P. W. Sy and T. Sunada
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DISCRETE SCHRÖDINGER OPERATORS ON A GRAPH

POLLY WEE SY* AND TOSHIKAZU SUNADA

In this paper, we study some spectral properties of the *discrete* Schrödinger operator $-\Delta + q$ defined on a locally finite connected graph with an automorphism group whose orbit space is a finite graph.

The discrete Laplacian and its generalization have been explored from many different viewpoints (for instance, see [2] [4]). Our paper discusses the discrete analogue of the results on the bottom of the spectrum established by T. Kobayashi, K. Ono and T. Sunada [3] in the Riemannian-manifold-setting.

§ 1. Discrete Laplacians

Let $X = (V, E)$ be a locally finite connected graph without loops and multiple edges. Here V and E are, respectively, the set of *vertices* and the set of *unoriented edges* of X. In a natural manner, X is regarded as a one-dimensional CW complex. We assign a positive *weight* to each vertex and also to each edge by giving mappings $m: V \to \mathbb{R}_+$ and $w: E \to \mathbb{R}_+$. Let $C_0(V)$ and $C_0(E)$ be the space of all complex-valued functions on V and E with finite support, respectively. Define inner products on $C_0(V)$ and $C_0(E)$ by

$$\langle f, g\rangle = \sum_{x\in V} f(x)\overline{g(x)}m(x) \tag{1.1}$$

$$\langle \omega, \eta\rangle = \sum_{e\in E} \omega(e)\overline{\eta(e)}w(e)\,. \tag{1.2}$$

The completions of $C_0(V)$ and $C_0(E)$ with respect to those inner products will be denoted by $L^2(V)$ and $L^2(E)$, respectively.

Each edge has two orientations. We use the symbol E^{or} to represent the set of all *oriented* edges, so that forgetting orientation yields a two-to-one map $p: E^{\mathrm{or}} \to E$. Reversing orientation gives rise to an involution on E^{or}, which we denote by $e \mapsto \bar{e}$. We shall use the same symbol w for

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the composition $w \circ p$, which is a function on E^{or}. For an oriented edge e, $\mathfrak{o}(e)$ and $\mathfrak{t}(e)$ denote the origin and terminus point of e, respectively. Let $\mathcal{O}_x = \{e \in E^{or}; \mathfrak{o}(e) = x\}$.

We fix an orientation on each edge by giving a subset E_0 of E^{or} such that $E^{or} = E_0 \cup \bar{E}_0$ (disjoint) and we identify E_0 with E by the map p. Define the operator $d : C_0(V) \to C_0(E)$ by

$$df(e) = f(\mathfrak{t}(e)) - f(\mathfrak{o}(e)), \tag{1.3}$$

which is a natural analogue of the exterior derivation on a manifold.

A simple calculation gives the following formula for the formal adjoint d^* of d:

$$d^*\omega(x) = m(x)^{-1}\{\sum_{\substack{e \in E_0 \\ \mathfrak{t}(e)=x}} \omega(e)w(e) - \sum_{\substack{e \in E_0 \\ \mathfrak{o}(e)=x}} \omega(e)w(e)\}.$$

The *discrete Laplacian* $\Delta = \Delta_X$ is now defined by

$$\Delta f(x) = -d^*df(x) = m(x)^{-1}\{\sum_{e \in \mathcal{O}_x} f(\mathfrak{t}(e))w(e) - (\sum_{e \in \mathcal{O}_x} w(e))f(x)\}. \tag{1.4}$$

Note that Δ is independent of the choice of orientation on edges.

Remark 1. Let $h : V \to \mathbb{R}$ be a function defined by

$$h(x) = (1/m(x)) \sum_{e \in \mathcal{O}_x} w(e).$$

Then the operator Δ is bounded as an operator acting in $L^2(V)$ if and only if h is bounded. For the sake of completeness, we shall give a proof. Suppose that h is bounded. Then for any $f \in C_0(V)$,

$$\begin{aligned}
\|df\|^2 &\leq 2 \sum_{e \in E_0} (|f(\mathfrak{t}(e))|^2 + |f(\mathfrak{o}(e))|^2)w(e) \\
&= 2\{\sum_{x \in V} \sum_{\substack{e \in E_0 \\ \mathfrak{t}(e)=x}} |f(\mathfrak{t}(e))|^2 w(e) + \sum_{x \in V} \sum_{\substack{e \in E_0 \\ \mathfrak{o}(e)=x}} |f(\mathfrak{o}(e))|^2 w(e)\} \\
&= 2\{\sum_{x \in V} |f(x)|^2 (\sum_{e \in \mathcal{O}_x} w(e))\} \\
&\leq c\|f\|^2,
\end{aligned}$$

where $c = 2 \sup_{x \in V}\{(1/m(x)) \sum_{e \in \mathcal{O}_x} w(e)\}$. Thus Δ is bounded. Conversely, assume that Δ is bounded. If h is unbounded, then for every positive real number K, there is an $x \in V$ such that $(1/m(x)) \sum_{e \in \mathcal{O}_x} w(e) \geq K$. We see that $\|d\delta_x\|^2 = \sum_{e \in \mathcal{O}_x} w(e) \geq Km(x) = K\|\delta_x\|^2$, where $\delta_x(y)$ equals 1 when $y = x$ and zero elsewhere. It follows that $\|\Delta\delta_x\|\|\delta_x\| \geq |(d^*d\delta_x, \delta_x)| = \|d\delta_x\|^2 \geq K\|\delta_x\|^2$. Thus Δ is unbounded. This contradicts our hypothesis that Δ is bounded.

Remark 2. The discrete Laplacian defined above is a bit generalized one of [2].

§ 2. Bottom of the spectrum

Let $M=(V, E)$ be a *finite* connected graph, and let $\pi: X \to M$ be a normal covering map as CW complexes with the covering transformation group Γ. The covering space X has a graph structure $(\tilde{V}, \tilde{E})$ such that π is a morphism of graphs. Then Γ acts freely on $\tilde{E}$ and $\tilde{V}$ and $\Gamma \backslash \tilde{E} \simeq E$, $\Gamma \backslash \tilde{V} \simeq V$. We assume that M has weights on vertices and edges. The weights on vertices and edges of X are naturally assigned by using the map π so that they are left invariant under the Γ-action. If we fix orientation on edges of M, then the induced one on $\tilde{E}$ is preserved by the Γ-action. Take any real-valued function $q_M \in C(V)$. We see that $q=q_M \circ \pi$ is invariant under the Γ-action. Since M is finite, $H_M=-\Delta_M+q_M$ is identified with a hermitian matrix of finite size and its spectrum consists of real eigenvalues.

The operator $H_X=-\Delta_X+q$ is just the lift of the operator H_M on M by the map π and is therefore bounded (see Remark 1) and self-adjoint. We denote by $\lambda_0(H)$ the greatest lower bound of the spectrum of a self-adjoint operator H. Note that $\lambda_0(H_M)$ is just the minimal eigenvalue of H_M.

LEMMA 1. *$\lambda_0(H_M)$ is simple and has a positive eigenfunction.*

Proof. Let $V=\{1, \cdots, n\}$. For $1 \leq i \leq n$, set

$$\begin{aligned} \varphi_i(x) &= \frac{1}{\sqrt{m(i)}} \qquad \text{if} \quad x=i \\ &= 0 \qquad\qquad \text{otherwise}. \end{aligned}$$

Then $\{\varphi_i\}$ is an orthonormal basis of $L^2(V)$. Let $A=(a_{ij})$ be the matrix of Δ_M with respect to this basis. If (i, j) is an edge of M with $i \neq j$, then $a_{ij}=(\Delta_M \varphi_j, \varphi_i)=(1/\sqrt{m(i)m(j)})\, w(i, j)$. Hence the off-diagonal entries of the matrix A are nonnegative real numbers. Let $A'=(a'_{ij})$ be the matrix with $a'_{ij}=a_{ij}$ for $i \neq j$ and $a'_{ii}=0$. Since M is connected, the matrix A' is irreducible. Thus the operator Δ_M-q_M has the form $A'+D$, where D is a diagonal matrix with entries $d_{ii} \in \mathbb{R}$. The facts that the maximal eigenvalue $-\lambda_0(A'+D)(=-\lambda_0(H_M))$ is simple and there exists a positive eigenfunction associated with it, follow readily by applying the Perron-Frobenius Theorem [5] to the matrix $A'+D+xI$ for large enough $x \in \mathbb{R}$.

THEOREM 1. $\lambda_0(H_M) \leq \lambda_0(H_X)$. *The equality holds if and only if the covering transformation group Γ is amenable.*

To prove this, we will employ a representation-theoretic technique. We fix orientation $\tilde{E}_0$ on $\tilde{E}$ induced from an orientation of edges of M. We also identify $\tilde{E}_0$ with $\tilde{E}$.

Let ρ be a unitary representation of Γ on a Hilbert space W and $L^2_\rho(V) = \{s : \tilde{V} \mapsto W;\ s(\sigma x) = \rho(\sigma)s(x) \text{ for all } x \in \tilde{V} \text{ and } \sigma \in \Gamma\}$ with the natural inner product

$$\langle s_1, s_2 \rangle = \sum_{x \in \mathscr{D}_V} \langle s_1(x), s_2(x) \rangle_W\, m(x)\,,$$

where $\mathscr{D}_V$ is a finite fundamental subset in $\tilde{V}$ for the Γ-action; i.e., $\mathscr{D}_V$ is a subset of $\tilde{V}$ such that for every $x \in \tilde{V}$, there exists a unique pair $(\sigma, x') \in \Gamma \times \mathscr{D}_V$ satisfying $\sigma x = x'$. Note that $\tilde{V} = \bigcap_{\gamma\in\Gamma} \gamma\mathscr{D}_V$ and $\gamma\mathscr{D}_V \cap \mathscr{D}_V = \phi$ for $\gamma \neq \mathrm{id}$. One can easily check that the inner product is independent of the choice of $\mathscr{D}_V$. Let $L^2_\rho(E) = \{\varphi : \tilde{E} \mapsto W;\ \varphi(\sigma e) = \rho(\sigma)\varphi(e) \text{ for all } e \in \tilde{E}$ and $\sigma \in \Gamma\}$ with the following inner product

$$\langle \varphi_1, \varphi_2 \rangle = \sum_{e \in \mathscr{D}_E} \langle \varphi_1(e), \varphi_2(e) \rangle_W\, w(e)\,,$$

where $\mathscr{D}_E$ is a finite fundamental subset in $\tilde{E}$ for the Γ-action. This definition also does not depend on the choice of $\mathscr{D}_E$.

The bounded operator $d_\rho : L^2_\rho(V) \to L^2_\rho(E)$ is defined by

$$d_\rho s(e) = s(\mathfrak{t}(e)) - s(\mathfrak{o}(e))\,.$$

LEMMA 2. *The adjoint operator of d_ρ is given by*

$$(d^*_\rho\varphi)(x) = m(x)^{-1}\Big(\sum_{\substack{e\in\tilde{E}_0 \\ \mathfrak{t}(e)=x}} \varphi(e)w(e) - \sum_{\substack{e\in\tilde{E}_0 \\ \mathfrak{o}(e)=0}} \varphi(e)w(e)\Big)\,.$$

Proof. First note that the correspondences

$$\begin{aligned} d_1 &: s \longmapsto \varphi_1 \qquad \varphi_1(e) = s(\mathfrak{t}(e)) \\ d_2 &: s \longmapsto \varphi_2 \qquad \varphi_2(e) = s(\mathfrak{o}(e)) \end{aligned}$$

give rise to operators of $L^2_\rho(V)$ into $L^2_\rho(E)$, and $d_\rho = d_1 - d_2$. Let $\mathscr{D}_V$ be a fundamental set in $\tilde{V}$, and put

$$\mathscr{D}_E = \{e \in \tilde{E}_0;\ \mathfrak{t}(e) \in \mathscr{D}_V\}\,.$$

Then $\mathscr{D}_E$ is a fundamental set in $\tilde{E} = \tilde{E}_0$, and

$$\begin{aligned} \langle d_1 s, \varphi \rangle &= \sum_{e\in\mathscr{D}_E} \langle s(\mathfrak{t}(e)), \varphi(e) \rangle_W\, w(e) \\ &= \sum_{x\in\mathscr{D}_V} \sum_{\substack{e\in\tilde{E}_0 \\ \mathfrak{t}(e)=x}} \langle s(x), \varphi(e) \rangle_W\, w(e)\,. \end{aligned}$$

Thus we have

$$d_1^*\varphi(x) = m(x)^{-1} \sum_{\substack{e\in \bar{E}_0 \\ t(e)=x}} \varphi(e)w(e)\,.$$

Similarly, we obtain

$$d_2^*\varphi(x) = m(x)^{-1} \sum_{\substack{e\in \bar{E}_0 \\ o(e)=x}} \varphi(e)w(e)\,.$$

This completes the proof.

The Laplacian Δ_ρ acting on $L^2_\rho(V)$ is now defined by $-d_\rho^* d_\rho$ which is equal to

$$\Delta_\rho s(x) = m(x)^{-1}\{\sum_{e\in \mathcal{O}_x} s(t(e))w(e) - (\sum_{e\in \mathcal{O}_x} w(e))s(x)\}\,.$$

The twisted discrete Schrödinger operator is then defined as the self-adjoint operator $H_\rho = -\Delta_\rho + q$.

LEMMA 3. *If ρ is the right regular representation of Γ, then $(H_\rho, L^2_\rho(V))$ is unitarily equivalent to $(H_X, L^2(\tilde{V}))$; and if ρ is the trivial representation 1, then $(H_\rho, L^2_\rho(V))$ is unitarily equivalent to $(H_M, L^2(V))$.*

Proof. Let $W = L^2(\Gamma) = \{\varphi : \Gamma \to \mathbb{C} \mid \sum_{\sigma\in\Gamma} |\varphi(\sigma)|^2 < \infty\}$ and ρ be the right regular representation ρ_r of Γ on W. From now on, we simply write ρ for ρ_r. To prove that H_ρ and H_X are unitarily equivalent to each other, we have to show that there exists a unitary map $\Phi : L^2(\tilde{V}) \mapsto L^2_\rho(V)$ such that $H_\rho \circ \Phi = \Phi \circ H_X$.

Define the map $\Phi : C_0(\tilde{V}) \mapsto L^2_\rho(V)$ by

$$\Phi(f) = s\,,$$

where the function s is defined to be $s(x)(\sigma) = f(\sigma x)$ for $x \in \tilde{V}$, $\sigma \in \Gamma$. One can check that $s(\mu x) = \rho(\mu)s(x)$ for any $\mu \in \Gamma$, $x \in \tilde{V}$. By the definition of fundamental set, we have

$$\begin{aligned} \|s\|^2 &= \sum_{x\in\mathcal{D}_V} \|s(x)\|_W^2 m(x) \\ &= \sum_{x\in\mathcal{D}_V} \sum_{\sigma\in\Gamma} |f(\sigma x)|^2 m(x) \\ &= \|f\|^2 \end{aligned}$$

for any $f \in C_0(\tilde{V})$. Thus $s \in L^2_\rho(V)$. Hence the map Φ is extended uniquely to an isometry of $L^2(\tilde{V})$ into $L^2_\rho(V)$.

Next, we claim that Φ is onto. Take any $s \in L^2_\rho(V)$, define $f : \tilde{V} \to \mathbb{C}$

by $f(x) = s(x)1$, where 1 is the identity element of Γ. Since

$$\begin{aligned}\sum_{x\in\tilde{V}}|f(x)|^2 m(x) &= \sum_{x\in\tilde{V}}|s(x)1|^2 m(x)\\ &= \sum_{x\in\mathscr{D}_V}\sum_{\sigma\in\Gamma}|\rho(\sigma)s(x)1|^2 m(x)\\ &= \sum_{x\in\mathscr{D}_V}\sum_{\sigma\in\Gamma}|s(x)\sigma|^2 m(x)\\ &= \sum_{x\in\mathscr{D}_V}\|s(x)\|^2 m(x)\,,\end{aligned}$$

therefore $f\in L^2(\tilde{V})$. Put $s'=\Phi(f)$. Then $s'(x)(\sigma)=f(\sigma x)=s(\sigma x)1=[\rho(\sigma)s(x)]1 = s(x)(\sigma)$ for every $x\in\tilde{V}$ and $\sigma\in\Gamma$. Hence $\Phi(f) = s' = s$.

For any $f\in L^2(\tilde{V})$, we have

$$\begin{aligned}&(\{H_\rho\circ\Phi(f)\}(x))(\sigma) = (\{H_\rho\circ s(x)\}(\sigma))\\ &= -\frac{1}{m(x)}\{\sum_{e\in\mathscr{O}_x}s(\mathfrak{t}(e))\sigma w(e) - (\sum_{e\in\mathscr{O}_x}w(e))s(x)\sigma\} + q(x)s(x)\sigma\\ &= -\frac{1}{m(x)}\{\sum_{e\in\mathscr{O}_x}f(\sigma\mathfrak{t}(e))w(e) - (\sum_{e\in\mathscr{O}_x}w(e))f(\sigma x)\} + q(x)f(\sigma x)\\ &= -\frac{1}{m(x)}\{\sum_{e\in\mathscr{O}_{\sigma x}}f(\mathfrak{t}(e))w(e) - (\sum_{e\in\mathscr{O}_x}w(e))f(\sigma x)\} + q(\sigma x)f(\sigma x)\\ &= H_X f(\sigma x)\\ &= (\{\Phi\circ H_X(f)\}(x))\sigma\,.\end{aligned}$$

This proves the first part of the theorem.

The second part of the theorem is easy to prove.

The Kazhdan distance $\delta(\rho, \mathbf{1})$ (or $\delta_A(\rho, \mathbf{1})$) between ρ and $\mathbf{1}$ is defined by

$$\delta(\rho, \mathbf{1}) = \inf_{\substack{v\in w\\ \|v\|=1}}\sup_{\sigma\in A}\|\rho(\sigma)v - v\|\,,$$

where A is a fixed finite set of generators of Γ. The following lemma shows that the distance does not depend essentially on the choice of A.

LEMMA 4. *Suppose that A and B are any finite sets of generators of Γ. Then there exist positive constants k_1 and k_2 such that*

$$k_1\delta_B(\rho, \mathbf{1}) \le \delta_A(\rho, \mathbf{1}) \le k_2\delta_B(\rho, \mathbf{1})\,.$$

Proof. Let $C = A\cup B$. Choose an integer N large enough such that every $\sigma\in C$ can be expressed as

$$\sigma = \mu_1\mu_2\cdots\mu_n\,,$$

where $\mu_i\in A$ and $n\le N$. Then

$$\begin{aligned}\|\rho(\sigma)v - v\| &\leq \|\rho(\mu_1)\cdots\rho(\mu_n)v - \rho(\mu_1)\cdots\rho(\mu_{n-1})v\| \\ &\quad + \|\rho(\mu_1)\cdots\rho(\mu_{n-1})v - v\| \\ &\leq \|\rho(\mu_n)v - v\| + \|\rho(\mu_1)\cdots\rho(\mu_{n-1})v - v\| \\ &\leq \sum_{i=1}^{n}\|\rho(\mu_i)v - v\| \\ &\leq N\sup_{\mu\in A}\|\rho(\mu)v - v\| .\end{aligned}$$

It follows that $\delta_A \geq c_1\delta_C$ for some constant c_1. Similarly, one can also show that $\delta_B \geq c_2\delta_C$ for some constant c_2. On the other hand, since $A, B \subset C$, we have $\delta_C \geq \delta_A, \delta_B$. These inequalities together prove the result.

To prove Theorem 1, it suffices to establish the following Theorem (cf. [4] [7]). For, in the next theorem, when ρ is the right regular representation ρ_r, Theorem 1 follows from the fact that $\delta(\rho_r, \mathbf{1}) = 0$ if and only if Γ is amenable.

THEOREM 2. *There exist positive constants c_1 and c_2 such that*

$$c_1\delta(\rho, \mathbf{1})^2 \leq \lambda_0(H_\rho) - \lambda_0(H_{\mathbf{1}}) \leq c_2\delta(\rho, \mathbf{1})^2$$

for all ρ. In particular, $\lambda_0(H_\rho) = \lambda_0(H_{\mathbf{1}})$ if and only if $\delta(\rho, \mathbf{1}) = 0$.

Proof. Note that

$$\lambda_0(H_\rho) = \inf_{s\in L^2_\rho(V)} \frac{\langle H_\rho s, s\rangle}{\|s\|^2} .$$

By Lemma 1, we may take a positive solution $f \in L^2(V)$ to the equation $H_M f = \lambda_0(H_{\mathbf{1}})f$. We have

$$\begin{aligned}(2.1)\qquad \langle \Delta_\rho(fs), fs\rangle = \sum_{x\in\mathscr{D}_V}\langle \sum_{e\in\mathcal{O}_x} f(\mathfrak{t}(e))s(\mathfrak{t}(e))w(e) \\ - (\sum_{e\in\mathcal{O}_x} w(e))f(x)s(x), f(x)s(x)\rangle .\end{aligned}$$

Substituting the following equality

$$\sum_{e\in\mathcal{O}_x} w(e)f(x) = \lambda_0(H_{\mathbf{1}})f(x)m(x) - q(x)f(x)m(x) + \sum_{e\in\mathcal{O}_x} f(\mathfrak{t}(e))w(e)$$

into (2.1), we obtain

$$\begin{aligned}(2.2)\qquad \langle \Delta_\rho(fs), fs\rangle = \sum_{x\in\mathscr{D}_V}\langle \sum_{e\in\mathcal{O}_x} f(\mathfrak{t}(e))(s(\mathfrak{t}(e)) - s(x))w(e), f(x)s(x)\rangle \\ - \lambda_0(H_{\mathbf{1}})\langle fs, fs\rangle + \langle qfs, fs\rangle .\end{aligned}$$

We now set $\mathscr{D} = \{e \in \tilde{E}^{\mathrm{or}}; e \in \mathcal{O}_x$ for some $x \in \mathscr{D}_V\}$. It is easy to check that $\mathscr{D}$ and $\bar{\mathscr{D}} = \{\bar{e}; e \in \mathscr{D}\}$ are fundamental sets in $\tilde{E}^{\mathrm{or}}$ for the natural Γ-action.

Note that, if $g_i(\sigma e) = \rho(\sigma) g_i(e)$, $i = 1, 2$, for every $\sigma \in \Gamma$ and $e \in \tilde{E}^{\mathrm{or}}$, then the summation

$$\sum_{e \in \mathscr{D}} \langle g_1(e), g_2(e) \rangle$$

does not depend on the choice of a fundamental set $\mathscr{D}$. Therefore we find

$$\begin{aligned}
&\sum_{x \in \mathscr{D}_V} \langle \sum_{e \in \mathcal{O}_x} f(\mathfrak{t}(e))(s(\mathfrak{t}(e)) - s(x)) w(e), f(x) s(x) \rangle \\
&\quad = \sum_{e \in \mathscr{D}} \langle f(\mathfrak{t}(e))(s(\mathfrak{t}(e)) - s(\mathfrak{o}(e))) w(e), f(\mathfrak{o}(e)) s(\mathfrak{o}(e)) \rangle \\
&\quad = \sum_{\bar{e} \in \bar{\mathscr{D}}} \langle f(\mathfrak{t}(\bar{e}))(s(\mathfrak{t}(\bar{e})) - s(\mathfrak{o}(\bar{e}))) w(\bar{e}), f(\mathfrak{o}(\bar{e})) s(\mathfrak{o}(\bar{e})) \rangle \\
&\quad = \sum_{e \in \mathscr{D}} \langle f(\mathfrak{o}(e))(s(\mathfrak{o}(e)) - s(\mathfrak{t}(e))) w(e), f(\mathfrak{t}(e)) s(\mathfrak{t}(e)) \rangle \\
&\quad = \sum_{x \in \mathscr{D}_V} \langle \sum_{e \in \mathcal{O}_x} f(x)(s(x) - s(\mathfrak{t}(e))) w(e), f(\mathfrak{t}(e)) s(\mathfrak{t}(e)) \rangle ,
\end{aligned}$$

so that

$$\begin{aligned}
&\sum_{x \in \mathscr{D}_V} \sum_{e \in \mathcal{O}_x} f(\mathfrak{t}(e)) f(x) \| s(\mathfrak{t}(e)) - s(x) \|_W^2 w(e) \\
&\quad = \sum_{x \in \mathscr{D}_V} \sum_{e \in \mathcal{O}_x} \{ \langle f(x)(s(\mathfrak{t}(e)) - s(x)) w(e), f(\mathfrak{t}(e)) s(\mathfrak{t}(e)) \rangle \\
&\qquad - \langle f(x)(s(\mathfrak{t}(e)) - s(x)) w(e), f(\mathfrak{t}(e)) s(x) \rangle \} \\
&\quad = -2 \sum_{x \in \mathscr{D}_V} \langle \sum_{e \in \mathcal{O}_x} f(\mathfrak{t}(e))(s(\mathfrak{t}(e)) - s(x)) w(e), f(x) s(x) \rangle .
\end{aligned}$$

Combining this with (2.2), we deduce

$$\frac{\langle -\Delta_\rho fs, fs \rangle + \langle q(fs), fs \rangle}{\| fs \|^2} = \lambda_0(H_1) + \frac{1}{2} P ,$$

where

$$P = \frac{\sum_{x \in \mathscr{D}_V} \sum_{e \in \mathcal{O}_x} f(\mathfrak{t}(e)) f(x) \| d_\rho s(e) \|_W^2 w(e)}{\sum_{x \in \mathscr{D}_V} f(x)^2 \| s(x) \|_W^2 m(x)} .$$

There are positive constants k_1, k_2 such that

$$k_1 P' \leq \inf_{fs \in L_\rho^2(V)} P \leq k_2 P' ,$$

where

$$P' = \frac{\sum_{x \in \mathscr{D}_V} \sum_{e \in \mathcal{O}_x} \| d_\rho s(e) \|_W^2 w(e)}{\sum_{x \in \mathscr{D}_V} \| s(x) \|_W^2 m(x)} .$$

Thus, it is enough to show that

$$c_1 \delta(\rho, \mathbf{1})^2 \leq \inf P' \leq c_2 \delta(\rho, \mathbf{1})^2 .$$

We now let $\mathscr{U}(\mathscr{D})$ be the set of vertices $x \in \tilde{V}$ such that there exists $e \in \mathscr{D}$ with $\mathfrak{t}(e) = x$. It follows from the definition of fundamental set that for every $y \in \mathscr{U}(\mathscr{D})$, there is a unique $\sigma_y \in \Gamma$ with $y \in \sigma_y \mathscr{D}_V$. Consider $B = \{\sigma_y ; y \in \mathscr{U}(\mathscr{D})\} \cup A$, another finite set of generators of Γ. From the definition of $\delta_B(\rho, \mathbf{1})$, it follows that for every $\varepsilon > 0$, there exists a $v \in W$ with $\|v\| = 1$ such that $\|\rho(\sigma)v - v\| \leq \delta_B(\rho, \mathbf{1}) + \varepsilon$ for all $\sigma \in B$. For this fixed v, we define a function $s : V \to W$ by setting $s(x) = v$ for all $x \in \mathscr{D}_V$ and $s(\sigma x) = \rho(\sigma)v$ for every $\sigma x \in \sigma\mathscr{D}_V$. It is clear that $s \in L^2_\rho(V)$. Thus

$$\sum_{x \in \mathscr{D}_V} \|s(x)\|^2 m(x) = \sum_{x \in \mathscr{D}_V} m(x)$$

and

$$\sum_{x \in \mathscr{D}_V} \sum_{e \in \mathscr{O}_x} \|d_\rho s(e)\|^2 w(e) \leq \{\max_{e \in E} w(e)\} \sum_{\sigma \in B} \|\rho(\sigma)v - v\|^2 \leq C\{\delta_B(\rho, \mathbf{1}) + \varepsilon\}^2 .$$

Since ε is arbitrary, we obtain

$$\inf P' \leq c_2 \delta(\rho, \mathbf{1})^2$$

for some positive constant c_2.

We next show the inequality $c_1 \delta(\rho, \mathbf{1})^2 \leq \inf P'$ for some positive constant c_1. Since for a unit vector v,

$$\delta(\rho, \mathbf{1})^2 \leq \sum_{\sigma \in A} \|\rho(\sigma)v - v\|^2 ,$$

by substituting $v = s(x)/\|s(x)\|$, we have

$$\delta(\rho, \mathbf{1})^2 \sum_{x \in \mathscr{D}_V} \|s(x)\|^2 m(x) \leq \sum_{x \in \mathscr{D}_V} \sum_{\sigma \in A} \|s(\sigma x) - s(x)\|^2 m(x) \tag{2.3}$$

for every $s \in L^2_\rho(V)$. For each $x \in \mathscr{D}_V$ and $\sigma \in \Gamma$, we choose a path $C(x, \sigma x)$ in X joining x and σx. Let $|C(x, \sigma x)| = \#$ {edges in the path $C(x, \sigma x)$} and $K = \max_{x \in \mathscr{D}_V} \max_{\sigma \in A} |C(x, \sigma x)|$. The inequality (2.3) and

$$\|s(\sigma x) - s(x)\|^2 \leq K \sum_{e \in C(x, \sigma x)} \|s(\mathfrak{t}(e)) - s(\mathfrak{o}(e))\|^2$$

imply

$$\delta(\rho, \mathbf{1})^2 \sum_{x \in \mathscr{D}_V} \|s(x)\|^2 m(x) \leq c(\# \mathscr{D}) K^2 \sum_{x \in \mathscr{D}_V} \sum_{e \in \mathscr{D}_x} \|d_\rho s(e)\|^2 w(e) ,$$

where $c = \max_{x \in V} m(x) \times (\min_{e \in E} w(e))^{-1} \times (\# A)$. Thus the proof of the theorem is complete.

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Polly Wee Sy
Department of Mathematics
University of the Philippines
Diliman, Quezon City
Philippines

T. Sunada
Department of Mathematics
University of Tokyo
Tokyo 113
Japan