P. Jothilingam
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A NOTE ON GRADE

P. JOTHILINGAM

All rings that occur in this note will be assumed to be commutative with unity and all modules will be finitely generated and unitary.

The grade of a module M over a noetherian local ring R is defined to be the length of a maximal R-sequence contained in the annihilator of M. If M has finite projective dimension it is well-known that grade $M \leq \text{proj. dim}\, M$. We can say more when R is a regular local ring. We state the

THEOREM. *Let R be a regular local ring and M a given R-module. Let N be any other R-module such that* $\operatorname{Hom}(M,N) \neq (0)$. *Let p be the least integer such that* $\operatorname{Ext}_R^p(M,N) = (0)$. *Then* $\operatorname{grade} M \leq \inf(p-1, \text{proj. dim}\, N)$. *If q is the least integer such that* $\operatorname{Ext}_R^q(M,M) = (0)$, *then projective dimension of M equals $q-1$.*

Remark. Taking $N = k$, we get $\operatorname{grade} M \leq \text{proj. dim}\, M$, the result mentioned in the introduction.

The proof of theorem depends on the following

LEMMA. *Let R be a regular local ring; let M, N be any two R-modules. If* $\operatorname{Ext}_R^p(M,N) = (0)$ *for some integer $p \geq 1$, then there exists a natural isomorphism* $\operatorname{Ext}_R^{p-1}(M,R) \otimes_R N \cong \operatorname{Ext}_R^{p-1}(M,N)$.

Proof. Define $\Omega^0 = M$ and for $p \geq 1$, define Ω^p to be the pth syzygy module of M taken with respect to a fixed minimal resolution of M,

$$\to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0 \tag{1}$$

Taking the R-dual sequence of (1) and using * to denote R-duals we define $D\Omega^p = \text{cokernel}\,(F_p^* \to F_{p+1}^*)$ for $p \geq 0$. According to [3] for every integer $p \geq 0$, there exists an exact sequence

$$\operatorname{Tor}_2^R(D\Omega^p, N) \to \operatorname{Ext}_R^p(M,R) \otimes N \to \operatorname{Ext}_R^p(M,N) \to \operatorname{Tor}_1^R(D\Omega^p, N) \to 0 \tag{2}$$

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where the maps are certain natural homomorphisms. Suppose $\mathrm{Ext}_R^p(M, N)$ is the zero module. Applying (2) we obtain $\mathrm{Tor}_1^R(D\Omega^p, N) = (0)$. Using [6] this implies $\mathrm{Tor}_j^R(D\Omega^p, N) = (0)$ for $j \geq 1$. The second application of (2) yields $\mathrm{Ext}_R^p(M, R) \otimes N = (0)$. Since $N \neq (0)$, we conclude that $\mathrm{Ext}_R^p(M, R) = (0)$, i.e. $\mathrm{Ext}_R^1(\Omega^{p-1}, R) = (0)$. Taking R-duals in the exact sequence $0 \to \Omega^p \to F_{p-1} \to \Omega^{p-1} \to 0$ and using the fact that $\mathrm{Ext}_R^1(\Omega^{p-1}, R) = (0)$, we obtain the following exact sequence

$$0 \to (\Omega^{p-1})^* \to F_{p-1}^* \to (\Omega^p)^* \to 0 . \tag{3}$$

Using the definition of $D\Omega^p$, we get an exact sequence

$$0 \to (\Omega^p)^* \to F_p^* \to F_{p+1}^* \to D\Omega^p \to 0 . \tag{4}$$

Putting (2) and (3) together and making use of the definition of $D\Omega^{p-1}$, we get the following exact sequence,

$$0 \to D\Omega^{p-1} \to F_{p+1}^* \to D\Omega^p \to 0 . \tag{5}$$

The exact sequence (5) gives $\mathrm{Tor}_j^R(D\Omega^{p-1}, N) = \mathrm{Tor}_{j+1}^R(D\Omega^p, N) = (0)$ for $j \geq 1$. The lemma follows after using this information in the exact sequence (2) with p replaced by $p - 1$.

Proof of the theorem: If $\operatorname{grade} M > \operatorname{proj.dim} N$, then clearly $\operatorname{depth} N > \operatorname{Krull\ dim} M$ and so applying [4] we find that $\mathrm{Hom}(M, N) = (0)$, a contradiction. Hence $\operatorname{grade} M \leq \operatorname{proj.dim} N$. The lemma gives an isomorphism $\mathrm{Ext}_R^{p-1}(M, R) \otimes N \cong \mathrm{Ext}_R^{p-1}(M, N)$. Now if $\operatorname{grade} M \geq p$, it is well-known that $\mathrm{Ext}_R^i(M, R) = (0)$ for $0 \leq i \leq p - 1$, so that $\mathrm{Ext}_R^{p-1}(M, N) = (0)$, a contradiction to the minimality of p. Hence $\operatorname{grade} M \leq p - 1$. Combining with the inequality $\operatorname{grade} M \leq \operatorname{proj.dim} N$ established before we find that $\operatorname{grade} M \leq \inf(p - 1, \operatorname{proj.dim} N)$. This proves the first part of the theorem. As for the second part we observe that $\mathrm{Ext}_R^q(M, M) = (0)$ implies, as in the lemma above that $\mathrm{Tor}_j^R(D\Omega^{q-1}, M) = (0)$ for $j \geq 1$. Using this in the exact sequence $0 \to \Omega^{q-1} \to F_{q-2} \to \cdots \to F_0 \to M \to 0$ we get $\mathrm{Tor}_j^R(D\Omega^{q-1}, \Omega^{q-1}) = (0)$ for $j \geq 1$. An application of (2) with $p = 0$ and M replaced by Ω^{q-1} shows that the natural map $(\Omega^{q-1})^* \otimes \Omega^{q-1} \to \mathrm{Hom}(\Omega^{q-1}, \Omega^{q-1})$ is an isomorphism. Hence Ω^{q-1} is projective, i.e. $\operatorname{proj.dim} M \leq q - 1$. The minimality of q implies that $\operatorname{proj.dim} M = q - 1$.

The theorem is proved.

We recall the following conjecture of M. Auslander

TOR CONJECTURE: If M is a module of finite projective dimension over a noetherian local ring R and N any other R-module such that

$$\mathrm{Tor}_1^R(M,N) = (0)\ , \quad \text{then} \quad \mathrm{Tor}_j^R(M,N) = (0) \quad \text{for} \quad j \geq 1\ .$$

It is well-known that this conjecture is true if R is regular local [6] and trivially so if proj. $\dim M \leq 1$. We remark that if the above conjecture is true then the lemma is valid for any noetherian local ring provided N has finite projective dimension. Consequently the second part of the theorem is also valid for any noetherian local ring provided M has finite projective dimension.

M. Auslander and O. Goldman have proved that a reflexive module M over a regular local ring R is free if and only if $\mathrm{Hom}(M,M)$ is free [1]. In his article on the purity of the Branch locus [2] M. Auslander asks if this result is true for any noetherian local ring provided one assumes that M has finite projective dimension. We shall show that the answer is yes if the Tor conjecture mentioned above is true. In fact we prove the following

PROPOSITITION. *Let M, N be reflexive modules of finite projective dimensions over a noetherian local ring R such that* $\mathrm{Hom}(M,N)$ *is a nonzero free R-module. Then if the Tor conjecture is true M and N are both free modules.*

Proof. By induction on the Krull-dimension of R and [1, Lemma 4.8] we easily find that $\mathrm{Ext}_R^1(M,N) = (0)$. As in the proof of the lemma we get an isomorphism $M^* \otimes N \cong \mathrm{Hom}(M,N)$. Hence $M^* \otimes N$ is a nonzero free module. From this it is easy to conclude that both M^* and N are free. Since M is reflexive, M and N are both free modules.

REFERENCES

[1] M. Auslander and O. Goldman: Maximal order, Trans. Amer. Math. Soc. Vol. **97** (1960).

[2] M. Auslander: On the purity of the branch locus, American Journal of Mathematics Vol. **84** (1962).

[3] M. Auslander: Coherent functors, Proceedings of a conference on categorical algebra, La Jolla 1965, Springer-Verlag Berlin, Heidelberg, New York 1966.

[4] F. Ischebeck: Eine Dualitat zwischen den Funktoren Ext und Tor, J. Algebra **11** (1969), 510–531.

[5] D. Rees: The grade of an ideal or module, Proc. Camb. Phil. Soc. **53** (1957).

[6] S. Lichtenbaum: On the vanishing of Tor in regular local rings Illinois. J. of Math. Vol. **10** (1966).

Tata Institute of Fundamental Research