER'S VARIATION IN THE CLASS OF UNIVALENT FUNCTIONS IN THE UNIT DISC

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1. Let S denote the class of univalent functions $f(z)$ in the unit disc $D: |z|<1$ with the following expansion:

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots + a_n z^n + \cdots. \tag{1}$$

We denote by $f_n(z)$ the extremal function in S which gives the maximum value of the real part of a_n and by D_n the image of D under $w = f_n(z)$. Schiffer proved in his papers [1] and [2] by using his variational method that the boundary of D_n consists of analytic slits $w = w(t)$, t being a real parameter, satisfying

$$\left(\frac{dw}{dt}\right)^2 \frac{1}{w} \sum_{k=2}^{n} \frac{a_n^{(k)}}{w^k} < 0, \tag{2}$$

where $a_n^{(k)}$ is the nth coefficient of $f_n(z)^k = \sum_{\nu=k}^{\infty} a_\nu^{(k)} z^\nu$, so that follows from the Schwarz reflection principle

$$\frac{z^2 f_n'(z)^2}{f_n(z)} \sum_{k=2}^{n} \frac{a_n^{(k)}}{f_n(z)^k} = (n-1)a_n + \sum_{k=1}^{n-1} k\left(\frac{a_k}{z^{n-k}} + \bar{a}_k z^{n-k}\right) \tag{3}$$

in the z-plane. Thus the left-hand side of (3) is due to a variation of the range D_n. In this note, we shall show that the right-hand side of (3) is due to a variation of the domain D.

2. For a complex number γ, a real number τ and a sufficiently small $r>0$, we consider the finite w-plane slit along the segment $S(\gamma\,;r,\tau)$ with end points $\gamma - re^{i\tau}$ and $\gamma + re^{i\tau}$ and denote it by $\Omega(\gamma\,;r,\tau)$. For ω, $-1<\omega<1$, let $\Lambda^+(\gamma\,;r,\tau,\omega)$ and $\Lambda^-(\gamma\,;r,\tau,\omega)$ be the circular arcs with end points $\gamma - re^{i\tau}$ and $\gamma + re^{i\tau}$ where they make with $S(\gamma\,;r,\tau)$ inner angles being equal to $\pi\omega$. We denote by $\Delta(\gamma\,;r,\tau,\omega)$ the domain which is

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obtained from the finite w-plane when we delete the closure of the domain bounded by $\Lambda^{+}(\gamma\,;r,\tau,\omega)\cup\Lambda^{-}(\gamma\,;r,\tau,\omega)$. Then the mapping function which maps $\Omega(\gamma\,;r,\tau)$ conformally onto $\Delta(\gamma\,;r,\tau,\omega)$ is obtained by

$$\eta=re^{i\tau}\frac{(w-\gamma+re^{i\tau})^{1-\omega}+(w-\gamma-re^{i\tau})^{1-\omega}}{(w-\gamma+re^{i\tau})^{1-\omega}-(w-\gamma-re^{i\tau})^{1-\omega}}+\gamma, \tag{4}$$

and hence it has the following expansion with respect to r:

$$\eta=\frac{w-\gamma}{1-\omega}\Big(1-\frac{\omega(2-\omega)e^{2i\tau}}{3(w-\gamma)^2}r^2+o(r^2)\Big)+\gamma. \tag{5}$$

3. For a real $\delta>0$, we consider the mapping function which maps $\Delta(\gamma\,;r,\tau,1/2)$ conformally onto $\Omega(\gamma\,;r,\tau+\delta)$. This is obtained by

$$\xi=\frac{\eta+\gamma}{2}+\frac{e^{2i(\tau+\delta)}}{2(\eta-\gamma)}r^2. \tag{6}$$

Now we set $\omega=1/2$ in (5) and substitute the resulting right-hand side of (5) for η of (6). Then we have

$$\xi=w-\frac{(1-e^{2i\delta})e^{2i\tau}}{4(w-\gamma)}r^2+o(r^2), \tag{7}$$

which maps $\Omega(\gamma\,;r,\tau)$ conformally onto $\Omega(\gamma\,;r,\tau+\delta)$.

4. We note that the extremal function $f_n(z)$ can be continued analytically in some neighborhood of each $\varepsilon=e^{i\theta_0}$ on $C:|z|=1$, except for finitely many points, because of the analyticity of the boundary curve of D_n. Let $\varepsilon=e^{i\theta_0}$ be such a point on C. Now we set $\gamma=f_n(\varepsilon)$ and $e^{i\tau}r=i\varepsilon f_n'(\varepsilon)\rho+o(\rho)$, $\rho=\theta-\theta_0$, in (7) and then substitute $f_n(z)$ for w there. We have

$$\xi=g(z)=f_n(z)+\frac{(1-e^{2i\delta})\varepsilon^2f_n'(\varepsilon)^2}{4(f_n(z)-f_n(\varepsilon))}\rho^2+o(\rho^2). \tag{8}$$

Normalizing $g(z)$ so that the resulting function vanishes and its derivative is 1 at the origin, we see that there is a function $f^*(z)$ in S with the following form:

$$f^*(z)=f_n(z)+\frac{(1-e^{2i\delta})\varepsilon^2f_n'(\varepsilon)^2f_n(z)^2}{4f_n(\varepsilon)^2(f_n(z)-f_n(\varepsilon))}\rho^2+o(\rho^2), \tag{9}$$

and hence

$$f^*(z)=z+\sum_{\nu=2}^{\infty}\Big\{a_\nu-\frac{(1-e^{2i\delta})\varepsilon^2f_n'(\varepsilon)^2}{4f_n(\varepsilon)}\Big(\sum_{k=2}^{\nu}\frac{a_\nu^{(k)}}{f_n(\varepsilon)^k}\Big)\rho^2+o(\rho^2)\Big\}z^\nu. \tag{10}$$

Since $f_n(z)$ is the extremal function, we have

$$\mathscr{R}\left\{\frac{(1-e^{2i\delta})\varepsilon^2 f_n'(\varepsilon)^2}{4f_n(\varepsilon)}\sum_{k=2}^{n}\frac{a_n^{(k)}}{f_n(\varepsilon)^k}\right\}\geqq 0, \tag{11}$$

where δ is an arbitrary real number. Hence we have a result of Schiffer [1]:

$$\frac{\varepsilon^2 f_n'(\varepsilon)^2}{f_n(\varepsilon)}\sum_{k=2}^{n}\frac{a_n^{(k)}}{f_n(\varepsilon)^k}\geqq 0. \tag{12}$$

5. For $\theta_0 (0\leqq\theta_0<2\pi)$, $\varphi(-1/2<\varphi<1)$ and a small $\rho>0$, $C(\theta_0\,;\,\rho)$ denotes the complement of the subarc $r=1$, $\theta_0-\rho<\theta<\theta_0+\rho$ $(z=re^{i\theta})$ of C and $\Gamma(\theta_0\,;\,\rho,\varphi)$ the circular arc with end points $e^{i(\theta_0-\rho)}$ and $e^{i(\theta_0+\rho)}$ where it makes with $C(\theta_0\,;\,\rho)$ inner angles being equal to $(1+\varphi)\pi$. We denote by $D(\theta_0\,;\,\rho,\varphi)$ the domain bounded by $C(\theta_0\,;\,\rho)\cup\Gamma(\theta_0\,;\,\rho,\varphi)$. Then the mapping function $\zeta=\zeta(z)$ with $\zeta(0)=0$ and $\zeta'(0)>0$ which maps conformally $D(\theta_0\,;\,\rho,\varphi)$ onto the unit disc $|\zeta|<1$, is obtained by

$$\zeta=\varepsilon e^{-i\rho/(1+\varphi)}\times \tag{13}$$
$$\times\frac{\left\{\left(i-\dfrac{\bar\varepsilon z-\cos\rho/(1+\sin\rho)}{1-\bar\varepsilon z\cos\rho/(1+\sin\rho)}\right)\Big/\left(1-i\dfrac{\bar\varepsilon z-\cos\rho/(1+\sin\rho)}{1-\bar\varepsilon z\cos\rho/(1+\sin\rho)}\right)\right\}^{1/(1+\varphi)}-e^{i\rho/(1+\varphi)}}{\left\{\left(i-\dfrac{\bar\varepsilon z-\cos\rho/(1+\sin\rho)}{1-\bar\varepsilon z\cos\rho/(1+\sin\rho)}\right)\Big/\left(1-i\dfrac{\bar\varepsilon z-\cos\rho/(1+\sin\rho)}{1-\bar\varepsilon z\cos\rho/(1+\sin\rho)}\right)\right\}^{1/(1+\varphi)}-e^{-i\rho/(1+\varphi)}},$$

where $\varepsilon=e^{i\theta_0}$. Hence the inverse function is obtained by

$$z=\varepsilon\frac{(i+e^{i\rho})(1-\bar\varepsilon e^{i\rho/(1+\varphi)}\zeta)^{1+\varphi}-(1+ie^{-i\rho})(e^{i\rho/(1+\varphi)}-\bar\varepsilon\zeta)^{1+\varphi}}{(1+ie^{-i\rho})(1-\bar\varepsilon e^{i\rho/(1+\varphi)}\zeta)^{1+\varphi}-(i+e^{i\rho})(e^{i\rho/(1+\varphi)}-\bar\varepsilon\zeta)^{1+\varphi}}, \tag{14}$$

so that we have the following expansion with respect to ρ:

$$z=\zeta\left(1+\frac{\varphi(2+\varphi)(1+\bar\varepsilon\zeta)}{6(1+\varphi)^2(1-\bar\varepsilon\zeta)}\rho^2+o(\rho^2)\right). \tag{15}$$

6. Substitute $2\omega/(1-\omega)$ for φ in (15) and the resulting right-hand side of (15) for z of $w=f_n(z)$. Now compose this with (5), where $\gamma=f_n(\varepsilon)$ and $e^{i\tau}r=i\varepsilon f_n'(\varepsilon)\rho+o(\rho)$, and normalize the composite function so that the resulting one vanishes and its derivative is 1 at the origin $\zeta=0$. Then we see that there exists a function $f^*(\zeta)$ in S with the following form:

$$(16)\quad f^*(\zeta) = f_n(\zeta) + \left\{\frac{2\omega}{3(1+\omega)^2}(f'_n(\zeta)\zeta\frac{1+\bar{\varepsilon}\zeta}{1-\bar{\varepsilon}\zeta} - f_n(\zeta)) + \right.$$
$$\left. + \frac{\omega(2-\omega)\varepsilon^2 f'_n(\varepsilon)^2 f_n(\zeta)^2}{3f_n(\varepsilon)^2(f_n(\zeta)-f_n(\varepsilon))}\right\}\rho^2 + o(\rho^2).$$

Since $f_n(\zeta)$ is the extremal function, we have for each ω with sufficiently small $|\omega|$,

$$(17)\quad \frac{\omega(2-\omega)\varepsilon^2 f'_n(\varepsilon)^2}{3f_n(\varepsilon)}\sum_{k=2}^{n}\frac{a_n^{(k)}}{f_n(\varepsilon)^k} - \frac{2\omega}{3(1+\omega)^2}\{(n-1)a_n + 2\,\mathscr{R}\sum_{k=1}^{n-1}k\bar{\varepsilon}^{n-k}a_k\} \geqq 0.$$

Thus we see that for the extremal function $f_n(z)$ in S which gives the maximum value of the real part of a_n,

$$(18)\quad (n-1)a_n + 2\,\mathscr{R}\sum_{k=1}^{n-1}k\bar{\varepsilon}^{n-k}a_k = \frac{\varepsilon^2 f'_n(\varepsilon)^2}{f_n(\varepsilon)}\sum_{k=2}^{n}\frac{a_n^{(k)}}{f_n(\varepsilon)^k}$$

on C.

By (12) the function $q(z) = (z^2 f'_n(z)^2/f_n(z))\sum_{k=2}^{n}(a_n^{(k)}/f_n(z)^k)$ is real on C, and hence we see by the Schwarz reflection principle that $q(z)$ is a rational function. By (18) the value of $q(z)$ is equal to that of the rational function $(n-1)a_n + \sum_{k=1}^{n-1}k(a_k/z^{n-k} + \bar{a}_k z^{n-k})$ on C, so that we have the following result of Schiffer [1] : For the extremal function $f_n(z)$,

$$(19)\quad \frac{z^2 f'_n(z)^2}{f_n(z)}\sum_{k=2}^{n}\frac{a_n^{(k)}}{f_n(z)^k} = (n-1)a_n + \sum_{k=1}^{n-1}k\left(\frac{a_k}{z^{n-k}} + \bar{a}_k z^{n-k}\right).$$

References

[1] Schiffer, M.: A method of variation within the family of simple functions, Proc. London Math. Soc., **44** (1938), 432–449.
[2] Schiffer, M.: On the coefficients of simple functions, ibid., 450–452.

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