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THE NORM ESTIMATE OF THE DIFFERENCE BETWEEN THE KAC OPERATOR AND THE SCHRÖDINGER SEMIGROUP: A UNIFIED APPROACH TO THE NONRELATIVISTIC AND RELATIVISTIC CASES

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Dedicated to Professor Rentaro Agemi on his sixtieth birthday

Abstract. An L^p operator norm estimate of the difference between the Kac operator and the Schrödinger semigroup is proved and used to give a variant of the Trotter product formula for Schrödinger operators in the L^p operator norm. The method of the proof is probabilistic based on the Feynman–Kac formula. The problem is discussed in the relativistic as well as nonrelativistic case.

§1. Introduction

In [8], we have given an estimate in the L^p operator norm of the difference between the Kac operator $e^{-tV/2}e^{-tH_0}e^{-tV/2}$ and the Schrödinger semigroup $e^{-tH} = e^{-t(H_0+V)}$, where $H = H_0 + V \equiv -\frac{1}{2}\Delta + V$ is the nonrelativistic Schrödinger operator with mass 1 and scalar potential $V(x)$, a real–valued continuous function bounded below, in the space $L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$, and also in the Banach space $C_\infty(\mathbf{R}^d)$ of the continuous functions in $\mathbf{R}^d$ vanishing at infinity. Here as the Kac operator we mention the transfer matrix/operator for a Kac model [12] in statistical mechanics associated with a potential $V(x)$. The operator norm of this difference is estimated by a power of small $t > 0$ with order greater than or equal to 1. As a by-product a variant of the Trotter product formula for the nonrelativistic Schrödinger operator in the L^p operator norm is obtained.

Helffer ([5],[6]) was the first to treat this problem in L^2, when $V(x)$ is a C^∞–function in $\mathbf{R}^d$ bounded below by a constant b and satisfying

$$|\partial^\alpha V(x)| \leq C_\alpha(1 + x^2)^{(2-|\alpha|)_+/2}$$

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for every multi-index α where $a_+ = \max\{a, 0\} = a \vee 0$, in order to relate in some asymptotic limit the spectral properties of the Kac operator to those of the nonrelativistic Schrödinger operator $-\frac{1}{2}\Delta + V$. We have used in [8] probabilistic methods to extend his result to the case of more general scalar potentials $V(x)$. For the related L^2 result with operator-theoretic methods, we also refer to Doumeki–Ichinose–Tamura [2], where the problem in the trace norm is also treated.

The aim of this paper is to present a unified approach to the relativistic case as well as nonrelativistic case for this problem. The new result of the paper is to give the proof for the relativistic case, which employs slightly modified, though probabilistic, arguments of [8]. However, the present method unifies the idea of proof in both the nonrelativistic and relativistic cases. In fact, it enables us not only to improve slightly our previous result in [8] for the nonrelativistic Schrödinger operator $H = H_0 + V \equiv -\frac{1}{2}\Delta + V$, but also to obtain an analogous new result for the relativistic Schrödinger operator $H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V$. In this case the operator norm of the difference between the Kac operator and the relativistic Schrödinger semigroup reveals a slightly different behavior for small $t > 0$, compared with the nonrelativistic case.

In Section 2 we state our results, Theorems 2.1 and 2.2 for the nonrelativistic case and Theorems 2.3 and 2.4 for the relativistic case. They are proved in Sections 3 and 4, respectively, in a unified way.

§2. Statement of the results

To formulate our theorems we want to consider the nonrelativistic Schrödinger operator

$$H = H_0 + V \equiv -\frac{1}{2}\Delta + V \tag{2.1}$$

and the relativistic Schrödinger operator

$$H^r = H_0^r + V \equiv \sqrt{-\Delta + 1} - 1 + V \tag{2.2}$$

with mass 1 and scalar potential $V(x)$, not only in $L^2 = L^2(\mathbf{R}^d)$ but also in $L^p = L^p(\mathbf{R}^d)$, $1 \leq p \leq \infty$, and also in the Banach space $C_\infty = C_\infty(\mathbf{R}^d)$ of the continuous functions in $\mathbf{R}^d$ vanishing at infinity, equipped with L^∞ norm.

If $V(x)$ is a real-valued locally square-integrable function in $\mathbf{R}^d$ and bounded below, both H and H^r are essentially selfadjoint on $C_0^\infty = C_0^\infty(\mathbf{R}^d)$, which is shown by use of Kato's inequality (Kato [13], Ichinose–Tsuchida [11]). So their unique selfadjoint extensions are also denoted by the same $H = H_0 + V$ and $H^r = H_0^r + V$.

Then their semigroups e^{-tH} and e^{-tH^r} have the following path integral representations (e.g. Simon [15], Ichinose–Tamura [10]):

$$(2.3) \qquad \left(e^{-tH}f\right)(x) = E_x[\exp\left(-\int_0^t V(X(s))ds\right)f(X(t))],$$

$$(2.4) \qquad \left(e^{-tH^r}f\right)(x) = E_x^r[\exp\left(-\int_0^t V(X(s))ds\right)f(X(t))],$$

for $f \in L^2$. Here E_x (*resp.* E_x^r) means the expectation or integral with respect to the probability measure μ_x (*resp.* λ_x) on the space of the continuous (*resp.* right-continuous) paths $X : [0, \infty) \to \mathbf{R}^d$ starting at $X(0) = x$ such that

$$(2.5) \qquad E_x[e^{ip(X(t)-x)}] = \exp(-\frac{1}{2}tp^2),$$

$$(2.6) \qquad E_x^r[e^{ip(X(t)-x)}] = \exp(-t(\sqrt{p^2+1}-1)).$$

The measure μ_x is the Wiener measure and (2.3) is called the Feynman–Kac formula, while the measure λ_x is the probability measure associated with a Lévy process with characteristic function (2.6).

We can see via (2.3) and (2.4) that the operators e^{-tH} and e^{-tH^r} defined as bounded operators on L^2 extend from $L^p \cap L^2$ to bounded operators on L^p for $1 \le p < \infty$ (cf. Simon [16]). Both e^{-tH} and e^{-tH^r} are strongly continuous semigroups obeying

$$(2.7) \qquad \|e^{-tH}f\|_p \le e^{-tb}\|f\|_p,$$

$$(2.8) \qquad \|e^{-tH^r}f\|_p \le e^{-tb}\|f\|_p,$$

for $f \in L^p$, $1 \le p < \infty$. We denote also the generators H_p and H_p^r of these semigroups in L^p by the same $H = H_0 + V$ and $H^r = H_0^r + V$. When $p = \infty$, e^{-tH} and e^{-tH^r} are defined on L^∞ as the duals of the L^1 operators. They are not strongly continuous, but (2.7) and (2.8) hold for the $p = \infty$ operators. The $p = \infty$ operators H_∞ and H_∞^r are the adjoints of the $p = 1$ operators H_1 and H_1^r, respectively.

In addition, if $V(x)$ is continuous, (2.3) and (2.4) define as well the strongly continuous semigroups e^{-tH} and e^{-tH^r} on C_∞ obeying (2.7) and (2.8).

In the following, $\|\cdot\|_{p\to p}$ stands for the operator norm of bounded operators on L^p, $1 \le p \le \infty$, or on C_∞.

THEOREM 2.1. (The nonrelativistic case) *Let $0 < \delta \le 1$. Let m be a nonnegative integer such that $m\delta \le 1$. Suppose that $V(x)$ is a C^m-function in $\mathbf{R}^d$ bounded below by a constant b which satisfies that*

$$|\partial^\alpha V(x)| \le C(V(x) - b + 1)^{1-|\alpha|\delta}, \quad 0 \le |\alpha| \le m, \tag{2.9}$$

with a constant $C > 0$, and further that $\partial^\alpha V(x), |\alpha| = m$, are Hölder-continuous:

$$|\partial^\alpha V(x) - \partial^\alpha V(y)| \le C|x-y|^\kappa, \quad x, y \in \mathbf{R}^d, \tag{2.10}$$

with constants $C > 0$ and $0 \le \kappa \le 1$ (By $\kappa = 0$ we understand $\partial^\alpha V(x)$, $|\alpha| = m$, bounded). Then it holds that, as $t \downarrow 0$,

$$\|e^{-tV/2}e^{-tH_0}e^{-tV/2} - e^{-t(H_0+V)}\|_{p\to p} = \begin{cases} O(t^{1+\kappa/2}), & m = 0, \\ O(t^{1+2\delta\wedge\frac{1+\kappa}{2}}), & m = 1, \\ O(t^{1+2\delta}), & m \ge 2. \end{cases} \tag{2.11}$$

Here we write $\min\{a, b\} = a \wedge b$. Note that the condition (2.10) with $\kappa = 1$ is equivalent to that $\partial^\alpha V(x)$, $|\alpha| = m+1$, are essentially bounded.

An immediate consequence of Theorem 2.1 with telescoping is the following variant of the Trotter product formula.

THEOREM 2.2. (The nonrelativistic case) *For the same function $V(x)$ as in Theorem 2.1, it holds that, as $n \to \infty$,*

$$\begin{aligned} &\|(e^{-tV/2n}\, e^{-tH_0/n}e^{-tV/2n})^n - e^{-t(H_0+V)}\|_{p\to p} \\ &= \begin{cases} n^{-\kappa/2}O(t^{1+\kappa/2}), & m = 0, 0 < \kappa \le 1, \\ n^{-2\delta\wedge\frac{1+\kappa}{2}}O(t^{1+2\delta\wedge\frac{1+\kappa}{2}}), & m = 1, 0 \le \kappa \le 1, \\ n^{-2\delta}O(t^{1+2\delta}), & m \ge 2. \end{cases} \end{aligned} \tag{2.12}$$

EXAMPLES. The function $|x|^2$ (harmonic oscillator potential) satisfies the conditions (2.9) and (2.10) for $V(x)$ in Theorem 2.1 with $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or $(\frac{1}{2}, 2, 0)$, and the function $|x|^4 - |x|^2$ (double well potential) with $(\delta, m, \kappa) = (\frac{1}{4}, 3, 1)$ or $(\frac{1}{4}, 4, 0)$. The function $|x|^\rho$ satisfies the conditions (2.9) and (2.10) with $(\delta, m, \kappa) = (1, 0, \rho)$ for $0 < \rho \le 1$ and $(\delta, m, \kappa) = (1/\rho, [\rho], \rho - [\rho])$ for $\rho > 1$, where $[\rho]$ is the maximal integer that is not greater than ρ. But, for instance, $\exp(|x|^2 + 1)^a$, $a > 0$, and $\exp|x|^2$ do not satisfy these conditions.

Remark 1. It is Helffer [6] (cf. [5], [4]) that first proved (2.11) in the L^2 operator norm ($p = 2$), with $O(t^2)$ on the right-hand side of (2.11), by the pseudo-differential operator calculus, when $V(x)$ is a C^∞–function bounded below by b and satisfying $|\partial^\alpha V(x)| \leq C_\alpha(1+x^2)^{(2-|\alpha|)_+/2}$ for every multi-index α with constants C_α. In fact, as his condition implies that

$$|\partial^\alpha V(x)| \leq C(V(x) - b + 1)^{(1-|\alpha|/2)_+} \tag{2.13}$$

for the same α, so his result is included in the case $p = 2$ and $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or $(\delta, m, \kappa) = (\frac{1}{2}, 2, 0)$ in Theorem 2.1.

With the condition (2.13) Dia–Schatzman [1] also has recently given an operator-theoretical proof of Helffer's result.

Remark 2. Theorems 2.1 and 2.2 include [2, Theorem 2.1 and Lemma 2.2]. In fact, let $-\infty < \rho \leq 2$ and let b be a real constant. Suppose that $V(x)$ is a real–valued C^2–function satisfying

$$V(x) \geq b + C_0(1 + x^2)^{\rho/2}, \quad |\partial^\alpha V(x)| \leq C_\alpha(1 + x^2)^{(\rho-|\alpha|)_+/2}, \ |\alpha| = 1, 2.$$

Then if $1 < \rho \leq 2$, we have $|\partial^\alpha V(x)| \leq C(V(x) - b + 1)^{(1-|\alpha|/\rho)_+}$, for $|\alpha| = 1, 2$, so that $(\delta, m, \kappa) = (1/\rho, 1, 1)$, while if $-\infty < \rho \leq 1$, we have $|\partial^\alpha V(x)| \leq C_\alpha \leq C(V(x) - b + 1)^{1-|\alpha|/2}$, for $|\alpha| = 1, 2$, so that $(\delta, m, \kappa) = (\frac{1}{2}, 1, 1)$ or $(\frac{1}{2}, 2, 0)$. Therefore we have by Theorems 2.1 and 2.2

$$\|e^{-tV/2}e^{-tH_0}e^{-tV/2} - e^{-t(H_0+V)}\|_{p\to p} = O(t^2), \quad t \downarrow 0,$$

and

$$\|(e^{-tV/2n}e^{-tH_0/n}e^{-tV/2n})^n - e^{-t(H_0+V)}\|_{p\to p} = n^{-1}O(t^2), \quad n \to \infty.$$

Remark 3. Theorems 2.1 and 2.2 are valid with the operator H_0 replaced by the magnetic Schrödinger operator $H_0(A) = \frac{1}{2}(-i\partial - A(x))^2$ with vector potential $A(x)$ including the case of constant magnetic fields (see Ichinose–Takanobu [8], cf. Doumeki–Ichinose–Tamura [2]).

Remark 4. As for the Trotter product formula in operator norm, Rogava [14] proved for nonnegative selfadjoint operators A and B in a Hilbert space that, if the domain $D[A]$ of A is included in the domain $D[B]$ of B and $A + B$ is selfadjoint on $D[A + B] = D[A] \cap D[B] = D[A]$, then, as $n \to \infty$,

$$\|(e^{-tB/n}e^{-tA/n})^n - e^{-t(A+B)}\| = O(n^{-1/2}\ln n),$$

$$\|(e^{-tA/2n}e^{-tB/n}e^{-tA/2n})^n - e^{-t(A+B)}\| = O(n^{-1/2}\ln n).$$

In this case, B is A–bounded. Notice that in our Theorems 2.1 and 2.2, neither V is H_0–bounded nor H_0 is V–bounded.

For some complementary results to Rogava's we refer to Ichinose–Tamura [9].

THEOREM 2.3. (The relativistic case) *Let* $V(x)$ *be the same function as in Theorem* 2.1. *Then it holds that, as* $t \downarrow 0$,

$$\|e^{-tV/2}\, e^{-tH_0^r} e^{-tV/2} - e^{-t(H_0^r+V)}\|_{p\to p} \tag{2.14}$$
$$= \begin{cases} O(t^{1+\kappa}), & m = 0, 0 \le \kappa < 1, \\ O(t^2|\ln t|), & (m,\kappa) = (0,1), \\ O(t(t^{2\delta} \vee t|\ln t|)), & (m,\kappa) = (1,0), \\ O(t^{1+2\delta\wedge 1}), & m = 1, 0 < \kappa \le 1, \\ O(t^{1+2\delta}), & m \ge 2. \end{cases}$$

An immediate consequence of Theorem 2.3 is the following variant of the Trotter product formula.

THEOREM 2.4. (The relativistic case) *For the same function* $V(x)$ *as in Theorem* 2.1, *it holds that, as* $n \to \infty$,

$$\|(e^{-tV/2n}e^{-tH_0^r/n}e^{-tV/2n})^n - e^{-t(H_0^r+V)}\|_{p\to p} \tag{2.15}$$
$$= \begin{cases} n^{-\kappa}O(t^{1+\kappa}), & m = 0, 0 < \kappa < 1, \\ n^{-1}O(t^2|\ln(t/n)|), & (m,\kappa) = (0,1), \\ O((n^{-2\delta}t^{1+2\delta}) \vee (n^{-1}t^2|\ln(t/n)|)), & (m,\kappa) = (1,0), \\ n^{-2\delta\wedge 1}O(t^{1+2\delta\wedge 1}), & m = 1, 0 < \kappa \le 1, \\ n^{-2\delta}O(t^{1+2\delta}), & m \ge 2. \end{cases}$$

The method of our proof is probabilistic based on the Feynman–Kac formulas (2.3) and (2.4).

We need only to prove Theorems 2.1 and 2.3. To prove them we shall estimate the integral kernel of the difference between the Kac operator and the Schrödinger semigroup. They will be proved in Sections 3 and 4.

§3. Proof of Theorem 2.1

Put

$$Q(t) = e^{-tV/2}e^{-tH_0}e^{-tV/2} - e^{-t(H_0+V)}, \quad t > 0. \tag{3.1}$$

Without loss of generality, we may suppose that $V(x) \geq 1$, and the condition (2.9) holds with $b = 1$.

Since $Q(t)$ are uniformly bounded operators on L^p and C_∞ in $t > 0$, and since C_0^∞ is dense in $L^p, 1 \leq p < \infty$, and C_∞ , we have only to show that for $f \in C_0^\infty$ with $\|f\|_p = 1$, $\|Q(t)f\|_p$ has the order of the power of t as in (2.11). Here note that the L^∞ case follows as the dual of the L^1 case.

By the Feynman–Kac formula (2.3) we have for $f \in C_0^\infty$

$$\begin{aligned}(3.2)\quad &(Q(t)f)(x)\\ &= E_x\Big[\Big(\exp\Big(-\frac{t}{2}(V(x) + V(X(t)))\Big) - \exp\Big(-\int_0^t V(X(s))ds\Big)\Big)f(X(t))\Big].\end{aligned}$$

Let $p(t,x)$ be the heat kernel, the integral kernel of e^{-tH_0}:

$$p(t,x) = (2\pi t)^{-d/2}e^{-x^2/2t}. \tag{3.3}$$

We have

$$\int |x|^a p(t,x)dx = C(a)t^{a/2}, \tag{3.4}$$

with a constant $C(a)$ depending on $a > 0$ and the dimension d.

To avoid notational complexity, we shall assume $d = 1$; there is no essential change of the following proof in the multi-dimensional case.

We use the conditional expectation $E_x[\,\cdot \mid X(t) = y]$ to rewrite (3.2) as

$$\begin{aligned}(3.5\text{a})\quad &(Q(t)f)(x)\\ &= \int f(y)p(t,x-y)E_x\Big[\exp\Big(-\frac{t}{2}(V(x) + V(y))\Big)\\ &\qquad\qquad - \exp\Big(-\textstyle\int_0^t V(X(s))ds\Big) \mid X(t) = y\Big]dy\\ &\equiv \int f(y)p(t,x-y)d(t,x,y)dy,\end{aligned}$$

where

$$d(t,x,y) = E_x[v(t,x,y) \mid X(t) = y] \tag{3.5b}$$

with

$$(3.5c)\quad v(t,x,y)=\exp\Big(-\frac{t}{2}(V(x)+V(y))\Big)-\exp\Big(-\int_0^t V(X(s))ds\Big).$$

By Taylor's theorem

$$\begin{aligned} e^{-a}-e^{-b} &= -(1-e^{a-b})e^{-a} \\ &= -\sum_{j=1}^{m}\frac{1}{j!}(a-b)^j e^{-a} \\ &\qquad -\frac{1}{m!}(a-b)^{m+1}\int_0^1 d\theta(1-\theta)^m e^{-(1-\theta)a-\theta b}. \end{aligned}$$

Putting

$$\begin{aligned} (3.6)\quad w(t,x,y) &= \frac{t}{2}(V(x)+V(y))-\int_0^t V(X(s))ds \\ &= -\int_0^{t/2}(V(X(s))-V(x))ds-\int_{t/2}^t (V(X(s))-V(y))ds, \end{aligned}$$

this yields the following expansion of $v(t,x,y)$ in (3.5c).

(3.7)

$$\begin{aligned} &v(t,\ x,y) \\ &\quad= -w(t,x,y)e^{-\frac{t}{2}(V(x)+V(y))} \\ &\qquad -\sum_{j=2}^{m}\frac{1}{j!}w(t,x,y)^j e^{-\frac{t}{2}(V(x)+V(y))} \\ &\qquad -\frac{1}{m!}w(t,x,y)^{m+1} \\ &\qquad \times\int_0^1 d\theta(1-\theta)^m\exp\Big[-(1-\theta)\frac{t}{2}(V(x)+V(y))-\theta\int_0^t V(X(s))ds\Big] \\ &\quad\equiv \sum_{i=1}^{3} v_i(t,x,y). \end{aligned}$$

Note in (3.7) that if $m=1$, $v_2(t,x,y)$ is absent, and if $m=0$, both $v_1(t,x,y)$ and $v_2(t,x,y)$ are absent.

Put

$$(3.8a)\qquad d(t,x,y)=\sum_{i=1}^{3} d_i(t,x,y),$$

$$d_i(t,x,y) = E_x[v_i(t,x,y) \mid X(t) = y], \quad i = 1,2,3. \tag{3.8b}$$

Then the function

$$\begin{aligned} q(t,x,y) &= p(t,x-y)d(t,x,y) \\ &= \sum_{i=1}^{3} q_i(t,x,y) \equiv \sum_{i=1}^{3} p(t,x-y)d_i(t,x,y) \end{aligned} \tag{3.9}$$

is the integral kernel of the operator $Q(t)$ in (3.1). Here, in (3.8ab), if $m = 1$, $d_2(t,x,y)$ is absent, and if $m = 0$, both $d_1(t,x,y)$ and $d_2(t,x,y)$ are absent, and so are for $q_i(t,x,y)$ in (3.9).

For $m \geq 1$, we have by Taylor's theorem

$$\begin{aligned} &V(x+z) - V(x) \\ &= \sum_{k=1}^{m-1} \frac{1}{k!} z^k V^{(k)}(x) \\ &\qquad + \frac{1}{(m-1)!} z^m \int_0^1 d\tau (1-\tau)^{m-1} V^{(m)}(x+\tau z) \\ &= \sum_{k=1}^{m} \frac{1}{k!} z^k V^{(k)}(x) \\ &\qquad + \frac{1}{(m-1)!} z^m \int_0^1 d\tau (1-\tau)^{m-1} (V^{(m)}(x+\tau z) - V^{(m)}(x)). \end{aligned}$$

Applying this to the integrands of the last member of (3.6) yields

$$\begin{aligned} &w(t,x,y) \\ &= -\int_0^{t/2} (X(s)-x)V'(x)ds - \sum_{k=2}^{m} \frac{1}{k!} \int_0^{t/2} (X(s)-x)^k V^{(k)}(x)ds \\ &\quad - \frac{1}{(m-1)!} \int_0^1 d\tau (1-\tau)^{m-1} \\ &\quad \times \int_0^{t/2} (X(s)-x)^m \Big(V^{(m)}(x+\tau(X(s)-x)) - V^{(m)}(x) \Big) ds \\ &\quad - \int_{t/2}^{t} (X(s)-y)V'(y)ds - \sum_{k=2}^{m} \frac{1}{k!} \int_{t/2}^{t} (X(s)-y)^k V^{(k)}(y)ds \\ &\quad - \frac{1}{(m-1)!} \int_0^1 d\tau (1-\tau)^{m-1} \end{aligned} \tag{3.10}$$

$$\times \int_{t/2}^{t} (X(s)-y)^m \Big(V^{(m)}(y+\tau(X(s)-y)) - V^{(m)}(y)\Big) ds,$$

where the second and fifth terms $-\sum_{k=2}^{m}$ on the right are absent if $m=1$.

It follows that

$$w(t,x,y) = \sum_{i=1}^{3} w_i(t,x,y), \tag{3.11a}$$

with

$$\begin{aligned} &w_1(t,x,y) \\ &= -\Big\{V'(x)\int_0^{t/2}(X(s)-x)ds + V'(y)\int_{t/2}^{t}(X(s)-y)ds\Big\}, \end{aligned} \tag{3.11b}$$

$$\begin{aligned} &w_2(t,x,y) \\ &= -\sum_{k=2}^{m}\frac{1}{k!}\Big\{V^{(k)}(x)\int_0^{t/2}(X(s)-x)^k ds + V^{(k)}(y)\int_{t/2}^{t}(X(s)-y)^k ds\Big\}, \end{aligned} \tag{3.11c}$$

$$\begin{aligned} w_3(t,x,y) &= -\frac{1}{(m-1)!}\int_0^1 d\tau(1-\tau)^{m-1} \\ &\times\Big\{\int_0^{t/2}(X(s)-x)^m\Big(V^{(m)}(x+\tau(X(s)-x)) - V^{(m)}(x)\Big)ds \\ &\quad + \int_{t/2}^{t}(X(s)-y)^m\Big(V^{(m)}(y+\tau(X(s)-y)) - V^{(m)}(y)\Big)ds\Big\}, \end{aligned} \tag{3.11d}$$

where the second term $w_2(t,x,y) = -\sum_{k=2}^{m}\cdots$ is absent if $m=1$.

According to the decomposition (3.11a) of $w(t,x,y)$ rewrite $d_1(t,x,y)$ as

$$d_1(t,x,y) = \sum_{i=1}^{3} d_{1i}(t,x,y), \tag{3.12a}$$

$$\begin{aligned} &d_{1i}(t,x,y) \\ &= -E_x[w_i(t,x,y) \mid X(t)=y]e^{-\frac{t}{2}(V(x)+V(y))}, \quad 1\le i\le 3, \end{aligned} \tag{3.12b}$$

where $d_{12}(t,x,y)$ is absent if $m=1$.

We can use (2.9), (2.10) and that for $a \geq 0$

$$t^a e^{-t/2} \leq (\frac{2a}{e})^a, \quad t \geq 0, \tag{3.13}$$

which for $a = 0$ we understand as $e^{-t/2} \leq 1$, to show the following lemma.

LEMMA 3.1. a) *If* $m = 0$,

$$\begin{aligned} |v(t,x,y)| &\leq |w(t,x,y)| \\ &\leq C\Big(\int_0^{t/2} |X(s)-x|^\kappa ds + \int_{t/2}^t |X(s)-y|^\kappa ds\Big). \end{aligned} \tag{3.14}$$

b) *If* $m \geq 1$,

$$v_1(t,x,y) = \sum_{i=1}^3 v_{1i}(t,x,y) \equiv -\sum_{i=1}^3 w_i(t,x,y) e^{-\frac{t}{2}(V(x)+V(y))}, \tag{3.15a}$$

and

$$\begin{aligned} &v_{11}(t,x,y) \\ &= \Big\{V'(x)\int_0^{t/2} (X(s)-x)ds + V'(y)\int_{t/2}^t (X(s)-y)ds\Big\} \\ &\qquad \times e^{-\frac{t}{2}(V(x)+V(y))}; \end{aligned} \tag{3.15b}$$

if $m \geq 2$,

$$\begin{aligned} &|v_{12}(t,x,y)| \\ &\leq \sum_{k=2}^m O(t^{-1+k\delta})\Big(\int_0^{t/2} |X(s)-x|^k ds + \int_{t/2}^t |X(s)-y|^k ds\Big); \end{aligned} \tag{3.15c}$$

if $m \geq 1$,

$$\begin{aligned} &|v_{13}(t,x,y)| \\ &\leq \frac{C}{m!}\Big(\int_0^{t/2} |X(s)-x|^{m+\kappa} ds + \int_{t/2}^t |X(s)-y|^{m+\kappa} ds\Big). \end{aligned} \tag{3.15d}$$

If $m \geq 2$,

$$\begin{aligned} &|v_2(t,x,y)| \\ &\leq \sum_{j=2}^m \Big\{\sum_{k=1}^m O(t^{-1+jk\delta})\Big(\int_0^{t/2} |X(s)-x|^{jk} ds + \int_{t/2}^t |X(s)-y|^{jk} ds\Big) \\ &\quad + O(t^{j-1})\Big(\int_0^{t/2} |X(s)-x|^{j(m+\kappa)} ds + \int_{t/2}^t |X(s)-y|^{j(m+\kappa)} ds\Big)\Big\}. \end{aligned} \tag{3.16}$$

If $m \geq 1$,

$$(3.17)\quad |v_3(t,x,y)|$$
$$\leq \sum_{k=1}^{m} O(t^{-1+(m+1)k\delta})\Big(\int_0^{t/2} |X(s)-x|^{(m+1)k} ds + \int_{t/2}^{t} |X(s)-y|^{(m+1)k} ds\Big)$$
$$+O(t^m)\Big(\int_0^{t/2} |X(s)-x|^{(m+1)(m+\kappa)} ds + \int_{t/2}^{t} |X(s)-y|^{(m+1)(m+\kappa)} ds\Big).$$

We give here only a few words for the proof of Lemma 3.1. As mentioned before, we make use of (2.9), (2.10) with (3.13). (3.14) follows directly from (3.6). (3.15b) follows immediately from (3.11b). (3.15c) and (3.15d) follow from (3.11c) and (3.11d), respectively, while (3.16) and (3.17) are obtained by using the expression (3.10) of $w(t,x,y)$ to calculate $w(t,x,y)^j$ and $w(t,x,y)^{m+1}$.

Therefore, to prove Theorem 2.1, we need to calculate the conditional expectation $E_x[\cdot \mid X(t)=y]$ of some basic quantities as in the following lemma. We note here that for $0<s<t$

$$(3.18)\qquad \begin{aligned} &E_x[\varphi(X(s)) \mid X(t)=y] \\ &= p(t,x-y)^{-1}E_x[p(t-s,X(s)-y)\varphi(X(s))] \end{aligned}$$

for a nonnegative measurable function $\varphi(x)$.

LEMMA 3.2. *Let* $t>0$.

(i) *Let* $0<s<t$.

$$(3.19)\qquad \begin{aligned} E_x[X(s)-x \mid X(t)=y] &= \frac{s}{t}(y-x); \\ E_x[X(s)-y \mid X(t)=y] &= \frac{t-s}{t}(x-y). \end{aligned}$$

(ii) *Let $a > 0$.*

$$
\begin{aligned}
&\int_0^{t/2} E_x[|X(s)-x|^a \mid X(t)=y]ds \\
&\qquad \leq \begin{cases} \frac{1}{2}t|x-y|^a + \frac{1}{2}C(a)t^{a/2+1}, & 0 < a < 1, \\ 2^{a-1}(\frac{1}{2}t|x-y|^a + \frac{1}{2}C(a)t^{a/2+1}), & a \geq 1; \end{cases} \\
&\int_{t/2}^{t} E_x[|X(s)-y|^a \mid X(t)=y]ds \\
&\qquad \leq \begin{cases} \frac{1}{2}t|x-y|^a + \frac{1}{2}C(a)t^{a/2+1}, & 0 < a < 1, \\ 2^{a-1}(\frac{1}{2}t|x-y|^a + \frac{1}{2}C(a)t^{a/2+1}), & a \geq 1. \end{cases}
\end{aligned}
\tag{3.20}
$$

Proof. (i) We have by (3.18), using the Parseval equality and integrating by parts,

$$
\begin{aligned}
&E_x[X(s)-x \mid X(t)=y] \\
&\quad = p(t,x-y)^{-1}E_x[p(t-s,X(s)-y)(X(s)-x)] \\
&\quad = p(t,x-y)^{-1}\int (z-x)p(t-s,z-y)p(s,z-x)dz \\
&\quad = (2\pi)^{-1}p(t,x-y)^{-1}\int e^{-ipy}e^{-(t-s)p^2/2}\overline{e^{-ipx}(i\partial_p)e^{-sp^2/2}}dp \\
&\quad = (2\pi)^{-1}p(t,x-y)^{-1}\int e^{ip(x-y)}e^{-(t-s)p^2/2}(isp)e^{-sp^2/2}dp \\
&\quad = (2\pi)^{-1}p(t,x-y)^{-1}\int e^{ip(x-y)}\frac{s}{t}(-i\partial_p)e^{-tp^2/2}dp \\
&\quad = (2\pi)^{-1}\frac{s}{t}p(t,x-y)^{-1}\int (i\partial_p e^{ip(x-y)})e^{-tp^2/2}dp \\
&\quad = -\frac{s}{t}(x-y)p(t,x-y)^{-1}(2\pi)^{-1}\int e^{ip(x-y)}e^{-tp^2/2}dp \\
&\quad = \frac{s}{t}(y-x).
\end{aligned}
$$

Similarly we have

$$
\begin{aligned}
&E_x[X(s)-y \mid X(t)=y] \\
&\quad = p(t,x-y)^{-1}E_x[p(t-s,X(s)-y)(X(s)-y)] \\
&\quad = p(t,x-y)^{-1}\int (z-y)p(t-s,z-y)p(s,z-x)dz
\end{aligned}
$$

$$
\begin{aligned}
&= (2\pi)^{-1}p(t,x-y)^{-1}\int e^{-ipy}(i\partial_p e^{-(t-s)p^2/2})\overline{e^{-ipx}e^{-sp^2/2}}dp \\
&= -(2\pi)^{-1}p(t,x-y)^{-1}\int e^{ip(x-y)}\frac{t-s}{t}(-i\partial_p)e^{-tp^2/2}dp \\
&= -(2\pi)^{-1}\frac{t-s}{t}p(t,x-y)^{-1}\int (i\partial_p e^{ip(x-y)})e^{-tp^2/2}dp \\
&= \frac{t-s}{t}(x-y)p(t,x-y)^{-1}(2\pi)^{-1}\int e^{ip(x-y)}e^{-tp^2/2}dp \\
&= \frac{t-s}{t}(x-y).
\end{aligned}
$$

Thus we have shown (3.19).

(ii) For $0 < s \le t/2$, we have by (3.18) and (3.4)

$$
\begin{aligned}
(3.21)\quad & E_x[|X(s)-x|^a \mid X(t)=y] \\
&= p(t,x-y)^{-1}E_x[p(t-s,X(s)-y)|X(s)-x|^a] \\
&= p(t,x-y)^{-1}\int |z-x|^a p(t-s,z-y)p(s,z-x)dz \\
&= p(t,x-y)^{-1}\int |z|^a p(t-s,z+x-y)p(s,z)dz \\
&\le p(t,x-y)^{-1}\int (\frac{s}{t}|x-y|+|z+\frac{s}{t}(x-y)|)^a \\
&\qquad\qquad \times p(t-s,z+x-y)p(s,z)dz \\
&\le \max\{1,2^{a-1}\}p(t,x-y)^{-1}\int ((\frac{s}{t})^a|x-y|^a+|z+\frac{s}{t}(x-y)|^a) \\
&\qquad\qquad \times p(t-s,z+x-y)p(s,z)dz \\
&\le \max\{1,2^{a-1}\}(|x-y|^a+C(a)s^{a/2}).
\end{aligned}
$$

For the last inequality of (3.21) we have used

$$
\begin{aligned}
&\int |z+\frac{s}{t}(x-y)|^a p(t-s,z+x-y)p(s,z)dz \\
&= (2\pi(t-s)2\pi s)^{-1/2}\int |z+\frac{s}{t}(x-y)|^a \exp(-\frac{(z+x-y)^2}{2(t-s)}-\frac{z^2}{2s})dz \\
&= (2\pi(t-s)2\pi s)^{-1/2} \\
&\quad \times \int |z+\frac{s}{t}(x-y)|^a \exp(-\frac{t}{2(t-s)s}(z+\frac{s}{t}(x-y))^2)e^{-(x-y)^2/2t}dz \\
&= C(a)(\frac{s}{t})^{a/2}(t-s)^{a/2}p(t,x-y) \\
&\le C(a)s^{a/2}p(t,x-y).
\end{aligned}
$$

For $t/2 \leq s < t$, we have similarly

$$
\begin{aligned}
(3.22)\qquad & E_x[|X(s) - y|^a \mid X(t) = y] \\
&= p(t, x-y)^{-1} E_x[p(t-s, X(s) - y)|X(s) - y|^a] \\
&= p(t, x-y)^{-1} \int |z-y|^a p(t-s, z-y) p(s, z-x) dz \\
&= p(t, x-y)^{-1} \int |z|^a p(t-s, z) p(s, z+y-x) dz \\
&\leq \max\{1, 2^{a-1}\}(|x-y|^a + C(a)(t-s)^{a/2}).
\end{aligned}
$$

Integrating (3.21) and (3.22) yields (3.20). This ends the proof of Lemma 3.2.

Now we complete the proof of Theorem 2.1, using Lemmas 3.1 and 3.2. To do so first we use (3.4) to get that for $a > 0$,

$$
(3.23)\quad \| \int p(t, \bullet - y) | \bullet - y|^a |f(y)| dy \|_p \leq C(a) t^{a/2} \|f\|_p, \qquad f \in L^p,
$$

for $1 \leq p \leq \infty$. This is obvious for $p = \infty$ and seen for $1 \leq p < \infty$ by the Hölder inequality.

Then for $m = 0$ we have from Lemma 3.1, (3.14), with Lemma 3.2 (ii), (3.20)

$$
\begin{aligned}
(3.24)\qquad |d(t, x, y)| &= |E_x[v(t, x, y) \mid X(t) = y]| \\
&\leq C\Big(\int_0^{t/2} E_x[|X(s) - x|^\kappa \mid X(t) = y] ds \\
&\qquad + \int_{t/2}^t E_x[|X(s) - y|^\kappa \mid X(t) = y] ds\Big) \\
&= |x-y|^\kappa O(t) + O(t^{1+\kappa/2}).
\end{aligned}
$$

Hence for $m = 0$ it follows with (3.23) that

$$
(3.25)\qquad \| \int |q(t, \bullet, y)||f(y)| dy \|_p \leq O(t^{1+\kappa/2}) \|f\|_p.
$$

For $m \geq 1$ we note here by Lemma 3.1, (3.15b) and Lemma 3.2 (i), (3.19) that

(3.26) $d_{11}(t, x, y)$

$$
\begin{aligned}
&= E_x[v_{11}(t,x,y) \mid X(t)=y] \\
&= \Big\{V'(x)\int_0^{t/2} E_x[X(s)-x \mid X(t)=y]ds \\
&\qquad\qquad + V'(y)\int_{t/2}^{t} E_x[X(s)-y \mid X(t)=y]ds\Big\}e^{-\frac{t}{2}(V(x)+V(y))} \\
&= -\frac{t}{8}(x-y)(V'(x)-V'(y))e^{-\frac{t}{2}(V(x)+V(y))}.
\end{aligned}
$$

Here it is crucial in this paper that we don't take the absolute values of $X(s)-x$ and $X(s)-y$ inside the conditional expectation $E_x[\,\cdot \mid X(t)=y]$ in the third member of (3.26), so as to use Lemma 3.2 (i), (3.19).

For $m=1$, we have $d(t,x,y)=d_1(t,x,y)+d_3(t,x,y)$ with $d_1(t,x,y)=d_{11}(t,x,y)+d_{13}(t,x,y)$, so that (3.9) turns out $q(t,x,y)=q_1(t,x,y)+q_3(t,x,y)$. We have from (3.26) with (2.10)

$$
|d_{11}(t,x,y)| \le |x-y|^{1+\kappa}O(t). \tag{3.27a}
$$

We have from Lemma 3.1, (3.15d) and (3.17) with Lemma 3.2 (ii), (3.20)

$$
\begin{aligned}
&|d_{13}(t,x,y)| \\
&= |E_x[v_{13}(t,x,y) \mid X(t)=y]| \\
&\le C\Big(\int_0^{t/2} E_x[|X(s)-x|^{1+\kappa} \mid X(t)=y]ds \\
&\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^{1+\kappa} \mid X(t)=y]ds\Big) \\
&= |x-y|^{1+\kappa}O(t) + O(t^{1+(1+\kappa)/2}),
\end{aligned} \tag{3.27b}
$$

and

$$
\begin{aligned}
&|d_3(t,x,y)| \\
&= |E_x[v_3(t,x,y) \mid X(t)=y]| \\
&\le O(t^{-1+2\delta})\Big(\int_0^{t/2} E_x[|X(s)-x|^2 \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^2 \mid X(t)=y]ds\Big) \\
&\quad + O(t)\Big(\int_0^{t/2} E_x[|X(s)-x|^{2(1+\kappa)} \mid X(t)=y]ds
\end{aligned} \tag{3.28}
$$

$$+\int_{t/2}^{t} E_x[|X(s)-y|^{2(1+\kappa)} \mid X(t)=y]ds\Big)$$
$$= |x-y|^2 O(t^{2\delta}) + O(t^{1+2\delta}) + |x-y|^{2(1+\kappa)} O(t^2) + O(t^{3+\kappa}).$$

Thus for $m=1$ it follows from (3.27ab) and (3.28) by (3.9) and (3.23) that

$$\|\int |q(t,\bullet,y)||f(y)|dy\|_p \le O(t^{1+2\delta\wedge\frac{1+\kappa}{2}})\|f\|_p. \tag{3.29}$$

For $m \ge 2$ we have from (3.26) by Taylor's theorem with (2.9), (2.10) and (3.13)

$$\begin{aligned} |d_{11}(t,x,y)| &= |E_x[v_{11}(t,x,y) \mid X(t)=y]| \\ &\le \sum_{l=2}^{m} |x-y|^l O(t^{l\delta}) + |x-y|^{m+\kappa} O(t). \end{aligned} \tag{3.30a}$$

We have from Lemma 3.1, (3.15cd), (3.16) and (3.17) with Lemma 3.2 (ii), (3.20)

$$\begin{aligned} &|d_{12}(t,x,y)| \\ &= |E_x[v_{12}(t,x,y) \mid X(t)=y]| \\ &\le \sum_{k=2}^{m} O(t^{-1+k\delta})\Big(\int_0^{t/2} E_x[|X(s)-x|^k \mid X(t)=y]ds \\ &\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^k \mid X(t)=y]ds\Big) \\ &= \sum_{k=2}^{m}(|x-y|^k O(t^{k\delta}) + O(t^{(1+2\delta)k/2})); \end{aligned} \tag{3.30b}$$

$$\begin{aligned} &|d_{13}(t,x,y)| \\ &= |E_x[v_{13}(t,x,y) \mid X(t)=y]| \\ &\le \frac{C}{m!}\Big(\int_0^{t/2} E_x[|X(s)-x|^{m+\kappa} \mid X(t)=y]ds \\ &\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^{m+\kappa} \mid X(t)=y]ds\Big) \\ &= |x-y|^{m+\kappa} O(t) + O(t^{1+(m+\kappa)/2}); \end{aligned} \tag{3.30c}$$

and

$$
\begin{aligned}
(3.31)\qquad & |d_2(t,x,y)| \\
&= |E_x[v_2(t,x,y) \mid X(t)=y]| \\
&\le \sum_{j=2}^{m}\Big\{\sum_{k=1}^{m} O(t^{-1+jk\delta})\Big(\int_0^{t/2} E_x[|X(s)-x|^{jk} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^{jk} \mid X(t)=y]ds\Big) \\
&\quad + O(t^{j-1})\Big(\int_0^{t/2} E_x[|X(s)-x|^{j(m+\kappa)} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^{j(m+\kappa)} \mid X(t)=y]ds\Big)\Big\} \\
&= \sum_{j=2}^{m}\Big\{\sum_{k=1}^{m}(|x-y|^{jk}O(t^{jk\delta}) + O(t^{(1+2\delta)jk/2})) \\
&\qquad\qquad + |x-y|^{j(m+\kappa)}O(t^{j}) + O(t^{(m+2+\kappa)j/2})\Big\};
\end{aligned}
$$

$$
\begin{aligned}
(3.32)\qquad & |d_3(t,x,y)| \\
&= E_x[v_3(t,x,y) \mid X(t)=y]| \\
&\le \sum_{k=1}^{m} O(t^{-1+(m+1)k\delta})\Big(\int_0^{t/2} E_x[|X(s)-x|^{(m+1)k} \mid X(t)=y]ds \\
&\qquad\qquad + \int_0^{t/2} E_x[|X(s)-y|^{(m+1)k} \mid X(t)=y]ds\Big) \\
&\quad + O(t^{m})\Big(\int_0^{t/2} E_x[|X(s)-x|^{(m+1)(m+\kappa)} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E_x[|X(s)-y|^{(m+1)(m+\kappa)} \mid X(t)=y]ds\Big) \\
&= \sum_{k=1}^{m}(|x-y|^{(m+1)k}O(t^{(m+1)k\delta}) + O(t^{(1+2\delta)(m+1)k/2})) \\
&\qquad\qquad + |x-y|^{(m+1)(m+\kappa)}O(t^{m+1}) + O(t^{(m+1)(1+(m+\kappa)/2)}).
\end{aligned}
$$

Thus for $m \ge 2$ it follows from (3.30abc), (3.31) and (3.32) by (3.9) and

(3.23) that

$$\begin{aligned}(3.33)\qquad &\|\int |q(t,\bullet,y)\||f(y)|dy\|_p \\ &\leq \Big\{\sum_{l=2}^{m} O(t^{(1+2\delta)l/2}) + O(t^{1+(m+\kappa)/2}) \\ &\quad + \sum_{k=2}^{m} O(t^{(1+2\delta)k/2}) + O(t^{1+(m+\kappa)/2}) \\ &\quad + \sum_{j=2}^{m+1}(\sum_{k=1}^{m} O(t^{(1+2\delta)jk/2}) + O(t^{(m+2+\kappa)j/2}))\Big\}\|f\|_p \\ &= O(t^{1+2\delta})\|f\|_p.\end{aligned}$$

This ends the proof of Theorem 2.1.

Remark. In view of (3.9) we have obtained, with (3.24) for $m = 0$, (3.27ab)–(3.28) for $m = 1$ and (3.30abc)–(3.31)–(3.32) for $m \geq 2$ above, an estimate of the integral kernel $q(t, x, y)$ in terms of $p(t, x - y)$ times a finite positive linear combination of powers of $|x - y|$ and t.

§4. Proof of Theorem 2.3

Put

$$(4.1)\qquad Q^r(t) = e^{-tV/2}e^{-tH_0^r}e^{-tV/2} - e^{-t(H_0^r+V)}, \quad t > 0.$$

For f in C_0^∞, $Q^r(t)f$ has the same representation as $Q(t)f$ in (3.2) and (3.5ab) with E_x replaced by E_x^r and $p(t,x)$ replaced by

$$(4.2)\quad p^r(t,x) = 2(2\pi)^{-(d+1)/2}te^t(\sqrt{x^2+t^2})^{-(d+1)/2}K_{(d+1)/2}(\sqrt{x^2+t^2}),$$

which is the integral kernel of $e^{-tH_0^r}$ (e.g. Ichinose [7, (2.4a), p. 269]). Here $K_\nu(z)$ is the modified Bessel function of the third kind of order ν. It has the following integral representations:

$$\begin{aligned}K_\nu(t) &= \frac{1}{2}\int_0^\infty s^{-\nu-1}e^{-\frac{t}{2}(s+1/s)}ds \\ &= \frac{1}{2}t^\nu \int_0^\infty s^{-\nu-1}e^{-\frac{1}{2}(s+t^2/s)}ds \\ &= \pi^{1/2}\Gamma(\nu+1/2)^{-1}(2t)^{-\nu}e^{-t}\int_0^\infty e^{-s}s^{\nu-1/2}(s+2t)^{\nu-1/2}ds,\end{aligned}$$

where the first and second expressions hold for ν real and the last one for $\nu > -1/2$ (see [3, 7.12. (23), p. 82; 7.3.4. (16), p. 19]). Hence we have

$$
\begin{aligned}
(4.2)' \qquad p^r(t,x) &= (2\pi)^{-(d+1)/2} t e^t \int_0^\infty s^{-(d+3)/2} e^{-\frac{1}{2}(s+(x^2+t^2)/s)} ds \\
&= (2\pi)^{-1/2} t e^t \int_0^\infty p(s,x) s^{-3/2} e^{-\frac{1}{2}(s+t^2/s)} ds,
\end{aligned}
$$

where $p(t,x)$ is the heat kernel in (3.3), so that

$$
(4.3) \qquad \int |x|^a p^r(t,x)dx = C'(a) t^{(a+1)/2} e^t K_{(1-a)/2}(t),
$$

where $C'(a) = (2/\pi)^{1/2} C(a)$ with $C(a)$ in (3.4).

To avoid notational complexity we shall assume $d = 1$ again; there is no essential change in the proof in the multi-demensional case.

The proof proceeds in the same way as that of Theorem 2.1. The integral kernel $q^r(t,x,y)$ of $Q^r(t)$ in (4.1) is decomposed in the same way as in (3.9) with $p(t,x-y)$ on the right replaced by $p^r(t,x-y)$ in (4.2). We have the same statements as in Lemma 3.1, but have to replace Lemma 3.2 by the following lemma. We note again that the formula (3.18) holds with $E_x[\cdot \mid X(t) = y]$, $p(t,x-y)$ and $p(t-s, X(s)-y)$ replaced by $E_x^r[\cdot \mid X(t) = y]$, $p^r(t,x-y)$ and $p^r(t-s, X(s)-y)$, respectively.

LEMMA 4.1. *Let $t > 0$.*

(i) *Let $0 < s < t$.*

$$
(4.4) \qquad
\begin{aligned}
E_x^r[X(s) - x \mid X(t) = y] &= \tfrac{s}{t}(y-x); \\
E_x^r[X(s) - y \mid X(t) = y] &= \tfrac{t-s}{t}(x-y).
\end{aligned}
$$

(ii) *Let $a > 0$ and $0 < t \le 1$.*

$$
(4.5) \qquad
\begin{aligned}
&\int_0^{t/2} E_x^r[|X(s) - x|^a \mid X(t) = y] ds \\
&\qquad \le \begin{cases} t|x-y|^a + C_1(a) t^{a+1}, & 0 < a < 1, \\ t|x-y| + C_1(1) t^2 (|\ln t| + 1), & a = 1, \\ 2^{a-1}(t|x-y|^a + C_1(a) t^2), & a > 1; \end{cases} \\
&\int_{t/2}^t E_x^r[|X(s) - y|^a \mid X(t) = y] ds \\
&\qquad \le \begin{cases} t|x-y|^a + C_1(a) t^{a+1}, & 0 < a < 1, \\ t|x-y| + C_1(1) t^2 (|\ln t| + 1), & a = 1, \\ 2^{a-1}(t|x-y|^a + C_1(a) t^2), & a > 1. \end{cases}
\end{aligned}
$$

Here $C_1(a)$ is a constant depending on $a > 0$.

Proof. (i) By (3.18) we have, using the Parseval equality and integrating by parts,

$$\begin{aligned}
&E_x^r[X(s) - x \mid X(t) = y] \\
&\quad = p^r(t, x-y)^{-1} E_x[p^r(t-s, X(s)-y)(X(s)-x)] \\
&\quad = p^r(t, x-y)^{-1} \int (z-x) p^r(t-s, z-y) p^r(s, z-x) dz \\
&\quad = (2\pi)^{-1} p^r(t, x-y)^{-1} \\
&\qquad \times \int e^{-ipy} e^{-(t-s)(\sqrt{p^2+1}-1)} \overline{e^{-ipx}(i\partial_p) e^{-s(\sqrt{p^2+1}-1)}} dp \\
&\quad = (2\pi)^{-1} p^r(t, x-y)^{-1} \\
&\qquad \times \int e^{ip(x-y)} e^{-(t-s)(\sqrt{p^2+1}-1)} \frac{isp}{\sqrt{p^2+1}} e^{-s(\sqrt{p^2+1}-1)} dp \\
&\quad = (2\pi)^{-1} p^r(t, x-y)^{-1} \int e^{ip(x-y)} \frac{isp}{\sqrt{p^2+1}} e^{-t(\sqrt{p^2+1}-1)} dp \\
&\quad = (2\pi)^{-1} p^r(t, x-y)^{-1} \int e^{ip(x-y)} \frac{s}{t} (-i\partial_p) e^{-t(\sqrt{p^2+1}-1)} dp \\
&\quad = (2\pi)^{-1} \frac{s}{t} p^r(t, x-y)^{-1} \int (i\partial_p e^{ip(x-y)}) e^{-t(\sqrt{p^2+1}-1)} dp \\
&\quad = -\frac{s}{t}(x-y) p^r(t, x-y)^{-1} (2\pi)^{-1} \int e^{ip(x-y)} e^{-t(\sqrt{p^2+1}-1)} dp \\
&\quad = \frac{s}{t}(y-x).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&E_x^r[X(s) - y \mid X(t) = y] \\
&\quad = p^r(t, x-y)^{-1} E_x[p^r(t-s, X(s)-y)(X(s)-y)] \\
&\quad = p^r(t, x-y)^{-1} \int (z-y) p^r(t-s, z-y) p^r(s, z-x) dz \\
&\quad = (2\pi)^{-1} p^r(t, x-y)^{-1} \\
&\qquad \times \int e^{-ipy} (i\partial_p e^{-(t-s)(\sqrt{p^2+1}-1)}) \overline{e^{-ipx} e^{-s(\sqrt{p^2+1}-1)}} dp \\
&\quad = -(2\pi)^{-1} p^r(t, x-y)^{-1} \int e^{ip(x-y)} \frac{t-s}{t} (-i\partial_p) e^{-t(\sqrt{p^2+1}-1)} dp \\
&\quad = -(2\pi)^{-1} \frac{t-s}{t} p^r(t, x-y)^{-1} \int (i\partial_p e^{ip(x-y)}) e^{-t(\sqrt{p^2+1}-1)} dp
\end{aligned}$$

$$
\begin{aligned}
&= \frac{t-s}{t}(x-y)p^r(t,x-y)^{-1}(2\pi)^{-1}\int e^{ip(x-y)}e^{-t(\sqrt{p^2+1}-1)}dp \\
&= \frac{t-s}{t}(x-y).
\end{aligned}
$$

Thus we have shown (4.4).

(ii) First note by the integral representation of K_ν that for $0 < t \le 1$

$$
0 < K_\nu(t) = K_{-\nu}(t) = \begin{cases} O(|\ln t| + 1), & \nu = 0, \\ O(t^{-\nu}), & \nu > 0, \end{cases} \tag{4.6}
$$

and that for $\nu \ge 0$ and $t/2 \le \tau \le t$

$$
\begin{aligned}
K_\nu(\sqrt{x^2+t^2}) &\le K_\nu(\sqrt{x^2+\tau^2}) \\
&\le \max\{2^{1/2}, 2^\nu\}e^{t-\tau}K_\nu(\sqrt{x^2+t^2}).
\end{aligned} \tag{4.7}
$$

In fact, (4.6) is easy to see by the last expression of $K_\nu(t)$. The first half of (4.7), $K_\nu(\sqrt{x^2+t^2}) \le K_\nu(\sqrt{x^2+\tau^2})$, is evident because $K_\nu(t)$ is decreasing in t. For the second half of (4.7), we see for $\nu - 1/2 \ge 0$ that

$$
\begin{aligned}
&K_\nu(\sqrt{x^2+\tau^2}) \\
&\quad = \pi^{1/2}\Gamma(\nu+1/2)^{-1}(2\sqrt{x^2+\tau^2})^{-\nu}e^{-\sqrt{x^2+\tau^2}} \\
&\qquad\qquad \times \int_0^\infty e^{-s}s^{\nu-1/2}(s+2\sqrt{x^2+\tau^2})^{\nu-1/2}ds \\
&\quad \le \pi^{1/2}\Gamma(\nu+1/2)^{-1}(2\frac{\sqrt{x^2+t^2}}{2})^{-\nu}e^{\sqrt{x^2+t^2}-\sqrt{x^2+\tau^2}}e^{-\sqrt{x^2+t^2}} \\
&\qquad\qquad \times \int_0^\infty e^{-s}s^{\nu-1/2}(s+2\sqrt{x^2+t^2})^{\nu-1/2}ds \\
&\quad \le 2^\nu e^{t-\tau}K_\nu(\sqrt{x^2+t^2}),
\end{aligned}
$$

and for $\nu - 1/2 < 0$ that

$$
\begin{aligned}
&K_\nu(\sqrt{x^2+\tau^2}) \\
&\quad = \pi^{1/2}\Gamma(\nu+1/2)^{-1}(2\sqrt{x^2+\tau^2})^{-\nu}e^{-\sqrt{x^2+\tau^2}} \\
&\qquad\qquad \times \int_0^\infty e^{-s}s^{\nu-1/2}(s+2\sqrt{x^2+\tau^2})^{\nu-1/2}ds \\
&\quad \le \pi^{1/2}\Gamma(\nu+1/2)^{-1}(2\frac{\sqrt{x^2+t^2}}{2})^{-\nu}e^{\sqrt{x^2+t^2}-\sqrt{x^2+\tau^2}}e^{-\sqrt{x^2+t^2}}
\end{aligned}
$$

$$\times \int_0^\infty e^{-s} s^{\nu-1/2} (\frac{s + 2\sqrt{x^2+t^2}}{2})^{\nu-1/2} ds$$
$$\leq 2^{1/2} e^{t-\tau} K_\nu(\sqrt{x^2+t^2}),$$

because $\sqrt{x^2+t^2} - \sqrt{x^2+\tau^2} \leq t - \tau$. Thus we have confirmed (4.7).

Now we show (4.5). Let $0 < t \leq 1$. For $0 < s \leq t/2$, we have by (3.18), (4.3) and (4.6), noting that

$$\int p^r(t-s, z+x-y) p^r(s,z) dz = p^r(t, x-y),$$

$$\begin{aligned}
(4.8)\quad & E_x^r[|X(s)-x|^a \mid X(t) = y] \\
&= p^r(t,x-y)^{-1} E_x^r[p^r(t-s, X(s)-y)|X(s)-x|^a] \\
&= p^r(t,x-y)^{-1} \int |z-x|^a p^r(t-s,z-y) p^r(s,z-x) dz \\
&= p^r(t,x-y)^{-1} \\
&\qquad \times \Big(\int_{|z-y|\leq|x-y|} + \int_{|z-y|>|x-y|} \Big) |z-x|^a p^r(t-s,z-y) p^r(s,z-x) dz \\
&\leq p^r(t,x-y)^{-1} \\
&\qquad \times \Big(\int_{|z-y|\leq|x-y|} (|z-y|+|x-y|)^a p^r(t-s,z-y) p^r(s,z-x) dz \\
&\qquad + \int_{|z-y|>|x-y|} |z-x|^a p^r(t-s,z-y) p^r(s,z-x) dz \Big) \\
&\leq \max\{1, 2^{a-1}\} p^r(t,x-y)^{-1} \int_{|z-y|\leq|x-y|} (|z-y|^a + |x-y|^a) \\
&\qquad\qquad \times p^r(t-s,z-y) p^r(s,z-x) dz \\
&\qquad + p^r(t,x-y)^{-1} p^r(t-s,x-y) \int_{|z-y|>|x-y|} |z-x|^a p^r(s,z-x) dz \\
&\leq \max\{1, 2^{a-1}\} 2|x-y|^a + C'(a) \frac{p^r(t-s,x-y)}{p^r(t,x-y)} s^{(a+1)/2} e^s K_{(1-a)/2}(s) \\
&\leq \begin{cases} 2|x-y|^a + C_1(a) s^a, & 0 < a < 1, \\ 2|x-y| + C_1(a) s(|\ln s| + 1), & a = 1, \\ 2^{a-1}(2|x-y|^a + C_1(a) s), & a > 1. \end{cases}
\end{aligned}$$

Here $C_1(a)$ is a constant depending on a and we have used the fact that

$$\sup_{t/2 \leq \tau \leq t} \sup_z \frac{p^r(\tau, z)}{p^r(t,z)} \leq 2^d,$$

which we can see in virtue of (4.7), observing the expression (4.2) of $p^r(t,x)$.

For $t/2 \le s < t$, we have similarly

$$
\begin{aligned}
(4.9)\quad & E^r_x[|X(s)-y|^a \mid X(t)=y] \\
&= p^r(t,x-y)^{-1}E^r_x[p^r(t-s,X(s)-y)|X(s)-y|^a] \\
&= p^r(t,x-y)^{-1}\int |z-y|^a p^r(t-s,z-y)p^r(s,z-x)dz \\
&= p^r(t,x-y)^{-1} \\
&\qquad \times\Big(\int_{|z-x|\le|x-y|}+\int_{|z-x|>|x-y|}\Big)|z-y|^a p^r(t-s,z-y)p^r(s,z-x)dz \\
&\le \max\{1,2^{a-1}\}2|x-y|^a \\
&\qquad +C'(a)\frac{p^r(s,x-y)}{p^r(t,x-y)}(t-s)^{(a+1)/2}e^{t-s}K_{(1-a)/2}(t-s) \\
&\le \begin{cases} 2|x-y|^a + C_1(a)(t-s)^a, & 0<a<1, \\ 2|x-y| + C_1(a)(t-s)(|\ln(t-s)|+1), & a=1, \\ 2^{a-1}(2|x-y|^a + C_1(a)(t-s)), & a>1. \end{cases}
\end{aligned}
$$

Integrating (4.8) and (4.9) yields (4.5). This ends the proof of Lemma 4.1.

Now we can prove Theorem 2.3, using Lemmas 3.1 and 4.1.

To do so first we use (4.3) to get that for $a > 0$,

$$
\begin{aligned}
(4.10)\quad & \|\int p^r(t,\bullet-y)|\bullet-y|^a|f(y)|dy\|_p \\
&\qquad\qquad \le C'(a)t^{(1+a)/2}e^tK_{(1-a)/2}(t)\|f\|_p, \qquad f\in L^p,
\end{aligned}
$$

for $1 \le p \le \infty$. This is obvious for $p=\infty$ and seen for $1 \le p < \infty$ by the Hölder inequality.

Then for $m = 0$ we have from Lemma 3.1, (3.14), with Lemma 4.1 (ii), (4.5)

$$
\begin{aligned}
(4.11)\qquad |d(t,x,y)| &= |E^r_x[v(t,x,y) \mid X(t)=y]| \\
&\le C\Big(\int_0^{t/2} E^r_x[|X(s)-x|^\kappa \mid X(t)=y]ds \\
&\qquad + \int_{t/2}^t E^r_x[|X(s)-y|^\kappa \mid X(t)=y]ds\Big) \\
&= \begin{cases} |x-y|^\kappa O(t) + O(t^{1+\kappa}), & 0\le\kappa<1, \\ |x-y|O(t)+O(t^2|\ln t|), & \kappa=1. \end{cases}
\end{aligned}
$$

Hence for $m = 0$ it follows with (4.10) that

$$(4.12)\quad \|\int |q^r(t,\bullet,y)||f(y)|dy\|_p \le \begin{cases} O(t^{1+\kappa})\|f\|_p, & 0 \le \kappa < 1, \\ O(t^2|\ln t|)\|f\|_p, & \kappa = 1. \end{cases}$$

For $m \ge 1$ we note here by Lemma 4.1 (i), (4.4) that (3.26) again holds with $E_x[\cdot \mid X(t) = y]$ replaced by $E_x^r[\cdot \mid X(t) = y]$.

For $m = 1$, we have $d(t,x,y) = d_1(t,x,y) + d_3(t,x,y)$ with $d_1(t,x,y) = d_{11}(t,x,y)+d_{13}(t,x,y)$, so that (3.9) turns out again $q(t,x,y) = q_1(t,x,y)+q_3(t,x,y)$. We have from (3.26) with (2.10)

$$(4.13\text{a})\quad |d_{11}(t,x,y)| = |E_x^r[v_{11}(t,x,y) \mid X(t) = y]| \le |x-y|^{1+\kappa}O(t).$$

We have from Lemma 3.1, (3.15d) and (3.17) with Lemma 4.1 (ii), (4.5)

$$\begin{aligned}(4.13\text{b})\quad |d_{13}(t,x,y)| &= |E_x^r[v_{13}(t,x,y) \mid X(t) = y]| \\ &\le C\Big(\int_0^{t/2} E_x^r[|X(s)-x|^{1+\kappa} \mid X(t) = y]ds \\ &\qquad + \int_{t/2}^t E_x^r[|X(s)-y|^{1+\kappa} \mid X(t) = y]ds\Big) \\ &= \begin{cases} |x-y|O(t) + O(t^2|\ln t|), & \kappa = 0, \\ |x-y|^{1+\kappa}O(t) + O(t^2), & 0 < \kappa \le 1, \end{cases}\end{aligned}$$

and

$$\begin{aligned}(4.14)\quad & |d_3(t,x,y)| \\ &= |E_x^r[v_3(t,x,y) \mid X(t) = y]| \\ &\le O(t^{-1+2\delta})\Big(\int_0^{t/2} E_x^r[|X(s)-x|^2 \mid X(t) = y]ds \\ &\qquad + \int_{t/2}^t E_x^r[|X(s)-y|^2 \mid X(t) = y]ds\Big) \\ &\quad + O(t)\Big(\int_0^{t/2} E_x^r[|X(s)-x|^{2(1+\kappa)} \mid X(t) = y]ds \\ &\qquad + \int_{t/2}^t E_x^r[|X(s)-y|^{2(1+\kappa)} \mid X(t) = y]ds\Big) \\ &= |x-y|^2O(t^{2\delta}) + O(t^{1+2\delta}) + |x-y|^{2(1+\kappa)}O(t^2) + O(t^3).\end{aligned}$$

Thus for $m = 1$ it follows from (4.13ab) and (4.14) by (4.10) and (3.9) with $q(t, x, y)$ and $p(t, x - y)$ replaced by $q^r(t, x, y)$ and $p^r(t, x - y)$ that

$$\| \int |q^r(t, \bullet, y)| |f(y)| dy \|_p \leq \begin{cases} O(t^{1+2\delta} \vee t^2 |\ln t|) \|f\|_p, & \kappa = 0, \\ O(t^{1+2\delta \wedge 1}) \|f\|_p, & 0 < \kappa \leq 1. \end{cases} \tag{4.15}$$

For $m \geq 2$ we have from (3.26) by Taylor's theorem with (2.9), (2.10) and (3.13)

$$\begin{aligned} |d_{11}(t, x, y)| &= |E_x^r[v_{11}(t, x, y) \mid X(t) = y]| \\ &\leq \sum_{l=2}^{m} |x - y|^l O(t^{l\delta}) + |x - y|^{m+\kappa} O(t). \end{aligned} \tag{4.16a}$$

We have

$$\begin{aligned} &|d_{12}(t, x, y)| \\ &= |E_x^r[v_{12}(t, x, y) \mid X(t) = y]| \\ &\leq \sum_{k=2}^{m} O(t^{-1+k\delta}) \Big(\int_0^{t/2} E_x^r[|X(s) - x|^k \mid X(t) = y] ds \\ &\qquad\qquad + \int_{t/2}^{t} E_x^r[|X(s) - y|^k \mid X(t) = y] ds \Big) \\ &= \sum_{k=2}^{m} (|x - y|^k O(t^{k\delta}) + O(t^{1+k\delta})); \end{aligned} \tag{4.16b}$$

$$\begin{aligned} &|d_{13}(t, x, y)| \\ &= |E_x^r[v_{13}(t, x, y) \mid X(t) = y]| \\ &\leq \frac{C}{m!} \Big(\int_0^{t/2} E_x^r[|X(s) - x|^{m+\kappa} \mid X(t) = y] ds \\ &\qquad\qquad + \int_{t/2}^{t} E_x^r[|X(s) - y|^{m+\kappa} \mid X(t) = y] ds \Big) \\ &= |x - y|^{m+\kappa} O(t) + O(t^2); \end{aligned} \tag{4.16c}$$

and

(4.17) $|d_2(t, x, y)|$

$$
\begin{aligned}
&= |E^r_x[v_2(t,x,y) \mid X(t)=y]| \\
&\le \sum_{j=2}^{m}\Big\{\sum_{k=1}^{m} O(t^{-1+jk\delta})\Big(\int_0^{t/2} E^r_x[|X(s)-x|^{jk} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E^r_x[|X(s)-y|^{jk} \mid X(t)=y]ds\Big) \\
&\qquad +O(t^{j-1})\Big(\int_0^{t/2} E^r_x[|X(s)-x|^{j(m+\kappa)} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E^r_x[|X(s)-y|^{j(m+\kappa)} \mid X(t)=y]ds\Big)\Big\} \\
&= \sum_{j=2}^{m}\Big\{\sum_{k=1}^{m}(|x-y|^{jk}O(t^{jk\delta})+O(t^{1+jk\delta})) \\
&\qquad\qquad +|x-y|^{j(m+\kappa)}O(t^{j})+O(t^{j+1})\Big\};
\end{aligned}
$$

$$
\begin{aligned}
(4.18)\quad & |d_3(t,x,y)| \\
&= |E^r_x[v_3(t,x,y) \mid X(t)=y]| \\
&\le \sum_{k=1}^{m} O(t^{-1+(m+1)k\delta})\Big(\int_0^{t/2} E^r_x[|X(s)-x|^{(m+1)k} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E^r_x[|X(s)-y|^{(m+1)k} \mid X(t)=y]ds\Big) \\
&\qquad +O(t^{m})\Big(\int_0^{t/2} E^r_x[|X(s)-x|^{(m+1)(m+\kappa)} \mid X(t)=y]ds \\
&\qquad\qquad + \int_{t/2}^{t} E^r_x[|X(s)-y|^{(m+1)(m+\kappa)} \mid X(t)=y]ds\Big) \\
&= \sum_{k=1}^{m}(|x-y|^{(m+1)k}O(t^{(m+1)k\delta})+O(t^{1+(m+1)k\delta})) \\
&\qquad\qquad +|x-y|^{(m+1)(m+\kappa)}O(t^{m+1})+O(t^{m+2}).
\end{aligned}
$$

Thus for $m \ge 2$ it follows from (4.16abc), (4.17) and (4.18) by (4.10) and (3.9) with $q(t,x,y)$ and $p(t,x-y)$ replaced by $q^r(t,x,y)$ and $p^r(t,x-y)$ that

$$
(4.19)\qquad \|\int |q^r(t,\bullet,y)||f(y)|dy\|_p
$$

$$
\begin{aligned}
&\le \{\sum_{l=2}^{m} O(t^{1+l\delta}) + O(t^2) + \sum_{k=2}^{m} O(t^{1+k\delta}) + O(t^2) \\
&\qquad + \sum_{j=2}^{m}(\sum_{k=1}^{m} O(t^{1+jk\delta}) + O(t^{j+1})) \\
&\qquad + \sum_{k=1}^{m} O(t^{1+(m+1)k\delta}) + O(t^{m+2})\}\|f\|_p \\
&= O(t^{1+2\delta})\|f\|_p.
\end{aligned}
$$

This ends the proof of Theorem 2.3.

Remark. In view of (3.9) we have obtained, with (4.11) for $m = 0$, (4.13ab)–(4.14) for $m = 1$ and (4.16abc) – (4.17) – (4.18) for $m \ge 2$ above, an estimate of the integral kernel $q^r(t, x, y)$ in terms of $p^r(t, x - y)$ times a finite positive linear combination of $t^2|\ln t|$, powers of $|x - y|$ and t.

Finally, before closing this final section, we make a comment on the present approach compared with our previous work [8]. It dealt only with the nonrelativistic case and its calculation was done by using, via the conditional expectation $E_0[\cdot \mid X(t) = 0]$, the Brownian bridge to rewrite (3.5a). But this procedure cannot work out in the relativistic case. So in this paper we have used the conditional expectation $E_x[\cdot \mid X(t) = y]$ itself throughout, so that both the nonrelativistic and relativistic cases can be discussed in a unified way.

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