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MODULI SPACES OF STABLE VECTOR BUNDLES ON ENRIQUES SURFACES

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Abstract. We show that the image of the moduli space of stable bundles on an Enriques surface by the pull back map is a Lagrangian subvariety in the moduli space of stable bundles, which is a symplectic variety, on the covering K3 surface. We also describe singularities and some other features of it.

§0. Introduction

Moduli spaces of stable vector bundles on algebraic surfaces have been described by several authors. Vector bundles on rational surfaces ([Ba], [Hu]), ruled surfaces ([Br], [Q]), K3 surfaces ([Mu1, 2], [Ty1, 2]), elliptic surfaces ([F], [O,V]) and some surfaces of general type ([Bh], [D,K]) have been studied. In this paper we want to study the moduli spaces of stable bundles on Enriques surfaces. Every Enriques surface has a K3 surface as a universal covering space.

Mukai ([Mu1]) showed that the moduli space of stable vector bundles on any K3 surface has a symplectic structure. We will describe the moduli spaces of stable bundles on Enriques surfaces with relation to those on the corresponding K3 surfaces.

THEOREM. (1) *The image of the moduli space of stable bundles on an Enriques surface by the pull back map is a Lagrangian subvariety, exactly the fixed locus of the induced involution, in the moduli space of stable bundles, which is a symplectic variety, on the covering* K3 *surface.*

(2) *The singularities in the moduli space M of stable bundles on an Enriques surface are the images of finitely many union of the moduli spaces of the moduli spaces of stable bundles on the* K3 *surface and the dimension of singular locus is at most $\frac{1}{2}(\dim M + 3)$ (big codimensional singularity).*

(3) *The pull back map is two to one from the smooth locus of M to the moduli space of stable bundles on the K3 surface, with no branch locus.*

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§1. Preliminaries

1. An Enriques surface X is a minimal algebraic surface whose canonical divisor $K_X \not\sim 0$, but $2K_X \sim 0$, where $\sim$ denotes the linear equivalence.

2. Every Enriques surface has an elliptic structure over $\mathbf{P}^1$. It has exactly two multiple fibres of multiplicity 2, say them F_A, F_B. Then the canonical divisor can be expressed as a difference of two multiple fibres, that is $K_X \sim F_A - F_B$.

3. The fundamental group of any Enriques surface is $\mathbf{Z}_2$, so that the universal covering space is a K3 surface. Let the quotient map be π. That is an etale covering with respect to K_X. So $\pi_*(O_{\overline{X}}) \cong O_X \oplus K_X$, $\pi^*(K_X) \cong O_{\overline{X}}$.

4. An Enriques surface X is called nodal if there exists a smooth rational curve R. (In this case $R^2 = -2$.) Otherwise, it is called unnodal. In the 10 dimensional moduli space of Enriques surfaces, a generic one is unnodal, while the nodal ones form a 9 dimensional subvariety ([C,D]).

5. A nodal cycle N on an Enriques, or a K3 surface is a positive 1-cycle such that $h^1(O_N) = 0$. This is a tree of smooth rational curves ([Ar]).

6. We define the slope of E with respect to some ample divisor H, denoted by $\mu_H(E)$, as $(c_1(E) \cdot H)/\operatorname{rank}(E)$. A vector bundle E is called H(-semi)-stable if for every subsheaf F, with $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$,

$$\mu_H(F) < (\leq) \mu_H(E).$$

There exists a moduli space of stable vector bundles which is a quasi-projective algebraic variety.

7. Let X be an Enriques surface (K3 surface). Then the map c_1 : $\operatorname{Pic} X \to H^2(X, Z)$ is an isomorphism (injective). So, we identify $\operatorname{Pic} X$ with its image.

Now we fix the notations.

X is an Enriques surface and its universal covering space, which is a K3 surface, is denoted by $\overline{X}$ and the quotient map from $\overline{X}$ to X is π. Let $M_{X,H}(r, c_1, c_2)$ (resp. $M_{\overline{X},\pi^*H}(r, c_1, c_2)$) be the moduli space of stable vector bundles on X (resp. $\overline{X}$) with respect to H (resp. π^*H), where r is the rank of the bundles and c_i is the assignment of the i-th Chern class.

We denote by i the involution on $\overline{X}$ compatible to π and by i^* the induced involution on $M_{\overline{X}}$. We mean K_X by K and $E \otimes L$ by $E(L)$, where L is a line bundle and E is a vector bundle.

§2. General structure theorem

Here we interpret the results of Takemoto [Ta] in our Enriques surface X and the covering K3 surface $\overline{X}$.

THEOREM. [Ta] (1) *If a π^*H-stable bundle F on $\overline{X}$ is not isomorphic to π^*E for any bundle E on X, then $\pi_*(F)$ is H-stable. IfF is π^*H-semi-stable, then π_*F is H-semi-stable.*

(2) *If a simple bundle E on X is isomorphic to $E(K)$, then there exists a simple bundle F on X such that $\pi_*(F) \cong E$.*

Next we introduce the result of Mukai on the moduli spaces of stable bundles on K3 surfaces.

THEOREM. [Mu1] *The moduli space M of stable bundles on a* K3 *surface S is smooth and there is a line bundle $L = O_M$ and a skew-symmetric bilinear form $B : TM \times TM \to L$ such that $B \otimes k([F])$ is nondegenerate and canonically isomorphic to the natural pairing* $\mathrm{Ext}^1(F, F) \times \mathrm{Ext}^1(F, F) \to \mathrm{Ext}^2(F, F)$ *for any stable bundle F.*

Let us begin with a lemma.

LEMMA. *Let X be an Enriques surface and $\overline{X}$, be the universal covering space of X and F be a simple vector bundle on $\overline{X}$ such that $F \cong i^*F$. Then there exists a bundle E on X such that $\pi^*E \cong F$.*

Proof. It suffices to prove that there exists a map $f : F \to F$, over the involution such that $f^2 = \mathrm{id}$. For a given isomorphism $h : F \to i^*F$, let g be a composition of h followed by the natural map $j : i^*F \to F$. Then g is a map from F to F over the involution and $g^2 = \lambda Id$ for some $\lambda \neq 0$ in $\mathbf{C}$. Let f be $(\lambda)^{-1/2}g$. Then, f satisfies the required property. $\square$

Remark. This can be genralized to any bundle F whose endomorphism is $M_r(\mathbf{C})$, where r is the rank of E.

Before going into the main theorem, we recall the formula for the dimension of $M = M_X(r, c_1, c_2)$, the moduli space of stable bundles on an Enriques surface X, and the dimension of the tangent space $T_E M$ at $E \in M$.

$$\dim_E M \geq 2rc_2 - (r-1)c_1^2 - r^2 + 1$$
$$\dim T_E M = 2rC_2 - (r-1)c_1^2 - r^2 + 1 + h^2(\operatorname{End} E).$$

Here $h^2(\operatorname{End} E) = 0$ if $E \ncong E(K)$ and 1 if $E \cong E(K)$. This comes from the fact that for any non-trivial homomorphism between two stable bundles with the same slope is an isomorphism.([O,S,S]) The right hand side of the above inequality is called is the expected dimension of M.

Then we state our main result.

THEOREM. *Let $\overline{X}$ be a* K3 *surface which is the universal covering space of an Enriques surface X and M_X, (resp. $M_{\overline{X}}$) be a moduli space of stable vector bundles on X (resp. $\overline{X}$) (see §1).*

(1) *Then, M_X is singular at E if and only if $E \cong E(K)$ except the case that E belongs to a 0-dimensional component (exceptional bundle) or a two dimensional component, where every bundle E satisfies that $E \cong E(K)$. The singular locus of M_X is a union of the images by π of finitely many different components of $M^0_{\overline{X}}$ with possibly different Chern classes on $\overline{X}$, where $M^0_{\overline{X}} = \{F \in M_{\overline{X}} | F \ncong i^*F\}$ and its dimension is $\leq \frac{1}{2}(\dim M_X + 3)$. So, M_X is generically smooth. In particular, if the rank is odd, it is everywhere smooth, and if the rank $= 2$ then it can have only finitely many isolated singular points.*

(2) *The pull back map π^* from M^0_X to $M_{\overline{X}}$ is two to one with no branch, where $M^0_X = \{E \in M_X | E \ncong E(K)\}$.*

(3) *The image of M^0_X by π^* is a Lagrangian subvariety in $M_{\overline{X}}$ and is equal to the fixed locus by involution i^*.*

Proof of (1). If E is a singular point in M_X then $E \cong E(K)$, so that $E \cong \pi_* F$ for some stable bundle F on X. ([Ta], or §1) So, the rank is an even number, say $2k$. Then $\pi^* E \cong F \oplus i^* F$. Let the Chern polynomial of F be $1 + c_1(F)t + c_2(F)t^2$. Then, that of i^*F is $1 + (i^*c_1(F))t + c_2(F)t^2$. So, we have

$$\pi^* c_1(E) = c_1(F) + i^* c_1(F),$$
$$2c_2(E) = \pi^* c_2(E) = c_1(F) \cdot i^* c_1(F) + 2c_2(F).$$

Here $c_1(F)\cdot\pi^*H = i^*c_1(F)\cdot\pi^*H$ since $\pi^*H \sim i^*(\pi^*H)$. This implies that $(c_1(F)-i^*c_1(F))^2 \leq 0$ by the Hodge Index theorem (the equality holds if and only if $c_1(F) = i^*c_1(F)$). Now we can find a relationship between the dimensions of $M_X(2k, c_1(E), c_2(E))$, and $M_{\overline{X}}(k, c_1(F), c_2(F))$.

$$\begin{aligned}\dim M_X &= 4kc_2(E) - (2k-1)c_1^2(E) - 4k^2 + 1\\ &= 2k(c_1(F)\cdot i^*c_1(F)) + 4kc_2(F)\\ &\qquad -(2k-1)(c_1^2(F) + c_1(F)\cdot i^*c_1(F)) - 4k^2 + 1\\ &= 2(2kc_2(F) - (k-1)c_1^2(F) - 2k^2 + 2)\\ &\qquad +(c_1(F)\cdot i^*c_1(F) - c_1^2(F)) - 3.\end{aligned}$$

(This is a computation when M_X is of expected dimension. For the component not of expected dimension, where $E \cong E(K)$, we must add by 1.) However we know that

$$(c_1(F)\cdot i^*c_1(F) - c_1^2(F)) \geq 0,$$

where the equality holds if and only if $c_1(F) = i^*c_1(F)$. So, we can conclude that

$$\dim M_{\overline{X}}(k, c_1(F), c_2(F)) \leq \frac{1}{2}(\dim M_X(2k, c_1(E), c_2(E)) + 3),$$

where the equality holds if and only if $c_1(F) = i^*c_1(F)$. (We have $\dim M_{\overline{X}} \leq \frac{1}{2}(\dim M_X + 2)$, the equality holds if and only if $c_1(F) = i^*c_1(F)$ for the component not of expected dimension.)

From the above formula, we see that

$$c_1(F)^2 - c_1(F)\cdot i^*c_1(F) \geq -\dim M_X - 3 = B,$$

so that,

$$2B \leq (c_1(F) - i^*c_1(F))^2 \leq 0.$$

So, there can be only finitely many numbers for $(c_1(F)-i^*c_1(F))^2$ and for a fixed value there can be only finitely many choices for $c_1(F)$, since $(\pi^*H)^\perp$ is a negative definite lattice. However, π_*F is stable if and only if $F \not\cong i^*F$. So the singular locus of M_X is a finitely many union of direct images of $M^0_{\overline{X}}$. Here the map π_* from these $M_{\overline{X}}$ to the singular locus of M_X is 2 to 1 with no branch. In fact $\pi_*(F) \cong \pi_*(G)$ implies that $\pi^*(\pi_*(F)) \cong \pi^*(\pi_*(G))$. This means that $F\oplus i^*F \cong G \oplus i^*G$. So

$$\begin{aligned}&\mathrm{Hom}(F,G)\oplus\mathrm{Hom}(F,i^*G)\oplus\mathrm{Hom}(i^*F,G)\oplus\mathrm{Hom}(i^*F,i^*G)\\ &\cong \mathrm{Hom}(F\oplus i^*F, G\oplus i^*G) \neq 0.\end{aligned}$$

So, this forces that $F \cong G$, or $F \cong i^*G$. Obviously, $\pi_*F \cong \pi_*(i^*F)$.

If π_* is 1 to 1, then $F \cong i^*F$ and π_*F is not stable. The singular locus is of even dimension and smooth in itself. If the rank is odd, then M is everywhere smooth.

Conversely, if $E \cong E(K)$, then $E \cong \pi_*F$ for some F on X. If E is a smooth point in M_X, then $E \cong E(K)$ everywhere in the component M containing E. Then M is a finite union of the images of some components of $M_{\overline{X}}$ with possibly different Chern classes, which have the same dimension as M, call one of the components $\overline{M}$. From the previous formula, we get

$$\dim M = \dim \overline{M} \leq \frac{1}{2}(\dim M + 2).$$

(Note that $E \cong E(K)$ in M). So, the possible dimension of M is 0 or 2. If $\dim M = 2$, then for F in $\overline{M}, c_1(F) = i^*c_1(F)$, so that $c_1(F) = \frac{1}{2}\pi^*c_1(E)$ and if $\dim M = 0$, M has a unique bundle E and $\overline{M}$ has two bundles F and i^*F such that $c_1^2(F) = c_1(F) \cdot i^*c_1(F) - 2$. If the rank is 2, then the singular locus is the direct images of finitely many different line bundles. This completes the proof of (1). $\square$

Remark 1. In fact, the singularity of $M_X(2k, c_1, c_2)$ is closely related to the singularity of the curves in the linear system of c_1 (the splitting behaviour of the divisor of $\pi^*(c_1)$ on $\overline{X}$.)

Remark 2. There can be 3 different types of components in M_X, (1) a component M which has the expected dimension and is smooth everywhere, (2) a component M which has the expected dimension, but has some singularity, (3) a component M which has the dimension one bigger than the expected dimension (must be smooth everywhere). The singularity can exist only in the second type. The components with codimension one singularity can exist only for $\dim M = 1, 3$ or 5.

We will give those examples.

EXAMPLE 1. The simplest example of M_X with some singularities is $M_X(2, F_A, 1)$, where F_A is a half fibre. Then $M_X(2, F_A, 1) = F_B$, another half fibre. If F_B is singular with an ordinary double point, then the inverse image of F_B is a union of two smooth rational curves $R_1, R_2 = i^*(R_1)$ ($\overline{A}_1$ type). Then the bundle E corresponding to the singularity is just $\pi_*O_{\overline{X}}(R_1)$ (or R_2). Note that $\det(\pi_*O_{\overline{X}}(R_1)) \sim F_B + K \sim F_A$ ([Ha]).

EXAMPLE 2. We can find many examples of the moduli space M of dimension three whose singularity is a K3 surface. We can find a compact moduli space $M_{\overline{X}}(k, c_1, c_2)$ such that $c_1^2 = c_1 \cdot i^*c_1 - 2$ (this holds if and only if $c_1 = N + S$, where N is a nodal cycle with $N \cdot i^*N = 0$ and S is a divisor fixed by involution and $\dim M_{\overline{X}} = 2$. There are many examples with these conditions. Then $M_{\overline{X}}$ is a K3 surface and the dimension of the corresponding M_X is 3 and the singular locus is just the image of that K3 surface. (Note that $\pi_* M_{\overline{X}}(k, c_1, c_2) = \pi_* M_{\overline{X}}(k, i^*c_1, c_2)$ and $c_1 \neq i^*c_1$.)

EXAMPLE 3. If we choose $M_{\overline{X}}(k, c_1, c_2)$ of dimension 4 with $c_1 = i^*c_1$ then the image of $M^0_{\overline{X}}$ by π_* is the singular locus of 4 dimension in the 5-dimensional space, M_X.

Proof of (2). First we show that if E is H-stable and is not isomorphic to $E(K)$, then π^*E is π^*H-stable, From the fact that

$$H^0(\operatorname{End} \pi^*E) = H^0(\operatorname{End} E) \oplus H^0((\operatorname{End} E)(K)) = \mathbf{C},$$

π^*E is simple. π^*E is also a direct sum of stable bundles with the same slope (since the pull back of an Einstein-Hermitian bundle is still Einstein-Hermitian and an Einstein-Hermitian bundle is a direct sum of stable bundles with the same slope.) From these two facts π^*E must be π^*H-stable. From the above equation we conclude also that if E is isomorphic to $E(K)$, then π^*E is not simple, just a direct sum of stable bundles. (In fact, $\pi^*E \cong F \oplus i^*F$, for some F such that $\pi_*F \cong E$. ([Ta])) So, π^* is well defined from $M^0_X = \{E | E \in M_X, E \ncong E(K)\}$ to $M_{\overline{X}}$. M^0_X is the same as the smooth locus of M_X except two cases as we saw in the proof of (1). Next we show that π^* is 2 to 1 with no branch. If $\pi^*E \cong \pi^*E'$, then $H^0(\pi^*(E^* \otimes E')) \neq 0$. However,

$$H^0(\pi^*(E^* \otimes E')) = H^0(E^* \otimes E') \oplus H^0(E^* \otimes E'(K)).$$

So, either $H^0(E^* \otimes E') \neq 0$, or $H^0(E^* \otimes E'(K)) \neq 0$. The property of stability implies $E \cong E'$, or $E \cong E'(K)$. So, the map π^* is 2 to 1 with no branch if the rank is even and 1 to 1 if the rank is odd. □

Remark. (a) π^* restricted to $M^0_X(2k, D, c_2)$ or $M^0_X(2k, D + K, c_2)$ is still 2 to 1 with no branch. In general, $M_X(2k, D, c_2)$ is not isomorphic to $M_X(2k, D + K, c_2)$. For example, $M_X(2, F_A, 1) = F_B$ is not isomorphic to $M_X(2, F_B, 1) = F_A$. If an exceptional bundle E of even rank exists for $\det(E) = D$, then there is no exceptional bundle E' for $\det(E') = D + K$.

(b) However, π_* restricted to $M_X(2k+1, D, c_2)$ or $M_X(2k+1, D+K, c_2)$ is 1 to 1, so that $M_X(2k+1, D, c_2)$ $(= M_X(2k+1, D+K, c_2))$ is isomorphic to its image.

Proof of (3). First we show that the dimension of $M_X^0(r, c_1, c_2)$ is half of the dimension of $M_{\overline{X}}(r, \pi^* c_1, \pi^* c_2)$.

$$\begin{aligned} \dim M_{\overline{X}} &= 2r\pi^* c_2 - (r-1)(\pi^* c_1)^2 - 2r^2 + 2 \\ &= 2(2rc_2 - (r-1)c_1^2 - r^2 + 1) = 2 \dim M_X^0. \end{aligned}$$

Next we show that the pull back of the holomorphic two form ω on $M_{\overline{X}}$ to M_X^0 vanishes. The proof comes easily from the following commuting diagram,

$$\begin{array}{ccccc} \mathrm{Ext}^1(\pi^* E, \pi^* E) & \times & \mathrm{Ext}^1(\pi^* E, \pi^* E) & \to & \mathrm{Ext}^2(\pi^* E, \pi^* E) \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Ext}^1(E, E) & \times & \mathrm{Ext}^1(E, E) & \to & \mathrm{Ext}^2(E, E), \end{array}$$

and the fact that $\mathrm{Ext}^2(E, E) = H^2(\mathrm{End}\, E) = 0$, for any $E \in M_X^0$. (In the above diagram, $\mathrm{Ext}^1(E, E) = T_E M_X$ and $\mathrm{Ext}^1(\pi^* E, \pi^* E) = T_{\pi^* E} M_{\overline{X}}$.) So, we can also conclude that the image of M_X^0 is a Lagrangian subvariety in $M_{\overline{X}}$. That the image of M_X^0 is fixed by involution i^* is obvious. Another direction comes from the lemma easily. So the image is exactly the fixed locus by involution. This completes the proof of (3). □

Remark 1. We expect that $M_{\overline{X}}$ is birational to the cotangent bundle of the image of M_X^0 by π^*.

Remark 2. We know that the dimension of $M_X(2k+1, c_1, c_2)$ is even. We expect that $M_X(2k+1, D, c_2)$ is birational to a symmetric power of some Enriques surface. In fact we know many cases that $M_{\overline{X}}$ is birational to a symmetric power of some K3 surface. In this case, the image of M_X is just the fixed locus by involution, so that it is a symmetric power of an Enriques surface, the quotient of that K3 surface. Another example is $M_X(3, c_1, 3)$, where X is a fourfold covering of $\mathbf{P}^2$ and c_1 is a pull back of hyperplane of $\mathbf{P}^2$. Then $c_1^2 = 4$. In this case M is birational to the original Enriques surface X.

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