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ALGEBRAIC $K3$ SURFACES WITH FINITE AUTOMORPHISM GROUPS

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Introduction

The purpose of this paper is to give a proof to the result announced in [3]. Let X be an algebraic surface defined over $\boldsymbol{C}$. X is called a $K3$ *surface* if its canonical line bundle K_X is trivial and $\dim H^1(X, \mathcal{O}_X) = 0$. It is known that the automorphism group $\mathrm{Aut}(X)$ of X is isomorphic, up to a finite group, to the factor group $O(S_X)/W_X$, where $O(S_X)$ is the automorphism group of the Picard lattice of X (i.e. S_X is the Picard group of X together with the intersection form) and W_X is its subgroup generated by all reflections associated with elements with square (-2) of S_X ([11]). Recently Nikulin [8], [10] has completely classified the Picard lattices of algebraic $K3$ surfaces with finite automorphism groups.

Our goal is to compute the automorphism groups of such $K3$ surfaces. Let X be an algebraic $K3$ surface with finite automorphism group $\mathrm{Aut}(X)$. By definition, there exists a nowhere vanishing holomorphic 2-form ω_X on X. Since an automorphism g of X preserves ω_X, up to constants, $g^*\omega_X = \alpha_X(g)\cdot\omega_X$ where $\alpha_X(g) \in \boldsymbol{C}^*$. Therefore we have an exact sequence

$$1 \longrightarrow G_X \longrightarrow \mathrm{Aut}(X) \xrightarrow{\alpha_X} \boldsymbol{Z}/m \longrightarrow 1 \tag{1}$$

where $\boldsymbol{Z}/m$ is a cyclic group of m-th root of unity in $\boldsymbol{C}^*$ and G_X is the kernel of α_X. Moreover the representation of the cyclic group $\boldsymbol{Z}/m$ in $T_X \otimes \boldsymbol{Q}$ is isomorphic to a direct sum of irreducible representations of the cyclic group $\boldsymbol{Z}/m$ over $\boldsymbol{Q}$ of maximal rank $\phi(m)$, where T_X is a transcendental lattice of X and ϕ is the Euler function. In paticular $\phi(m) \leq \mathrm{rank}\, T_X$ and hence $m \leq 66$ ([6], Theorem 3.1 and Corollary 3.2).

An algebraic $K3$ surface X is called *general* if the image of α_X is of order at most 2, and X is called *special* if it is not general. The meaning of this definition is as follows: Let X be an algebraic $K3$ surface with

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a Picard lattice S_X. Let S be an abstract lattice which is isomorphic to S_X. Denote by M_S the moduli space for algebraic $K3$ surfaces whose Picard lattices are isomorphic to S. Then the dimension of M_S is equal to $20 - \mathrm{rank}(S)$. A *general* $K3$ surface Y with $S_Y = S$ corresponds to a point of the complement of hypersurfaces in M_S.

THEOREM. *Let X be an algebraic $K3$ surface with finite automorphism group* $\mathrm{Aut}(X)$.

(i) *If X is general, then* $\mathrm{Aut}(X)$ *is as in the following table*:

Table 1.

S_X	$\mathrm{Aut}(X)$
$U \oplus E_8 \oplus E_8 \oplus A_1$	$\mathfrak{S}_3 \times Z/2$
$U \oplus E_8 \oplus E_8$, $U \oplus E_8 \oplus E_7$ $U \oplus E_8 \oplus D_6$, $U \oplus E_8 \oplus D_4 \oplus A_1$ $U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus A_1^4$ $U \oplus E_7 \oplus A_1^4$, $U \oplus D_6 \oplus A_1^4$ $U \oplus D_4 \oplus A_1^5$ $U(2) \oplus D_4 \oplus D_4$, $U \oplus A_1^8$ $U(2) \oplus A_1^7$	$Z/2 \times Z/2$
otherwise	$Z/2$ *or* $\{1\}$

where U (resp. $U(2)$) is the lattice of rank 2 with the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\left(\textit{resp. } \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}\right)$, *$A_m$, D_n and E_k are negative definite lattices associated with the Dynkin diagrams of type A_m, D_n and E_k respectively and A_1^k denotes the direct sum $A_1 \oplus A_1 \oplus \cdots \oplus A_1$ (k times).*

(ii) *If X is special, then* $\mathrm{Aut}(X)$ *is a cyclic extension of the group in the above table.*

We remark here that there exists a *special* $K3$ surface X with $\mathrm{Aut}(X) \simeq Z/66$. This automorphism acts on the Picard group of X as identity. In [4], we studied automorphisms with this property.

Also for Enriques surfaces with finite automorphism groups, we refer the reader to [2], [9].

To prove the above theorem we use the following phenomenon: In the exact sequence (1), if $\operatorname{rank}(S_X)$ becomes smaller, then G_X too becomes smaller, and the group Z/m grows bigger.

In Section 1, we recall the Picard lattices of algebraic $K3$ surfaces with finite automorphism groups. Section 2 is devoted to the results on finite automorphisms of $K3$ surfaces due to Nikulin [6] and Mukai [5]. In particular from these results we obtain all the possible cases of G_X (Lemma 2.3). In Sections 4 and 5 we prove the above theorem. In case $\operatorname{rank}(S_X) \geq 15$ we have the dual graph of all smooth rational curves on X ([8], Sect. 4, Part 5, Table 2) and hence we can compute $\operatorname{Aut}(X)$. In case $\operatorname{rank}(S_X) \leq 14$ it follows from the result in Section 2 that G_X is a subgroup of $Z/3$ or $Z/2 \times Z/2$. To determine $\operatorname{Aut}(X)$ we use the theory of symmetric bilinear forms (cf. [7]) and that of elliptic pencils due to Kodaira [1] and Shioda [12] (Sect. 3).

§ 1. Picard lattices of $K3$ surfaces with finite automorphism groups

In this section we recall the Nikulin's classification [8], [10] of Picard lattices of algebraic $K3$ surfaces with finite automorphism groups.

A *lattice* L is a free Z-module of finite rank endowed with an integral bilinear form $\langle\ ,\ \rangle$. By $L_1 \oplus L_2$ we denote the orthogonal direct sum of lattices L_1 and L_2. For a lattice L and an integer m we denote by $L(m)$ the lattice whose bilinear form is the one on L multiplied by m. Also we denote by U the lattice of rank 2 with the intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and by A_m, D_n and E_k the negative definite lattices associated with the Dynkin diagram of type A_m, D_n and E_k respectively. A lattice L is called *even* if $\langle x, x\rangle \in 2Z$ for all $x \in L$. Let S be a non degenerate lattice. We denote by $S^* = \operatorname{Hom}(S, Z)$ the dual of S. Put $A_S = S^*/S$. Then A_S is a finite abelian group which is called the *discriminant group* of S. We denote by $l(S)$ the number of minimal generators of A_S. A lattice S is called a 2-*elementary* if A_S is a 2-elementary abelian group. For a 2-elementary lattice S, we define a *parity* $\delta(S)$ of S as follows:

$$\delta(S) = \begin{cases} 0 & \text{if } q_S(x) = 0 \text{ for all } x \in A_S \\ 1 & \text{otherwise} \end{cases}$$

where q_S is the quadratic form on A_S induced from the one on S.

Proposition 1.1 ([8], Theorem 4.3.2). *An indefinite* 2*-elementary even lattice is determined, up to isomorphisms, by the invariants* ($\operatorname{rank}(S)$, $l(S)$,

$\delta(S)$).

The following tables give the description of Picard lattices of rank ≥ 9 of algebraic $K3$ surfaces with finite automorphism groups which we need for the proof of our theorem.

Table 2 (S_X is 2-elementary, rank $S_X \geq 9$).

rank(S_X)	S_X
19	$U\oplus E_8\oplus E_8\oplus A_1$
18	$U\oplus E_8\oplus E_8$
17	$U\oplus E_8\oplus E_7$
16	$U\oplus E_8\oplus D_6$
15	$U\oplus E_8\oplus D_4\oplus A_1$
14	$U\oplus E_8\oplus D_4$, $U\oplus D_8\oplus D_4$, $U\oplus E_8\oplus A_1^4$
13	$U\oplus E_8\oplus A_1^3$, $U\oplus E_7\oplus A_1^4$
12	$U\oplus E_8\oplus A_1^2$, $U\oplus E_7\oplus A_1^3$, $U\oplus D_6\oplus A_1^4$
11	$U\oplus E_8\oplus A_1$, $U\oplus E_7\oplus A_1^2$, $U\oplus D_6\oplus A_1^3$, $U\oplus D_4\oplus A_1^5$
10	$U\oplus E_8$, $U\oplus D_8$, $U\oplus D_4\oplus D_4$, $U(2)\oplus D_4\oplus D_4$, $U\oplus E_7\oplus A_1$, $U\oplus D_6\oplus A_1^2$, $U\oplus D_4\oplus A_1^4$, $U\oplus A_1^8$
9	$U\oplus E_7$, $U\oplus D_6\oplus A_1$, $U\oplus D_4\oplus A_1^3$, $U\oplus A_1^7$, $U(2)\oplus A_1^7$

Table 3 (S_X is not 2-elementary and rank$(S_X) \geq 9$).

rank(S_X)	S_X
13	$U\oplus E_8\oplus A_3$
12	$U\oplus E_8\oplus A_2$
11	$U\oplus E_6\oplus A_2$
9	$U\oplus A_7$, $U\oplus D_4\oplus A_3$, $U\oplus D_5\oplus A_2$, $U\oplus D_7$, $U\oplus E_6\oplus A_1$

§ 2. Finite automorphisms of $K3$ surfaces

Let X be an algebraic $K3$ surface. We denote by $\mathrm{Aut}(X)$ the group of automorphisms of X. Let G be a finite subgroup of $\mathrm{Aut}(X)$ and let ω_X be a nowhere vanishing holomorphic 2-form on X. Then for $g \in G$, $g^*\omega_X = \alpha_X(g) \cdot \omega_X$ where $\alpha_X(g) \in \boldsymbol{C}^*$. Therefore we have an exact sequence

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\alpha_X} \boldsymbol{Z}/m \longrightarrow 1$$

where $\boldsymbol{Z}/m$ is a cyclic group of m-th root of unity in $\boldsymbol{C}^*$ and K is the kernel of α_X. Moreover the representation of the cyclic group $\boldsymbol{Z}/m$ in $T_X \otimes \boldsymbol{Q}$ is isomorphic to a direct sum of irreducible representations of the cyclic group $\boldsymbol{Z}/m$ over $\boldsymbol{Q}$ of maximal rank $\phi(m)$, where ϕ is the Euler function. In particular $\phi(m) \leqq \mathrm{rank}(T_X)$ and hence $m \leq 66$ ([6], Theorem 3.1 and Corollary 3.2).

An automorphism g of X is called *symplectic* if $\alpha_X(g) = 1$. The classification of finite symplectic automorphism groups of $K3$ surfaces is recently given by S. Mukai [5], based on the study of abelian groups due to Nikulin [6].

PROPOSITION 2.1 ([6], § 5, [5], (0.1)). *Let g be a symplectic automorphism of finite order n of a $K3$ surface. Then $n \leq 8$ and the number of fixed points $f(n)$ depends only on n and is as follows:*

n	2	3	4	5	6	7	8
$f(n)$	8	6	4	4	2	3	2

Let G be a finite symplectic automorphism group of a $K3$ surface. Put $f(1) = 24$ and $\mu(G) = (1/|G|) \sum_{g \in G} f(|g|)$. By the Lefschetz fixed point formula and an elementary representation theory, we have

PROPOSITION 2.2 ([5], Proposition 2.4). $\mu(G) = 2 + \mathrm{rank}(L^G)$ *where* $L = H^2(X, \boldsymbol{Z})$ *and* $L^G = \{x \in L \mid g^*x = x$ *for any* $g \in G\}$.

In what follows we assume that $\mathrm{Aut}(X)$ is finite. Then we have an exact sequence

$$1 \longrightarrow G_X \longrightarrow \mathrm{Aut}(X) \xrightarrow{\alpha_X} \boldsymbol{Z}/m \longrightarrow 1$$

where G_X is the kernel of α_X. In Section 5 we shall need the following:

LEMMA 2.3. (i) *If* $\mathrm{rank}(S_X) \leq 14$, *then* G_X *is a subgroup of* $Z/3$ *or* $Z/2 \times Z/2$; (ii) *If* $\mathrm{rank}(S_X) \leq 12$, *then* G_X *is a subgroup of* $Z/2$; (iii) *If* $\mathrm{rank}(S_X) \leq 8$, *then* $G_X = \{1\}$.

Proof. It follows from [6], Theorem 1.1 that L^{G_X} contains T_X. Since G_X is finite, the signature of $S_X^{G_X}$ is equal to $(1, r)$, where r is a non negative integer. Hence $\mathrm{rank}(L^{G_X}) \geq \mathrm{rank}(T_X) + 1$. Note that $\mathrm{rank}(T_X) + \mathrm{rank}(S_X) = 22$. Now the assersions easily follows from Propositions 2.1 and 2.2.

PROPOSITION 2.4 ([6], § 10). *Assume that* $G = G_X$ *is a subgroup of* $Z/3$ *or* $Z/2 \times Z/2$. *Then the discriminant group* A_{L^G} *of* L^G *is as follows*:

G	$Z/2$	$Z/2 \times Z/2$	$Z/3$
A_{L^G}	$(Z/2)^8$	$(Z/2)^8 \times (Z/4)^2$	$(Z/3)^6$

§ 3. Elliptic pencils on $K3$ surfaces

Let X be a $K3$ surface. An *elliptic pencil* $\pi\colon X \to P^1$ is a holomorphic map π from X to P^1 whose general fibres are smooth elliptic curves. An effective divisor D is called a m-section of π if $D \cdot F = m$, where F is a fibre of π and $m \in N$. A 1-section is simply called a *section*. All singular fibres of an elliptic pencil were classified by Kodaira [1]. We use the terminology of singular fibres in [1]. The following lemma follows from [11], § 3, Corollary 3, the Riemann-Roch theorem and the classification of singular fibres of elliptic pencils [1].

LEMMA 3.1. *Let* X *be an algebraic* $K3$ *surface and let* S_X *be the Picard lattice of* X. *Assume that* $S_X = U \oplus K$, *where* K *is a negative definite lattice. Then*

(i) *there exists an elliptic pencil* $\pi\colon X \to P^1$ *with a section.*

(ii) *If* $K = K_1 \oplus N$, *where* K_1 *and* N *are negative definite lattices and* N *is generated by elements with square* (-2), *then* π *has a singular fibre* F *as in the following table*:

N	A_1	A_2	A_n $(n \geq 3)$	D_n $(n \geq 4)$	E_6	E_7	E_8
F	I_2 *or* III	I_3 *or* IV	I_{n+1}	I_{n-4}^*	IV*	III*	II*

The following will be used in the latter to prove the existence of symplectic automorphisms.

PROPOSITION 3.2 ([1], Theorem 12.2, [12], Corollaries 1.5, 1.7). *Let X be an algebraic $K3$ surface and S_X the Picard lattice of X. Let $\pi\colon X \to \boldsymbol{P}^1$ be an elliptic pencil with a section and let F_ν $(1 \leq \nu \leq k)$ be all singular fibres of π. We denote respectively by ε_ν, m_ν or μ_ν the Euler number of F_ν, the number of irreducible components of F_ν or the number of simple components of F_ν. Then*

(i) $\sum_{\nu=1}^{k} \varepsilon_\nu = 24$ (*= the Euler number of X*),

(ii) $\operatorname{rank}(S_X) = r + 2 + \sum_{\nu=1}^{k} (m_\nu - 1)$

where r is the rank of the group of sections of π,

(iii) *when $r = 0$, let n denote the order of the group of sections of π. Then we have*

$$|\det(S_X)| = \prod_{\nu=1}^{k} \mu_\nu / n^2 .$$

§ 4. Proof of the Theorem—the case when rank$(S_X) \geq 15$

In this section and the next we prove our theorem. By our proof in the following, we can see:

PROPOSITION. *Let X be an algebraic $K3$ surface with finite automorphism group* $\operatorname{Aut}(X)$. *Then the subgroup G_X of symplectic automorphisms of* $\operatorname{Aut}(X)$ *is uniquely determined by the isomorphism class of S_X.*

The assersion (ii) in Theorem follows from this Proposition and the exact sequence (1). For simplicity, in the following, we assume that X is a general algebraic $K3$ surface with finite automorphism group.

Let X be a *general* algebraic $K3$ surface with finite automorphism group and $\operatorname{rank}(S_X) \geq 15$. Then S_X is a 2-elementary lattice (see Table 2). By [8], Section 4, there exists an automorphism σ of order 2 such that $\sigma^* | S_X = 1_{S_X}$ and $\sigma^* | T_X = -1_{T_X}$. Therefore we have an exact sequence:

$$1 \longrightarrow G_X \longrightarrow \operatorname{Aut}(X) \xrightarrow{\alpha_X} \boldsymbol{Z}/2 \longrightarrow 1$$

where $\boldsymbol{Z}/2$ is generated by σ. Since $g^* | T_X = 1_{T_X}$ for all $g \in G_X$, $g^* \circ \sigma^* = \sigma^* \circ g^*$. It follows from the global Torelli theorem [11] that $g \circ \sigma = \sigma \circ g$. Hence the above exact sequence splits: $\operatorname{Aut}(X) \simeq G_X \times \boldsymbol{Z}/2$.

A *dual graph* of smooth rational curves is the following simplicial complex Γ: (i) the set of vertices is a set of smooth rational curves on

X; (ii) the vertices C, C' are joined by m-tuple line if $C \cdot C' = m$.

To determine the group G_X we use the dual graph of all smooth rational curves on X. Such graphs were found by Nikulin [8]. However for $S_X = U \oplus E_8 \oplus E_8 \oplus A_1$, his graph is not complete (compare the following graph in Figure 1 with the table 2 in [8], § 4, Part 5). It follows from [13], Proposition 1 and [14], Lemma 2.4 that the following graph represents all smooth rational curves on X.

Let Γ be the dual graph of all smooth rational curves on X (see Figures 1–5). Consider the natural homomorphism ρ: $\mathrm{Aut}(X) \to \mathrm{Aut}(\Gamma)$, where $\mathrm{Aut}(\Gamma)$ is the symmetry group of Γ. Since S_X is generated by the classes of smooth rational curves in Γ, the kernel of ρ acts on S_X as identity. Hence the symplectic group G_X is regarded as a subgroup of $\mathrm{Aut}(\Gamma)$.

(4.1) $S_X = U \oplus E_8 \oplus E_8 \oplus A_1$. The following diagram Γ is the dual graph of all smooth rational curves on X:

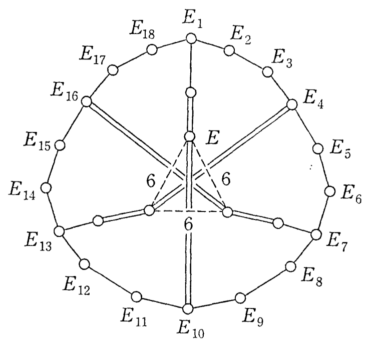


Figure 1

Obviously the symmetry group $\mathrm{Aut}(\Gamma)$ is isomorphic to $\mathfrak{S}_3$.

We now claim that $G_X \simeq \mathfrak{S}_3$. First consider the elliptic pencil $|\Delta_1| = |\sum_{i=1}^{18} E_i|$ which has a section and a singular fibre of type I_{18}. By the formulas in Proposition 3.2, we can see that $|\Delta_1|$ has only one reducible singular fibre of type I_{18} and the group of sections of $|\Delta_1|$ is isomorphic to $Z/3$. These sections act on X as a symplectic automorphism of order 3 which is a rotation of Γ of order 3. Next consider the elliptic pencil $|\Delta_2| = |E + E_{10}|$ which has a section and two singular fibres of type I_2 and of type I_{12}^*. Again it follows from the formulas in Proposition 3.2 that $|\Delta_2|$ has only two reducible singular fibres of type I_2 and of type

I_{12}^* and the group of sections of $|\Delta_2|$ is isomorphic to $Z/2$. Therefore $G_X \simeq \mathfrak{S}_3$.

(4.2) $S_X = U \oplus E_8 \oplus E_8$. The following diagram Γ is the dual graph of all smooth rational curves on X:

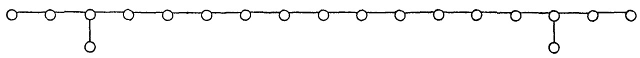


Figure 2

We claim that $G_X \simeq \mathrm{Aut}(\Gamma)$ $(\simeq Z/2)$. Let φ be an isometry of S_X defined by $\varphi((x, y, z)) = (x, z, y)$ where $(x, y, z) \in U \oplus E_8 \oplus E_8$. Note that the second cohomology lattice $L = H^2(X, Z)$ is the direct sum of S_X and T_X. Put $\tilde{\varphi} = (\varphi, 1_{T_X})$: $S_X \oplus T_X \to S_X \oplus T_X$. Then by the global Torelli theorem [11], there exists an automorphism g of X such that $g^* = \tilde{\varphi}$ on L. By construction, g is symplectic and generates $\mathrm{Aut}(\Gamma)$. Hence $G_X \simeq Z/2$ and $\mathrm{Aut}(X) \simeq Z/2 \times Z/2$.

(4.3) $S_X = U \oplus E_8 \oplus E_7$. The following diagram Γ is the dual graph of all smooth rational curves on X:

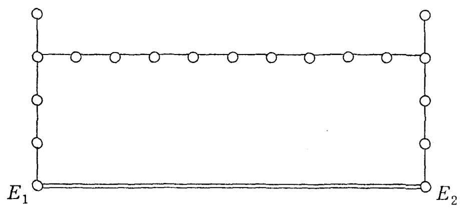


Figure 3

Obviously $\mathrm{Aut}(\Gamma) \simeq Z/2$. By considering the elliptic pencil $|E_1 + E_2|$ with a section, we have a symplectic automorphism of order 2 which acts on Γ as a symmetry of order 2. Hence we have $\mathrm{Aut}(X) \simeq Z/2 \times Z/2$.

(4.4) $S_X = U \oplus E_8 \oplus D_6$. The following diagram Γ is the dual graph of all smooth rational curves on X:

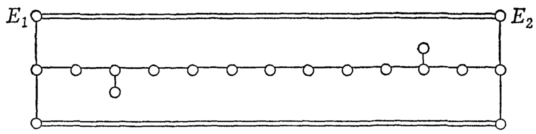


Figure 4

We can see $\mathrm{Aut}(\Gamma) \simeq \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$. We select a generator $\{\gamma_1, \gamma_2\}$ of $\mathrm{Aut}(\Gamma)$ as follows; γ_1 is the reflection of Γ with $\gamma_1(E_1) = E_2$ and γ_2 is the reflection with respect to the middle horizontal line. By considering the elliptic pencil $|E_1 + E_2|$ with a section, we have a symplectic automorphism g whose action on Γ coincides with γ_1. On the other hand, if γ_2 is represented by a symplectic automorphism g', then g' preserves 15 smooth rational curves respectively (see Figure 4). Hence the number of fixed points of g' is greater than 8 which is impossible (Proposition 2.1). Thus we have $G_X \simeq \boldsymbol{Z}/2$ and $\mathrm{Aut}(X) \simeq \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$.

(4.5) $S_X = U \oplus E_8 \oplus D_4 \oplus A_1$. The following diagram Γ is the dual graph of all smooth rational curves on X:

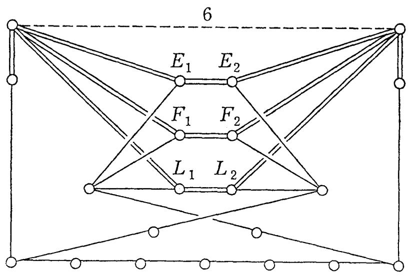


Figure 5

We can see that $\mathrm{Aut}(\Gamma) \simeq \mathfrak{S}_3 \times \boldsymbol{Z}/2$ where $\boldsymbol{Z}/2$ is generated by the reflection γ with $\gamma(E_1) = E_2$ and $\mathfrak{S}_3$ is the permutations of the set $\{E_1, F_1, L_1\}$. By considering the elliptic pencil $|E_1 + E_2|$ with a section, γ is represented by a symplectic automorphism of order 2. On the other hand, any element of $\mathfrak{S}_3$ is not represented by a symplectic automorphism because a symplectic automorphism of order 2 (resp. of order 3) has exactly 8 (resp. 6) isolated fixed points (Proposition 2.1). Therefore we have $G_X \simeq \boldsymbol{Z}/2$ and $\mathrm{Aut}(X) \simeq \boldsymbol{Z}/2 \times \boldsymbol{Z}/2$.

§ 5. Proof of the Theorem—the case when $\mathrm{rank}(S_X) \leq 14$

(5.1) First we remark that G_X is trivial if $\mathrm{rank}(S_X) \leq 8$ (Lemma 2.3, (iii)). Hence it suffices to consider the case that $9 \leq \mathrm{rank}(S_X) \leq 14$. In these cases, G_X is a subgroup of $\boldsymbol{Z}/2 \times \boldsymbol{Z}/2$ or $\boldsymbol{Z}/3$ (Lemma 2.3). Consider a primitive embedding $T_X \subset L^{G_X}$ and denote by $T_X^\perp$ the orthogonal complement of T_X in L^{G_X}. Then $T_X \oplus T_X^\perp$ is a sublattice of L^{G_X} of finite index and $A_{L^{G_X}}$ is a quotient group of $A_{T_X \oplus T_X^\perp}$, and hence $l(T_X \oplus T_X^\perp) \geq l(L^{G_X})$.

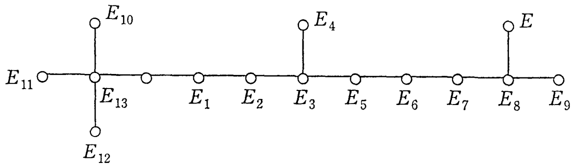
Since $\mathrm{rank}(T_X^{\perp}) \geq l(T_X^{\perp})$ and $l(T_X) = l(S_X)$, we have $l(S_X) + \mathrm{rank}(T_X^{\perp}) \geq l(L^{G_X})$. Therefore it follows from Proposition 2.4 that:

$G_X = \{1\}$ or $Z/2$ if $S_X = U \oplus E_8 \oplus D_4$, $U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus A_1^4$, $U \oplus E_7 \oplus A_1^4$, $U \oplus D_6 \oplus A_1^4$, $U \oplus D_4 \oplus A_1^6$, $U \oplus D_4 \oplus A_1^5$, $U \oplus E_8 \oplus A_1^3$, $U \oplus E_7 \oplus A_1^3$, $U(2) \oplus D_4 \oplus D_4$, $U \oplus D_6 \oplus A_1^3$, $U \oplus A_1^7$, $U \oplus D_4 \oplus A_1^4$, $U \oplus A_1^8$ or $U(2) \oplus A_1^7$ and $G_X = \{1\}$ if S_X is otherwise. Moreover, if $G_X = Z/2$ and $S_X = U \oplus E_8 \oplus A_1^3$, $U \oplus E_7 \oplus A_1^3$, $U \oplus D_6 \oplus A_1^3$, $U \oplus A_1^7$ or $U \oplus D_4 \oplus A_1^4$, then $A_{L^{G_X}} = A_{T \oplus T_X^{\perp}}$ and hence $L_X^{G_X} = T_X \oplus T_X^{\perp}$. This is a contradiction because $L_X^{G_X}$ is a 2-elementary lattice with $\delta_{L^{G_X}} = 0$ and, on the other hand, T_X is a 2-elementary lattice with $\delta_{T_X} = 1$. Also, if $S_X = U \oplus E_8 \oplus D_4$ and $G_X = Z/2$, then $l(L^{G_X}) = l(T_X) + l(T_X^{\perp})$ and hence $L^{G_X} = T_X \oplus T_X^{\perp}$. Hence $T_X^{\perp}$ is a 2-elementary lattice with $\mathrm{rank}(T_X^{\perp}) = 6$, $l(T_X^{\perp}) = 6$ and $\delta_{T_X^{\perp}} = 0$. However, by [7], Theorem 3.6.2, such lattice does not exist.

Hence $G_X = \{1\}$ if $S_X = U \oplus E_8 \oplus D_4$, $U \oplus E_8 \oplus A_1^3$, $U \oplus E_7 \oplus A_1^3$, $U \oplus D_6 \oplus A_1^3$, $U \oplus D_4 \oplus A_1^4$ or $U \oplus A_1^7$.

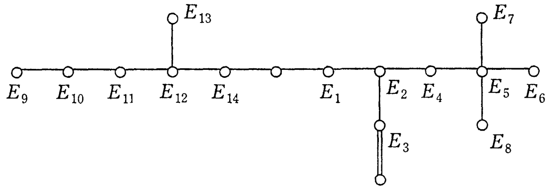
In the following we shall see that $G_X = Z/2$ if $S_X = U \oplus D_8 \oplus D_4$, $U \oplus E_8 \oplus A_1^4$, $U \oplus E_7 \oplus A_1^4$, $U \oplus D_6 \oplus A_1^4$, $U \oplus D_4 \oplus A_1^6$, $U \oplus D_4 \oplus A_1^5$, $U(2) \oplus D_4 \oplus D_4$, $U \oplus A_1^8$ or $U(2) \oplus A_1^7$.

(5.2) $S_X = U \oplus D_8 \oplus D_4$. Note that there exists an elliptic pencil with a section whose reducible singular fibres are of type I_0^* and of type I_4^* (Lemma 3.1). Hence we have the following dual graph of smooth rational curves on X:



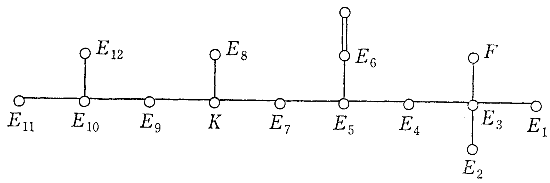
where E_1 is a section of this pencil and others are components of singular fibres. Let us consider the elliptic pencil $|\Delta| = |2E_1 + 4E_2 + 6E_3 + 3E_4 + 5E_5 + 4E_6 + 3E_7 + 2E_8 + E_9|$. Then E_{10}, E_{11}, E_{12} and E_{13} are components of a singular fibre F of this pencil $|\Delta|$. By Proposition 3.2, F is of type I_0^* and hence there exists a smooth rational curve E_{14} with $E_{10} + E_{11} + E_{12} + E_{14} + 2E_{13} \in |\Delta|$. Since E is a 2-section of $|\Delta|$, $E \cdot E_{14} = 2$. Then the elliptic pencil $|E_{14} + E|$ has two sections E_{13}, E_8 and these two sections define a symplectic automorphism. Therefore $G_X \simeq Z/2$.

(5.3) $S_X = U \oplus E_8 \oplus A_1^4$. First we remark that $U \oplus E_8 \oplus A_1^4$ is isomorphic to $U \oplus E_7 \oplus D_4 \oplus A_1$ (Proposition 1.1). Therefore there exists an elliptic pencil with a section which has three reducible singular fibres of type III*, I_0^* and I_2 (Lemma 3.1). Hence we have the following dual graph of smooth rational curves on X:



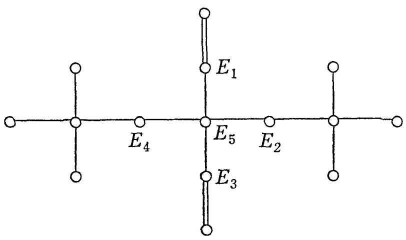
where E_2 is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|\varDelta| = |E_1 + E_3 + E_6 + E_7 + 2(E_2 + E_4$ $E_5)|$. Then E_j, $9 \leq j \leq 14$, are contained in some singular fibre F fo $|\varDelta|$. It follows from Proposition 3.2 that F is of type I_2^*. Hence there exists a smooth rational curve E with $E + E_9 + E_{13} + E_{14} + 2(E_{10} + E_{11} + E_{12}) \in |\varDelta|$. Since E_8 is a 2-section of $|\varDelta|$, $E \cdot E_8 = 2$. Then the elliptic pencil $|E + E_8|$ has two sections E_5 and E_{10} which define a symplectic automorphism of order 2. Therefore we have $G_X \simeq Z/2$.

(5.4) $S_X = U \oplus E_7 \oplus A_1^4$. First note that $U \oplus E_7 \oplus A_1^4 \simeq U \oplus D_6 \oplus D_4 \oplus A_1$ (Proposition 1.1). Since there exists an elliptic pencil with a section which has three reducible singular fibres of type I_2^*, I_0^* and I_2 (Lemma 3.1), we have the following dual graph:



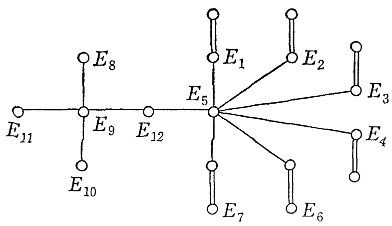
where E_5 is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|\varDelta| = |E_1 + E_2 + E_6 + E_7 + 2(E_3 + E_4 + E_5)|$. Then E_j, $8 \leq j \leq 12$, are components of singular fibres of $|\varDelta|$. Since K is a section of $|\varDelta|$ and $K \cdot E_8 = K \cdot E_9 = 1$, E_8 is not a component of a singular fibre containing E_9. It now follows from Proposition 3.2 that the reducible singular fibres of $|\varDelta|$ are of type I_2^*, I_0^* and I_2. Hence there exists a smooth rational curve E with $E + E_9 + E_{11} + E_{12} + 2E_{10} \in |\varDelta|$. Since F is a 2-section of $|\varDelta|$, $E \cdot F = 2$. The elliptic pencil $|E + F|$ has two sections E_3 and E_{10}, and hence $G_X \simeq Z/2$.

(5.5) $S_X = U \oplus D_6 \oplus A_1^4$. First note that $U \oplus D_6 \oplus A_1^4 \simeq U \oplus D_4 \oplus D_4 \oplus A_1^2$ (Proposition 1.1). Since there exists an elliptic pencil with a section which has 4 reducible singular fibres of type I_0^*, I_0^*, I_2 and I_2 (Lemma 3.1), we have the following dual graph:



where E_5 is a section of this pencil and others are components of singular fibres. Then the elliptic pencil $|E_1 + E_2 + E_3 + E_4 + 2E_5|$ has two sections. Hence $G_X \simeq Z/2$.

(5.6) $S_X = U \oplus D_4 \oplus A_1^6$. Since there exists an elliptic pencil with a section which has one singular fibre of type I_0^* and 6 singular fibres of type I_2 (Lemma 3.1), we have the following dual graph of smooth rational curves:



where E_5 is a section of this pencil and others are components of singular fibres. Consider the elliptic pencil $|\Delta| = |E_1 + E_2 + E_3 + E_4 + 2E_5|$. Then E_j, $6 \leq j \leq 11$, are components of singular fibres of $|\Delta|$. By Proposition 3.2, the following two cases occur: (α) $|\Delta|$ has reducible singular fibres of type I_0^*, I_0^*, I_2 and I_2; (β) $|\Delta|$ has two reducible singular fibres of type I_2^* and I_0^*. In case (α), we may assume that there exists a smooth rational curve E with $E + E_6 \in |\Delta|$. Since E_{12} is a 2-section of $|\Delta|$, we have $E \cdot E_{12} = 2$. Then the elliptic pencil $|E + E_{12}|$ has two sections E_5 and E_9, and hence $G_X \simeq Z/2$. In case (β), we may assume that there exists a smooth rational curve F with $E_6 + E_7 + E_8 + E_{11} + 2E_9 + 2E_{10} + 2F \in |\Delta|$. Then the elliptic pencil $|E_8 + E_{10} + E_{11} + E_{12} + 2E_9|$ has two sections E_5 and F, and hence $G_X \simeq Z/2$.

(5.7) $S_X = U \oplus D_4 \oplus A_1^5$. In this case, the same argument as in (5.6) shows $G_X \simeq Z/2$.

(5.8) $S_X = U(2) \oplus D_4 \oplus D_4$. First we claim that S_X is isomorphic to $U \oplus K$, where K is a negative definite lattice of rank 8. Let $\{e, f\}$ be a basis of $U(2)$ and $\{e_j\}$, $\{f_j\}$ the two copies of a basis of D_4 as in the following dual graphs:

Put $\delta = e + f + e_1 + f_1$. Then $\delta^2 = 0$ and $\langle \delta, e_4 \rangle = 1$. Hence δ and e_4 generate a sublattice of S_X isomorphic to U. So we have $S_X \simeq U \oplus K$. Therefore there exists an elliptic pencil $|\varDelta|$ with a section (Lemma 3.1). It follows from Proposition 3.2, (ii) that K has a sublattice K' of finite index which is generated by some components of singular fibres of $|\varDelta|$. Since K is a 2-elementary lattice with rank $K = 8$, det $K = 2^6$ and $\delta_K = 0$, we can see that $K \neq K'$. Hence the group of section of $|\varDelta|$ is not trivial (Proposition 3.2, (iii)). Therefore $G_X \simeq Z/2$.

(5.9) $S_X = U \oplus A_1^8$, $U(2) \oplus A_1^7$. In these cases, to prove $G_X \simeq Z/2$, we give a lattice theoretic construction of a symplectic automorphism.

In case $S_X = U \oplus A_1^8$, consider a sublattice $\langle 2 \rangle \oplus \langle -2 \rangle \oplus A_1^8$ of S_X. Since a 2-elementary lattices S is determined by $\mathrm{rank}(S_X)$, $l(S)$ and the parity of S, this sublattice is isomorphic to $\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$ (Proposition 1.1). By this isomorphism, we consider $\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$ as a sublattice of S_X. Let ι be an involution of $\langle 2 \rangle \oplus \langle -2 \rangle \oplus E_8(2)$ such that $\iota | \langle 2 \rangle \oplus \langle -2 \rangle = 1$ and $\iota | E_8(2) = -1$. Since $\langle 2 \rangle \oplus \langle -2 \rangle$ and $E_8(2)$ are 2-elementary, ι extends to an involution ι' of S_X. By construction, ι' acts on the discriminant group A_{S_X} as identity. Hence ι' extends to an involution $\tilde{\iota}$ of L_X with $\tilde{\iota} | T_X = 1$. $\tilde{\iota}$ preserves a period of X and the Kähler cone because $E_8(2)$ contains no (-2)-elements. Hence by the global Torelli theorem [11], ι is represented by a symplectic automorphism of order 2.

In case $S_X = U(2) \oplus A_1^7$, we define two involutions σ and g of L_X as follows: let $\{\alpha_i, \beta_i\}$ be a copy of a basis of U $(1 \leq i \leq 3)$ and let $\{e_j\}$, $\{f_j\}$ be copies of a basis of E_8 $(1 \leq j \leq 8)$. Then $\{\alpha_i, \beta_i, e_j, f_j | 1 \leq i \leq 3, 1 \leq j \leq 8\}$ is a basis of $L_X = U \oplus U \oplus U \oplus E_8 \oplus E_8$. Put $g | U \oplus U \oplus U = 1$ and

$g(e_j)=f_j$, $1\leq j\leq 8$, $\sigma(\alpha_1)=\beta_1$, $\sigma(\alpha_i)=-\alpha_i$, $\sigma(\beta_i)=-\beta_i$, $2\leq i\leq 3$, and $\sigma(e_j)=-f_j$, $1\leq j\leq 8$. Then $L^{\langle\sigma\rangle}$ is isomorphic to $\langle 2\rangle\oplus E_8(2)\simeq U(2)\oplus A_1^7$ which is generated by $\{\alpha_1+\beta_1,\ e_j-f_j\,|\,j=1,\cdots,8\}$. On the other hand $L^{\langle g\rangle}$ is isomorphic to $U\oplus U\oplus U\oplus E_8(2)$ which is generated by $\{\alpha_i,\ \beta_i,\ e_j+f_j\,|\,i=1,2,3,\ j=1,\cdots,8\}$. How we consider $L_X^{\langle\sigma\rangle}$ as a Picard lattices S_X. Then we can easily see that g preserves the Kähler cone of X and a period of X. Hence by the global Torelli theorem [11], g is represented by a symplectic automorphism. Thus we have $G_X\simeq Z/2$.

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